Time Series Analysis

Concepts, Methods, and Applications

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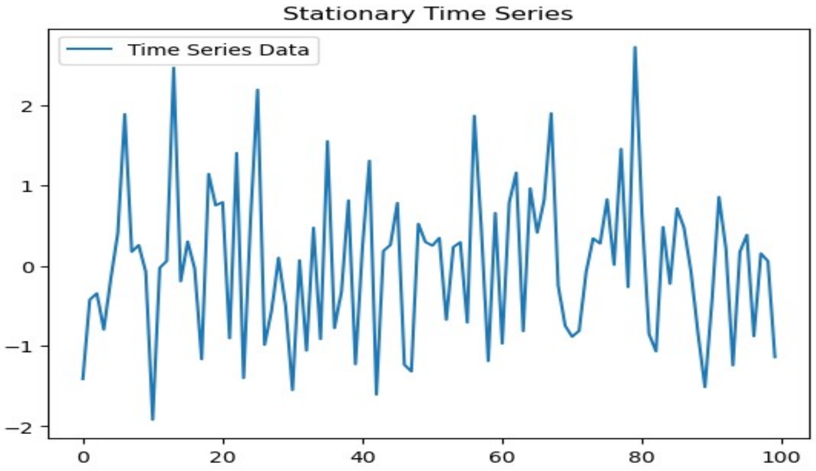
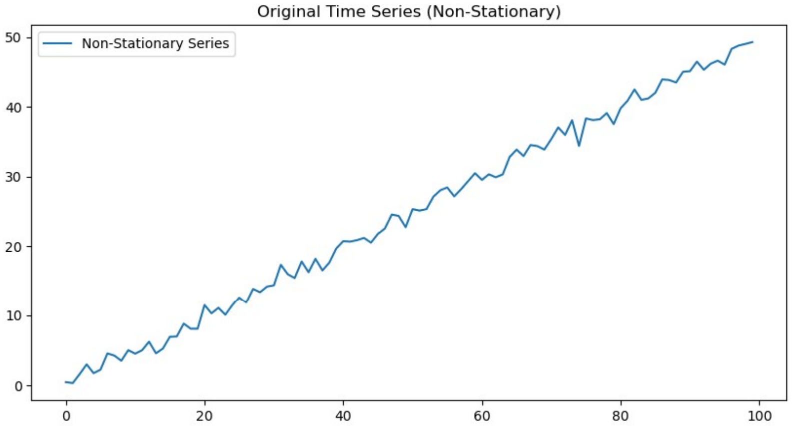
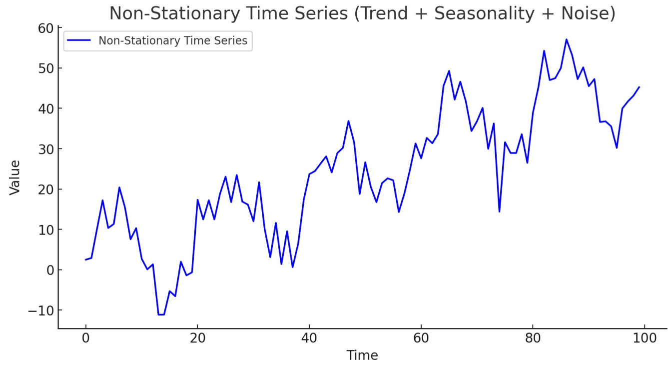
What is Time Series Analysis?

- A **time series** is a sequence of data points collected or recorded at successive time intervals. It represents how a variable changes over time.

Type of Time series







Importance of Time Series Analysis

Studying time series is important for several reasons, especially in fields like finance, economics, engineering, and science.

1. Understanding Trends and Patterns

- Time series analysis helps identify long-term trends, seasonal patterns, and cyclical behaviors in data. This is useful for making informed decisions based on historical data.

2. Forecasting and Prediction

- Time series models are used to predict future values based on past data. This is essential in stock market forecasting, weather prediction, demand forecasting, and many other applications.

3. Anomaly Detection

- Time series analysis helps detect unusual patterns or anomalies, which is important in fraud detection, network security, and quality control.

4. Causal Relationships and Dependencies

- It helps in understanding how different factors influence each other over time. For example, in economics, time series analysis can determine how interest rates affect inflation.

Importance of Time Series Analysis

5. Control and Optimization

– Time series models are used in control systems and optimization problems, such as managing inventory in supply chain management or adjusting parameters in industrial processes.

6. Risk Management

– In finance and insurance, time series models help assess risks, volatility, and uncertainties in markets.

7. Policy Making and Strategic Planning

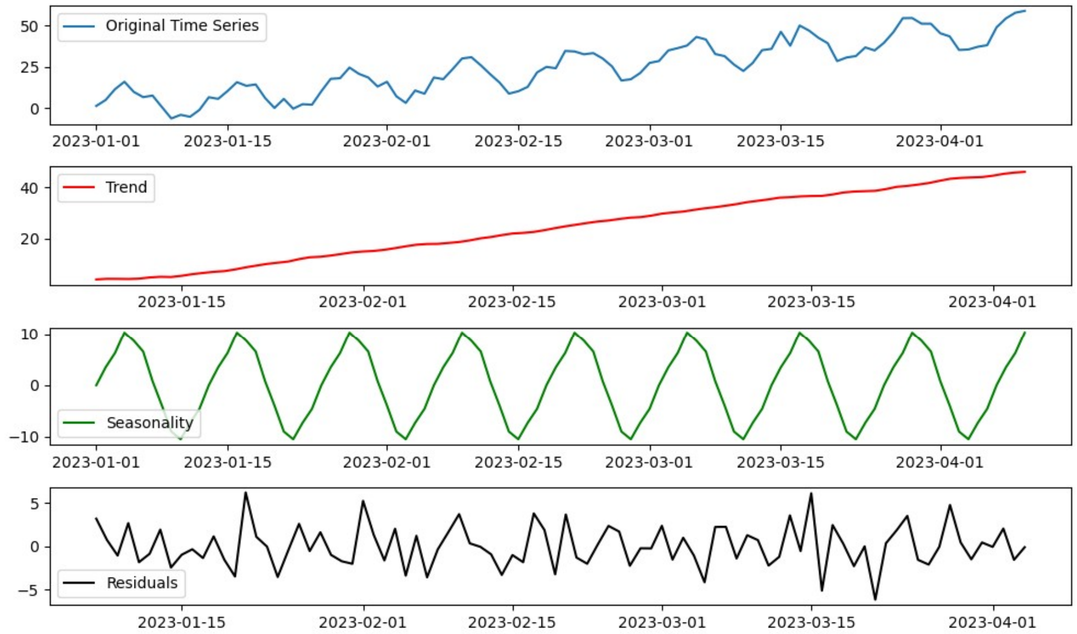
– Governments and businesses use time series analysis to make data-driven policy and strategic decisions.

8. Improving Machine Learning Models

– Time series data is widely used in machine learning for applications like speech recognition, energy consumption forecasting, and healthcare analytics.

Components of Time Series

- 1. Trend
- 2. Seasonality
- 3. Cyclic Patterns
- 4. Random Noise



Examples of Time Series Data

- Stock Prices,
- Weather Data,
- Sales Figures,
- Economic Indicators

Data Collection & Sources

- Financial,
- Weather,
- Economic,
- and Business Data

Data Preprocessing in time series

- Handling missing values,
- smoothing techniques,
- transformations.

Visualization of Time Series Data

- Line plots,
- Seasonal decomposition,
- Trends.

Smoothing models

- **Traditional Statistical Models**
  - **ARIMA (AutoRegressive Integrated Moving Average)**: A widely used model for forecasting stationary time series.
  - **SARIMA (Seasonal ARIMA)**: An extension of ARIMA that captures seasonality.
  - **VAR (Vector Autoregression)**: Used for multivariate time series forecasting.
  - **Holt-Winters (Exponential Smoothing)**: Effective for time series with trend and seasonality.

Smoothing models: Machine Learning Models

- **Random Forest / XGBoost / LightGBM**: These tree-based models can handle time series forecasting by treating the problem as a supervised learning task with lag features.
- **Support Vector Regression (SVR)**: Can be applied to time series forecasting but requires feature engineering.

Smoothing Techniques: Deep Learning Models

- **Recurrent Neural Networks (RNNs)**: Designed to handle sequential data.
- **Long Short-Term Memory (LSTM)**: A type of RNN that can capture long-term dependencies in time series.
- **Gated Recurrent Units (GRU)**: Similar to LSTMs but computationally more efficient.
- **Temporal Convolutional Networks (TCN)**: Uses convolutional layers instead of recurrent layers to capture time dependencies.
- **Transformers (e.g., Time Series Transformer, Informer)**: Adapted from NLP to handle long-range dependencies in time series data.

Hybrid and Specialized Models

- **Facebook Prophet**: Designed for business forecasting with automatic trend and seasonality detection.
- **DeepAR (Amazon)**: A probabilistic forecasting model using deep learning.
- **N-BEATS (Neural Basis Expansion Analysis for Time Series)**: A deep learning model specifically designed for time series forecasting.
- **Neural ODEs (Ordinary Differential Equations)**: A newer approach to modeling time series using continuous-time representations.

Autoregressive (AR) Model

- AR models the relationship between a variable and its past values.

AR(p) Model Equation:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

where:

- X_t = The observed time series at time t
- c = Constant term (optional)
- ϕ_i = Autoregressive coefficients (parameters)
- p = Order of the AR process (number of lagged terms)
- $\epsilon_t \sim N(0, \sigma^2)$ = White noise (random error)

Moving Average (MA) Model

- MA captures the relationship between a variable and past errors.

MA(q) Model Equation:

$$X_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + ... + \theta_q\epsilon_{t-q}$$

where:

- X_t = The observed time series at time t
- μ = Mean of the series (optional, often assumed to be 0 for simplicity)
- $\epsilon_t \sim N(0, \sigma^2)$ = White noise error terms
- θ_i = MA coefficients (parameters)
- q = Order of the MA process (number of lagged error terms included)

ARMA (Autoregressive Moving Average)

- Combination of AR(p) and MA(q) models.

ARMA(p, q) Model Equation:

$$X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t$$

where:

- X_t = Observed time series at time t
- c = Constant term (optional)
- ϕ_i = AR coefficients (how past values influence the present)
- θ_j = MA coefficients (how past error terms influence the present)
- p = Order of the **AR** part (number of lagged observations)
- q = Order of the **MA** part (number of lagged error terms)
- $\epsilon_t \sim N(0, \sigma^2)$ = White noise error term

ARIMA (Autoregressive Integrated Moving Average)

- ARIMA = AR + Differencing + MA.

ARIMA(p, d, q) Model Equation

$$\Phi_p(B)(1-B)^d X_t = c + \Theta_q(B)\epsilon_t$$

where:

- X_t = Observed time series at time t
- B = **Backshift operator** ($BX_t = X_{t-1}$)
- d = **Degree of differencing** (number of times differencing is applied to make the series stationary)
- p = **AR order** (number of lagged observations)
- q = **MA order** (number of lagged error terms)
- $\Phi_p(B)$ = Autoregressive polynomial $(1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p)$
- $\Theta_q(B)$ = Moving Average polynomial $(1 + \theta_1 B + \theta_2 B^2 + ... + \theta_q B^q)$
- $\epsilon_t \sim N(0, \sigma^2)$ = White noise

ACF and PACF Interpretation

Identification through ACF and PACF

Definition

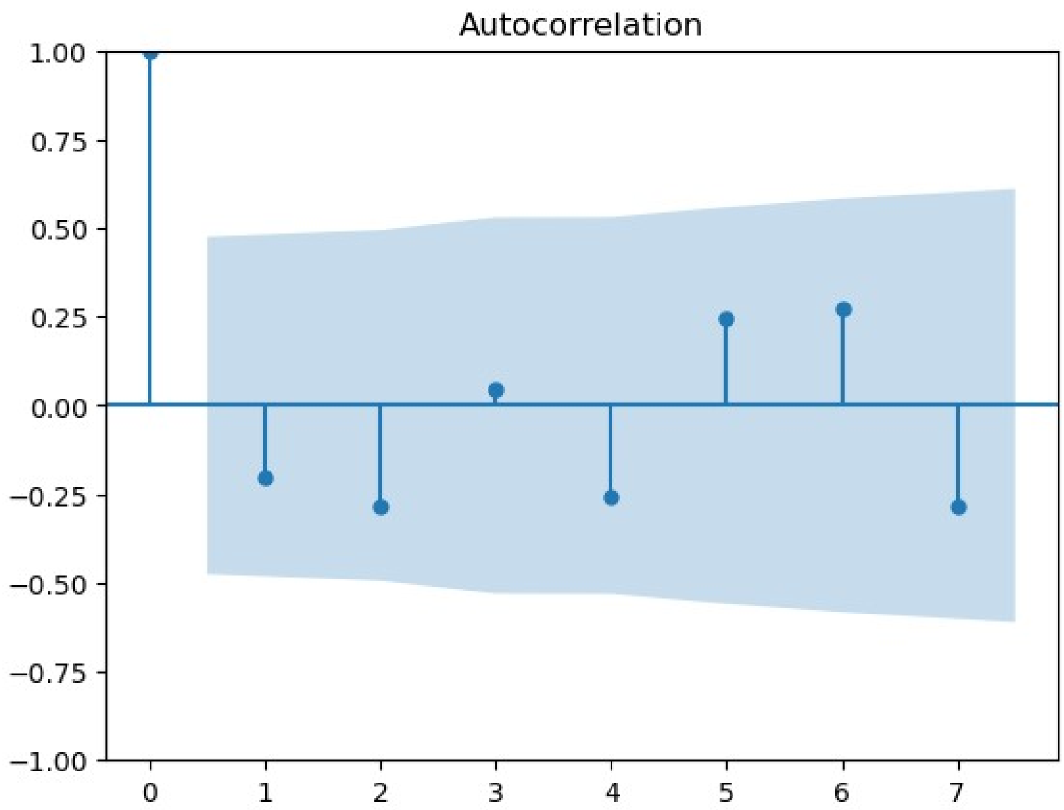
Model	*ACF*	*PACF*
$AR(1)$	Geometric Decay	Cutoff after Lag 1
$AR(p)$	Geometric Decay	Cutoff after Lag p
$MA(1)$	Cutoff after Lag 1	Geometric Decay
$MA(q)$	Cutoff after Lag q	Geometric Decay
$ARMA(p, q)$	Geometric Decay	Geometric Decay

ARMA model's parameters

- The **Autocorrelation Function (ACF)** and **Partial Autocorrelation Function (PACF)** help in selecting the appropriate values for **p** (AR order) and **q** (MA order) in an **ARMA(p, q) model**.

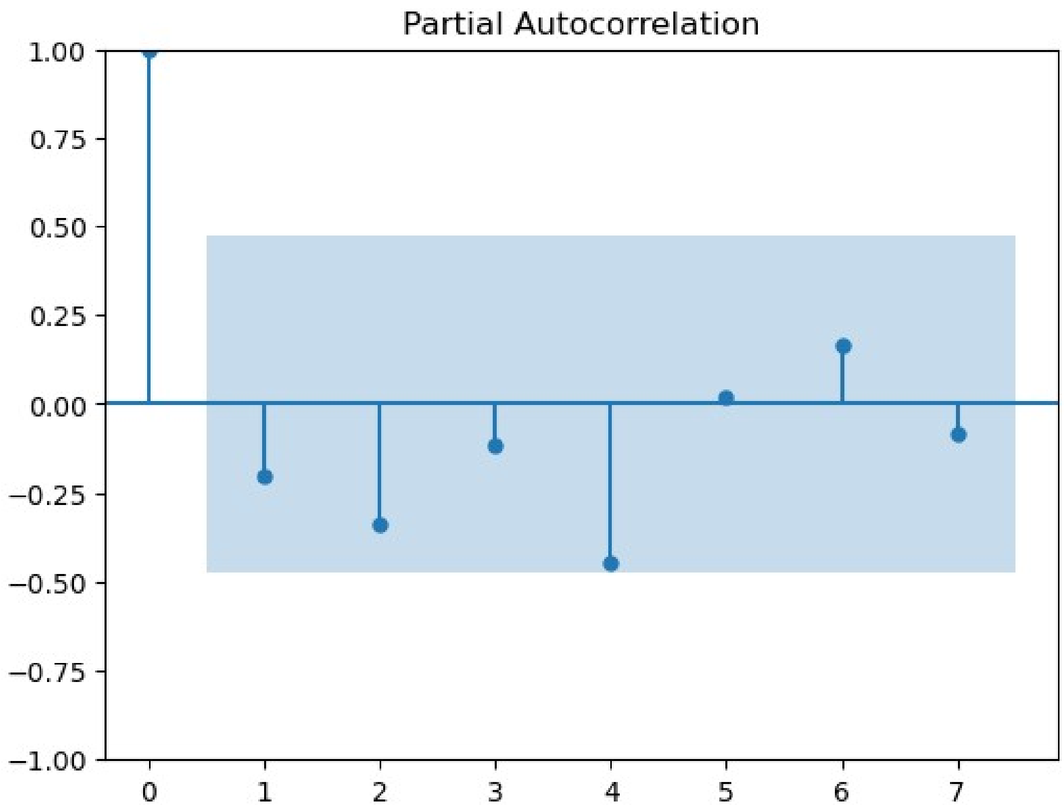
ACF

Autocorrelation Function (ACF): Measures the correlation between a time series and its past values at different lags. It helps to determine the order of a MA model



PACF

Partial Autocorrelation Function (PACF): Measures the direct correlation between a time series and its past values, removing the effects of intermediate lags. It hepls to dertmine the ordre of a AR model



Model Selection Based on ACF & PACF

- Examine ACF & PACF plots to determine if the time series follows **AR(p)**, **MA(q)**, or **ARMA(p, q)**.
- Fit different ARMA models and compare **AIC/BIC** for the best model.

AIC and BIC

- **AIC (Akaike Information Criterion):** Measures the relative quality of a statistical model, penalizing overfitting.
- **BIC (Bayesian Information Criterion):** Similar to AIC but penalizes complexity more strictly.
- The AIC and BIXC are used to compare two models, the model with small AIC and BIC is better

AIC and BIC

- The **Akaike Information Criterion (AIC)** and **Bayesian Information Criterion (BIC)** are used to assess the goodness of fit of statistical models while penalizing model complexity.

AIC

1. **Akaike Information Criterion (AIC)**

$$AIC = -2\ln(L) + 2k$$

- L = Maximum likelihood of the model
- k = Number of estimated parameters
- A lower **AIC** value indicates a better-fitting model.

$$\mathrm{AIC} = n \cdot \ln\left(\frac{\mathrm{SSE}}{n}\right) + 2k$$

Where:

- n = Number of observations (data points).
- SSE = Sum of Squared Errors or Residual Sum of Squares (RSS).
- k = Number of parameters (model parameters including the intercept).

BIC

2. **Bayesian Information Criterion (BIC)**

$$BIC = -2\ln(L) + k\ln(n)$$

- n = Number of observations
- k = Number of estimated parameters
- BIC penalizes complexity more than AIC, favoring simpler models when sample size n is large.

BIC (Bayesian Information Criterion):

$$\mathrm{BIC} = n \cdot \ln\left(\frac{\mathrm{SSE}}{n}\right) + k \cdot \ln(n)$$

Where:

- n = Number of observations.
- SSE = Sum of Squared Errors (RSS).
- k = Number of model parameters.
- $\ln$ = Natural logarithm.

Mathematical formulation of SSE and RSS

Mathematical Formulation of SSE and RSS

Given a set of **observed values** $y_1, y_2, \ldots, y_n$ and the **predicted values** from a model $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n$, the **SSE** (or **RSS**) is calculated as the sum of the squared differences between each observed value and its corresponding predicted value:

$$\text{SSE} = \text{RSS} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

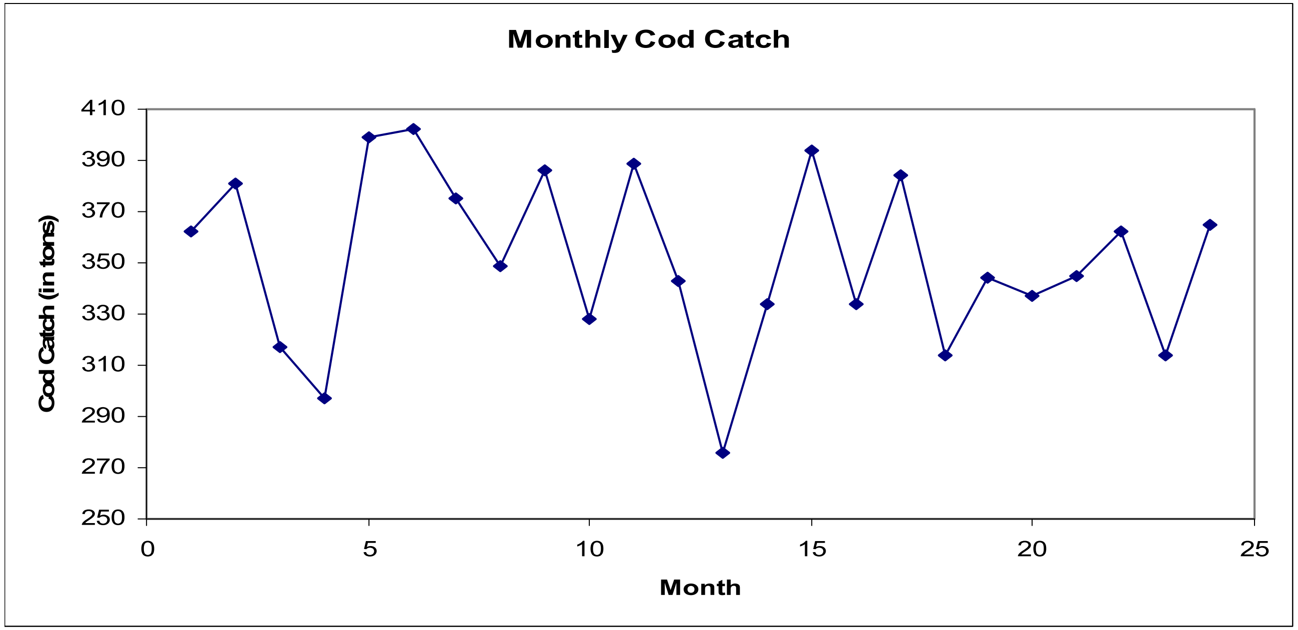
Where:

- y_i = The observed value (actual data point) at time i.
- $\hat{y}_i$ = The predicted value from the model at time i.
- n = The number of observations in the dataset.

Selection Models

- Model seclection based on:
  - p-values
  - Residual sum of squares and R^2=(TSS-RSS)/TSS
  - AIC
  - BIC
  - AICc

Simple Exponential Smoothing



Simple Exponential Smoothing

$F_{t+1} = F_t + \alpha(Y_t - F_t)$

$F_{t+1} = (1-\alpha)F_t + \alpha\, Y_t$

Where:

F_{t+1} is the forecasting of the period t+1

F_t: forecasting of the period t

Y_t: real observation of period t

α: *smoothing constant* in the range [0,1]

Holt's Trend Corrected Exponential Smoothing

- If a time series is increasing or decreasing approximately at a fixed rate, then it may be described by the LINEAR TREND model

  $$y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

  If the values of the parameters β_0 and β_1 are slowly changing over time, Holt's trend corrected exponential smoothing method can be applied to the time series observations.

  Note: When neither β_0 nor β_1 is changing over time, regression can be used to forecast future values of y_t.

- Level (or mean) at time T: $\beta_0 + \beta_1 T$

  Growth rate (or trend): β_1

Holt's Trend Corrected Exponential Smoothing

- Holt's model Augments SES by capturing a trend component

- Series has Levl (l_t)
- Trend b_t
- Noise : Unpredictable
- Forecast = estimated level + trend at most recent time point
  - $F_{t+k} = l_t + kb_t$

Updating the Level and trend

- Level estimate

$$\ell_T = \alpha y_T + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Trend estimate

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

where α = smoothing constant for the level ($0 \le \alpha \le 1$)

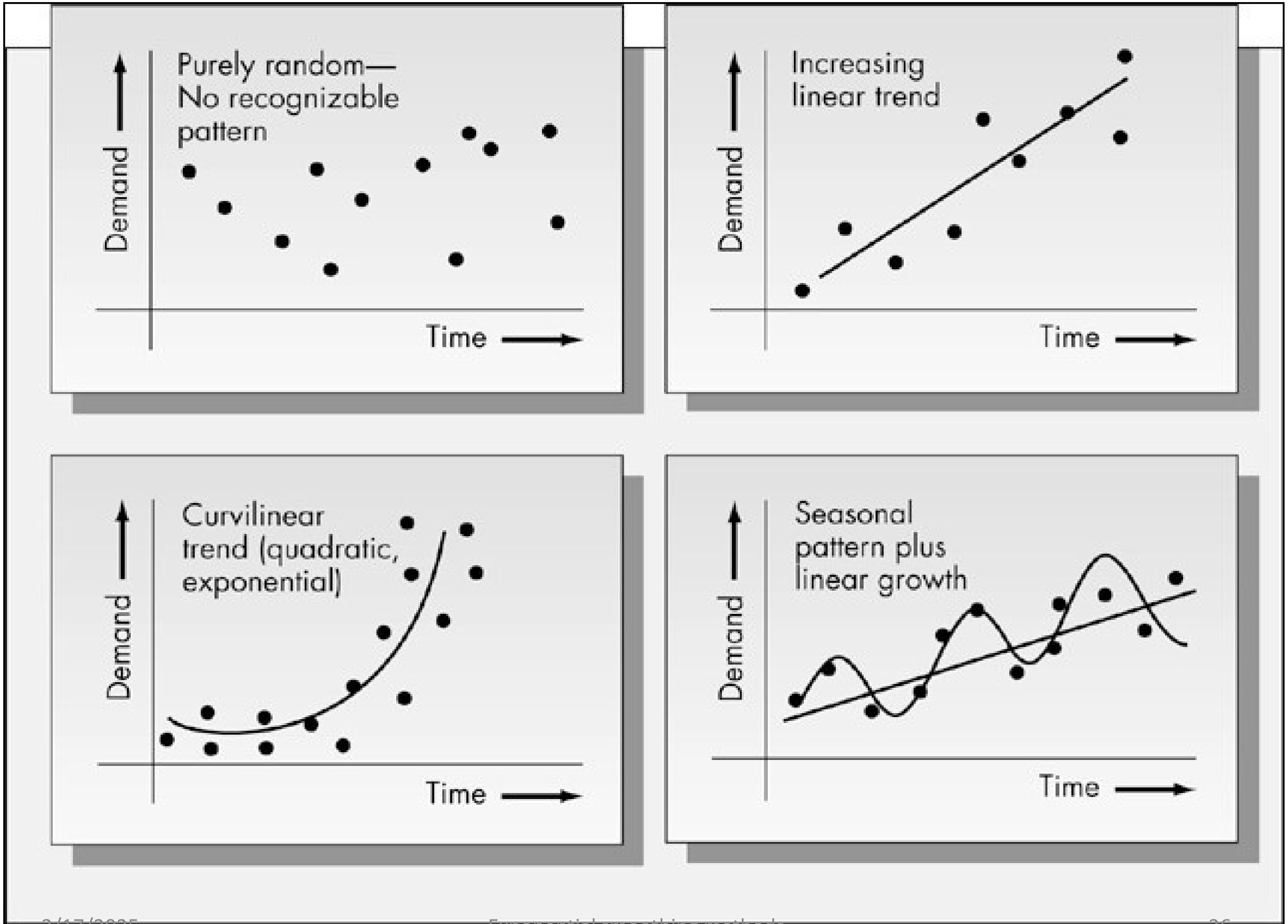
γ = smoothing constant for the trend ($0 \le \gamma \le 1$)

Hot’s model forecasting

- Additive Trend
- $F_{t+k} = l_t + k\ b_t$

- Multiplicative model:
- $F_{t+k} = lt * (b_t)^k$

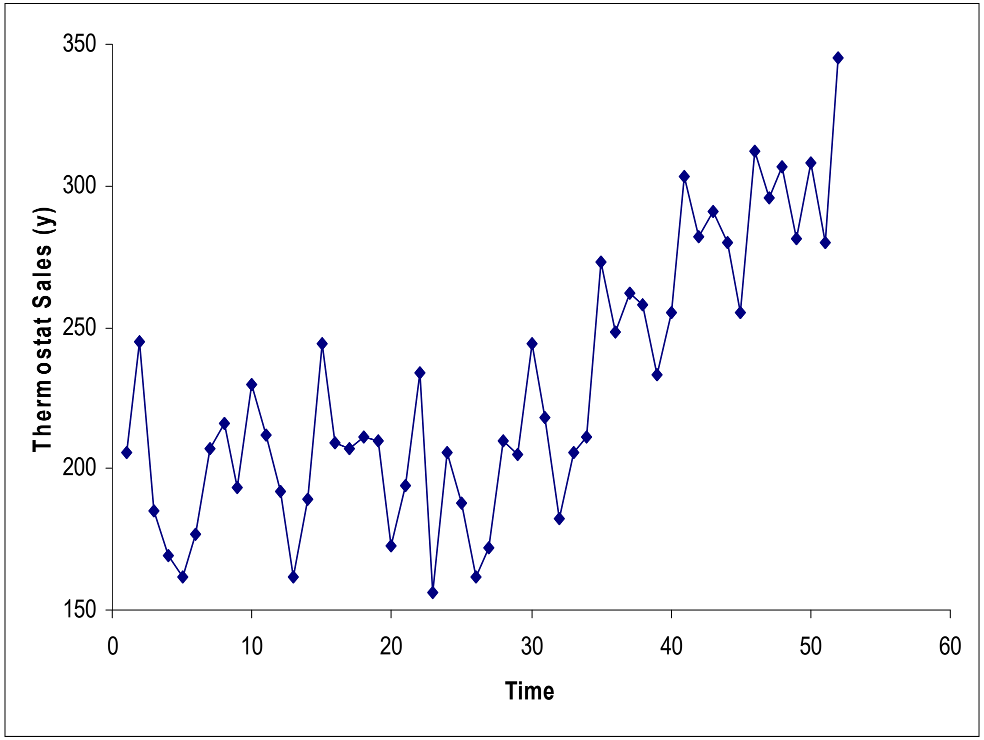
- Initialization:
  - l0 = intercept of the linear regression (others values can be considered)
  - b0 = trend of the linear regression (others values can be considered)


Purely random—
No recognizable
pattern
Demand
Time
Increasing
linear trend
Demand
Time
Curvilinear
trend (quadratic,
exponential)
Demand
Time
Seasonal
pattern plus
linear growth
Demand
Time


No seasonality
Additive Seasonality
Multiplicative Seasonality
No Trend
Additive Trend
Multiplicative Trend
Constant Trend Non-seasonal
Constant Trend with Additive Seasonality
Constant Trend with Multiplicative Seasonality
Upward Linear Trend Non-seasonal
Upward Linear Trend with Additive Seasonality
Upward Linear Trend with Multiplicative Seasonality
Upward Exponential Trend Non-seasonal
Upward Exponential Trend with Additive Seasonality
Upward Exponential Trend with Multiplicative Seasonality

Holt-Winter's model (Triple exponential smoothing)

- Augment holt's method by capturing the seasonal component



Holt-Winters Methods

- Two Holt-Winters methods are designed for time series that exhibit linear trend
  - Additive Holt-Winters method: used for time series with constant (additive) seasonal variations
  - Multiplicative Holt-Winters method: used for time series with increasing (multiplicative) seasonal variations
- Holt-Winters method is an exponential smoothing approach for handling SEASONAL data.
- The multiplicative Holt-Winters method is the better known of the two methods.

Multiplicative Holt-Winters Method

- It is generally considered to be best suited to forecasting time series that can be described by the equation:

  $$y_t = (\beta_0 + \beta_1 t) \times SN_t \times IR_t$$

  - SN_t: seasonal pattern
  - IR_t: irregular component
- This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern for which the level ($\beta_0 + \beta_1 t$), growth rate (β_1), and the seasonal pattern (SN_t) may be slowly changing over time.

Multiplicative Holt-Winters Method

- Estimate of the level

$$\ell_T = \alpha(y_T / sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Estimate of the growth rate (or trend)

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

- Estimate of the seasonal factor

$$sn_T = \delta(y_T / \ell_T) + (1-\delta)sn_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1,
L = number of seasons in a year (L = 12 for monthly data, and L = 4 for quarterly data)

Multiplicative Holt-Winters Method

- Point forecast made at time *T* for y_{T+p}

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- MSE and the standard errors at time *T*

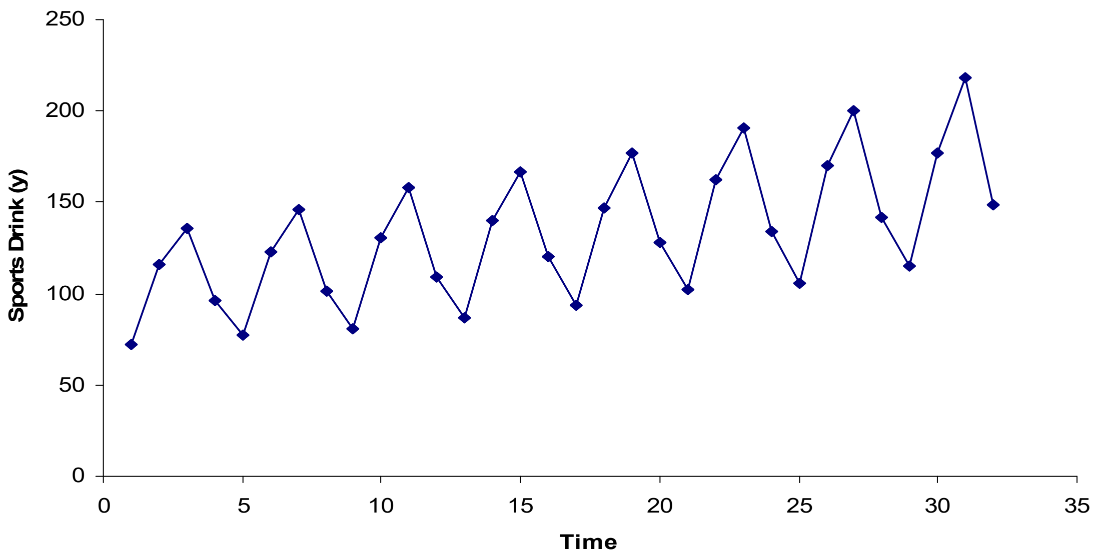
$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-3}, \qquad s = \sqrt{MSE}$$

Procedures of Multiplicative Holt-Winters Method

- Use the Sports Drink example as an illustration

Quarterly sales of Tiger Sports Drink								
	Year							
Quarter	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	72	77	81	87	94	102	106	115
2	116	123	131	140	147	162	170	177
3	136	146	158	167	177	191	200	218
4	96	101	109	120	128	134	142	149



Procedures of Multiplicative Holt-Winters Method

- Observations:
  - Linear upward trend over the 8-year period
  - Magnitude of the seasonal span increases as the level of the time series increases

  $\Rightarrow$ Multiplicative Holt-Winters method can be applied to forecast future sales

Procedures of Multiplicative Holt-Winters Method

- **Step 1**: Obtain initial values for the level ℓ_0, the growth rate b_0, and the seasonal factors sn_{-3}, sn_{-2}, sn_{-1}, and sn_0, by fitting a least squares trend line to at least four or five years of the historical data.
  - y-intercept = ℓ_0; slope = b_0

Procedures of Multiplicative Holt-Winters Method

- Example
  - Fit a least squares trend line to the first 16 observations
  - Trend line

$$\hat{y}_t = 95.2500 + 2.4706t$$

  - $\ell_0 = 95.2500$; $b_0 = 2.4706$

SUMMARY OUTPUT	
Regression Statistics	
Multiple R	0.403809754
R Square	0.163062318
Adjusted R Square	0.103281055
Standard Error	27.58325823
Observations	16
ANOVA	
	df
Regression	1
Residual	14
Total	15
	Coefficients
Intercept	95.25
X Variable 1	2.470588235

Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  1. Compute $\hat{y}_t$ for the in-sample observations used for fitting the regression. In this example, *t* = 1, 2, …, 16.

$$\hat{y}_1 = 95.2500 + 2.4706(1) = 97.7206$$
$$\hat{y}_2 = 95.2500 + 2.4706(2) = 100.1912$$
$$\ldots\ldots$$
$$\hat{y}_{16} = 95.2500 + 2.4706(16) = 134.7794$$

Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  2. Detrend the data by computing $S_t = y_t / \hat{y}_t$ for each time period that is used in finding the least squares regression equation. In this example, t = 1, 2, …, 16.

$$S_1 = y_1 / \hat{y}_1 = 72 / 97.7206 = 0.7368$$
$$S_2 = y_2 / \hat{y}_2 = 116 / 100.1912 = 1.1578$$
$$\ldots\ldots$$
$$S_{16} = y_{16} / \hat{y}_{16} = 120 / 134.7794 = 0.8903$$

Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors

  3. Compute the average seasonal values for each of the *L* seasons. The *L* averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

$$\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4}$$

$$= \frac{0.7368 + 0.7156 + 0.6894 + 0.6831}{4} = 0.7062$$

Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Multiply the average seasonal values by the normalizing constant

$$CF = \frac{L}{\sum_{i=1}^{L} \overline{S}_{[i]}}$$

such that the average of the seasonal factors is 1. The initial seasonal factors are

$$sn_{i-L} = \overline{S}_{[i]}(CF) \qquad (i = 1, 2, \ldots, L)$$

Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors

  4. Multiply the average seasonal values by the normalizing constant such that the average of the seasonal factors is 1.

     - Example

       CF = 4/3.9999 = 1.0000

$$sn_{-3} = sn_{1-4} = \bar{S}_{[1]}(CF) = 0.7062(1) = 0.7062$$

$$sn_{-2} = sn_{2-4} = \bar{S}_{[2]}(CF) = 1.1114(1) = 1.1114$$

$$sn_{-1} = sn_{3-4} = \bar{S}_{[3]}(CF) = 1.2937(1) = 1.2937$$

$$sn_{0} = sn_{4-4} = \bar{S}_{[1]}(CF) = 0.8886(1) = 0.8886$$

Procedures of Multiplicative Holt-Winters Method

- **Step 3**: Calculate a point forecast of y_1 from time 0 using the initial values

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (T = 0,\ p = 1)$$

$$\hat{y}_1(0) = (\ell_0 + b_0)sn_{1-4} = (\ell_0 + b_0)sn_{-3}$$

$$= (95.2500 + 2.4706)(0.7062)$$

$$= 69.0103$$

Procedures of Multiplicative Holt-Winters Method

- **Step 4**: Update the estimates ℓ_T, b_T, and sn_T by using some predetermined values of smoothing constants.
- Example: let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

$$\ell_1 = \alpha(y_1 / sn_{1-4}) + (1-\alpha)(\ell_0 + b_0)$$
$$= 0.2(72 / 0.7062) + 0.8(95.2500 + 2.4706) = 98.5673$$

$$b_1 = \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0$$
$$= 0.1(98.5673 - 95.2500) + 0.9(2.4706) = 2.5553$$

$$sn_1 = \delta(y_1 / \ell_1) + (1-\delta)sn_{1-4}$$
$$= 0.1(72 / 98.5673) + 0.9(0.7062) = 0.7086$$

$$\hat{y}_2(1) = (\ell_1 + b_1)sn_{2-4}$$
$$= (98.5673 + 2.5553)(1.1114) = 112.3876$$

$$
\begin{aligned}
\ell_2 &= \alpha\left(y_2 / sn_{2-4}\right) + (1-\alpha)\left(\ell_1 + b_1\right) \\
&= 0.2(116/1.1114) + 0.8(98.5673 + 2.5553) \\
&= 101.7727 \\
b_2 &= \gamma\left(\ell_2 - \ell_1\right) + (1-\gamma)b_1 \\
&= 0.1(101.7727 - 98.5673) + 0.9(2.5553) \\
&= 2.62031 \\
sn_2 &= \delta\left(y_2 / \ell_2\right) + (1-\delta)sn_{2-4} \\
&= 0.1(116/101.7727) + 0.9(1.1114) \\
&= 1.114239 \\
\hat{y}_3(2) &= \left(\ell_2 + b_2\right)sn_{3-4} \\
&= (101.7727 + 2.62031)(1.2937) \\
&= 135.053
\end{aligned}
$$

$$
\begin{aligned}
\ell_4 &= \alpha\left(y_4/sn_{4-4}\right) + (1-\alpha)(\ell_3 + b_3) \\
&= 0.2(96/0.8886) + 0.8(104.5393 + 2.6349) \\
&= 107.3464 \\
b_4 &= \gamma(\ell_4 - \ell_3) + (1-\gamma)b_3 \\
&= 0.1(107.3464 - 104.5393) + 0.9(2.6349) \\
&= 2.65212 \\
sn_4 &= \delta\left(y_4/\ell_4\right) + (1-\delta)sn_{4-4} \\
&= 0.1(96/107.3464) + 0.9(0.8886) \\
&= 0.889170 \\
\hat{y}_5(4) &= (\ell_4 + b_4)sn_{5-4} \\
&= (107.3464 + 2.65212)(0.7086) \\
&= 77.945
\end{aligned}
$$

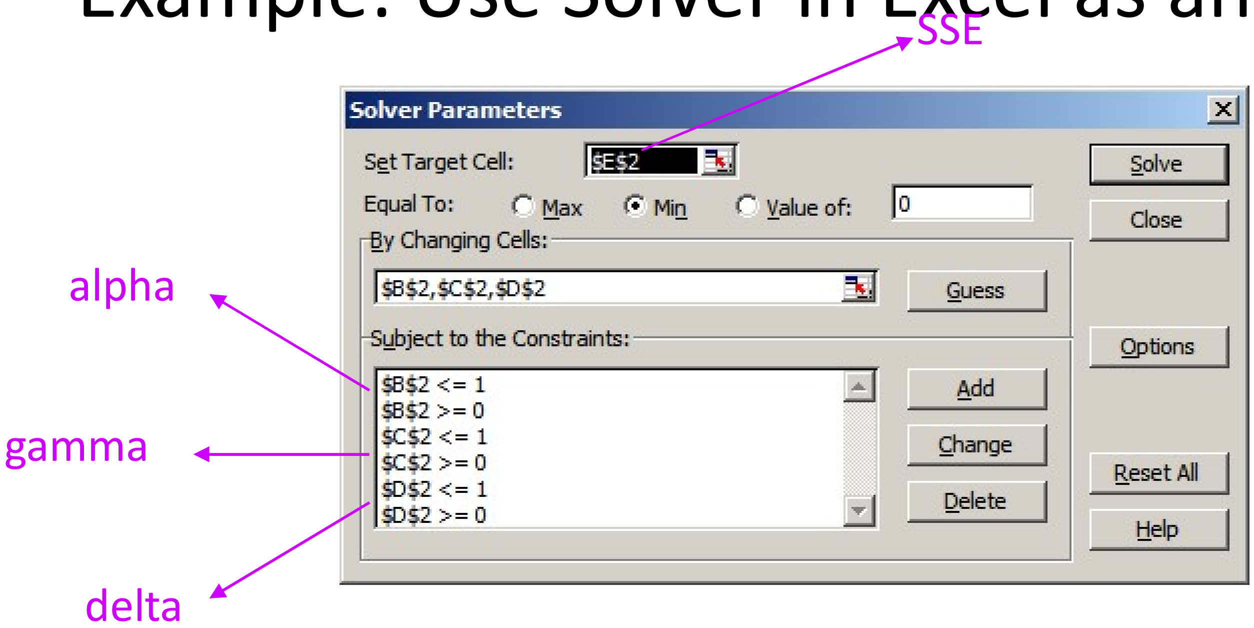
1	n	alpha	gamma	delta	SSE	MSE	s	
2	32	0.2	0.1	0.1	177.3223	6.1146	2.4728	
3								
4								
5						Forecast		Squared
6				Growth	Seasonal	Made Last	Forecast	Forecast
7	Time	y	Level	Rate	Factor	Period	Error	Error
8	-3				0.7062			
9	-2				1.1114			
10	-1				1.2937			
11	0		95.25	2.4706	0.8886			
12	1	72	98.56729	2.5553	0.7086	69.0103	2.9897	8.9384
13	2	116	101.7726	2.6203	1.1142	112.3876	3.6124	13.0494
14	3	136	104.5393	2.6349	1.2944	135.0531	0.9469	0.8967
15	4	96	107.3464	2.6521	0.8892	95.2350	0.7650	0.5853
16	5	77	109.731	2.6254	0.7079	77.9478	-0.9478	0.8984
17	6	123	111.9629	2.5860	1.1127	125.1919	-2.1919	4.8043
18	7	146	114.1974	2.5509	1.2928	148.2750	-2.2750	5.1755
19	8	101	116.1165	2.4877	0.8872	103.8091	-2.8091	7.8911
20	9	81	117.7668	2.4040	0.7059	83.9641	-2.9641	8.7858
21	10	131	119.6835	2.3552	1.1109	133.7108	-2.7108	7.3482
22	11	158	122.0734	2.3587	1.2930	157.7754	0.2246	0.0504
23	12	109	124.1164	2.3271	0.8863	110.4005	-1.4005	1.9615
24	13	87	125.8035	2.2631	0.7045	89.2593	-2.2593	5.1044
25	14	140	127.6589	2.2224	1.1094	142.2642	-2.2642	5.1268
26	15	167	129.7369	2.2079	1.2924	167.9337	-0.9337	0.8718

…… ……

38	27	200	156.1396	2.1752	1.2903	202.0396	-2.0396	4.1601
39	28	142	158.5505	2.1988	0.8908	140.9508	1.0492	1.1008
40	29	115	161.2803	2.2519	0.7047	113.1314	1.8686	3.4918
41	30	177	162.8178	2.1804	1.1046	180.9529	-3.9529	15.6252
42	31	218	165.7889	2.2595	1.2928	212.8988	5.1012	26.0220
43	32	149	167.8899	2.2437	0.8905	149.7057	-0.7057	0.4981

Procedures of Multiplicative Holt-Winters Method

- **Step 5**: Find the most suitable combination of α, γ, and δ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration



1	n	alpha	gamma	delta	SSE	MSE	s	
2	32	0.3356	0.0455	0.1342	168.4747	5.8095	2.4103	
3								
4								
5						Forecast		Squared
6				Growth	Seasonal	Made Last	Forecast	Forecast
7	Time	y	Level	Rate	Factor	Period	Error	Error
8	-3				0.7062			
9	-2				1.1114			
10	-1				1.2937			
11	0		95.25	2.4706	0.8886			
12	1	72	99.14144	2.5353	0.7089	69.0103	2.9897	8.9384
13	2	116	102.5816	2.5765	1.1140	113.0035	2.9965	8.9789
14	3	136	105.1469	2.5760	1.2937	136.0431	-0.0431	0.0019
15	4	96	107.8277	2.5808	0.8888	95.7226	0.2774	0.0769
16	5	77	109.8084	2.5534	0.7079	78.2674	-1.2674	1.6064
17	6	123	111.7076	2.5236	1.1123	125.1717	-2.1717	4.7164
18	7	146	113.7703	2.5027	1.2923	147.7768	-1.7768	3.1569
19	8	101	115.3868	2.4623	0.8870	103.3468	-2.3468	5.5075
20	9	81	116.7014	2.4100	0.7060	83.4207	-2.4207	5.8597
		......	......					
38	27	200	155.9042	2.2691	1.2906	202.1107	-2.1107	4.4552
39	28	142	158.5811	2.2876	0.8915	140.9173	1.0827	1.1721
40	29	115	161.7496	2.3278	0.7044	113.1540	1.8460	3.4078
41	30	177	162.7095	2.2655	1.1038	181.5085	-4.5085	20.3262
42	31	218	166.2957	2.3256	1.2934	212.9210	5.0790	25.7957
43	32	149	168.1213	2.3028	0.8908	150.3283	-1.3283	1.7643

Multiplicative Holt-Winters Method

- *p*-step-ahead forecast made at time *T*

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- Example

$$\hat{y}_{33}(32) = (\ell_{32} + b_{32})sn_{33-4} = (168.1213 + 2.3028)(0.7044) = 120.0467$$

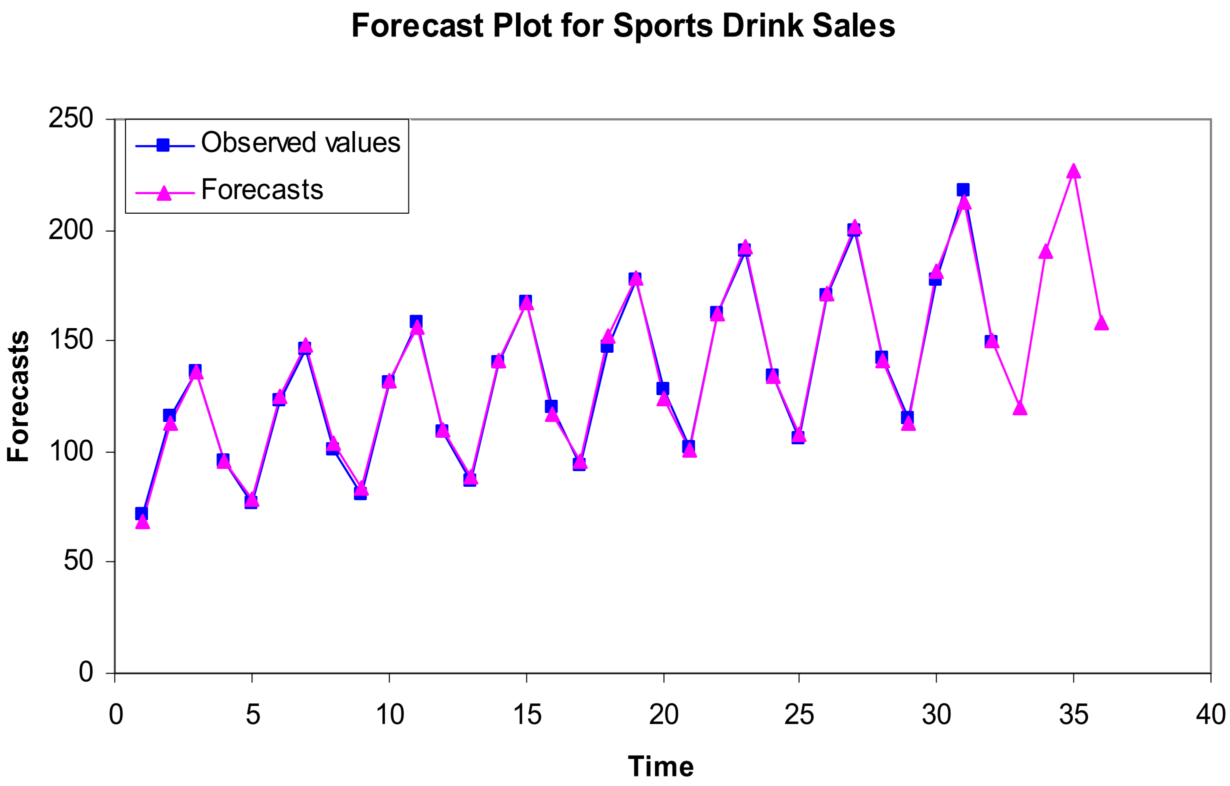
$$\hat{y}_{34}(32) = (\ell_{32} + 2b_{32})sn_{34-4} = [168.1213 + 2(2.3028)](1.1038) = 190.6560$$

$$\hat{y}_{35}(32) = (\ell_{32} + 3b_{32})sn_{35-4} = [(168.1213 + 3(2.3028)](1.2934) = 226.3834$$

$$\hat{y}_{36}(32) = (\ell_{32} + 4b_{32})sn_{36-4} = [(168.1213 + 4(2.3028)](0.8908) = 157.9678$$

Multiplicative Holt-Winters Method

- Example



Additive Holt-Winters Method

- It is generally considered to be best suited to forecasting a time series that can be described by the equation:

  $$y_t = (\beta_0 + \beta_1 t) + SN_t + IR_t$$

  – SN_t: seasonal pattern

  – IR_t: irregular component
- This method is appropriate when a time series has a linear trend with a constant (additive) seasonal pattern such that the level ($\beta_0 + \beta_1 t$), growth rate (β_1), and the seasonal pattern (SN_t) may be slowly changing over time.

Additive Holt-Winters Method

- Estimate of the level

$$\ell_T = \alpha(y_T - sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

- Estimate of the growth rate (or trend)

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

- Estimate of the seasonal factor

$$sn_T = \delta(y_T - \ell_T) + (1-\delta)sn_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1,

L = number of seasons in a year (*L* = 12 for monthly data, and *L* = 4 for quarterly data)

Additive Holt-Winters Method

- Point forecast made at time T for y_{T+p}

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- MSE and the standard error s at time T

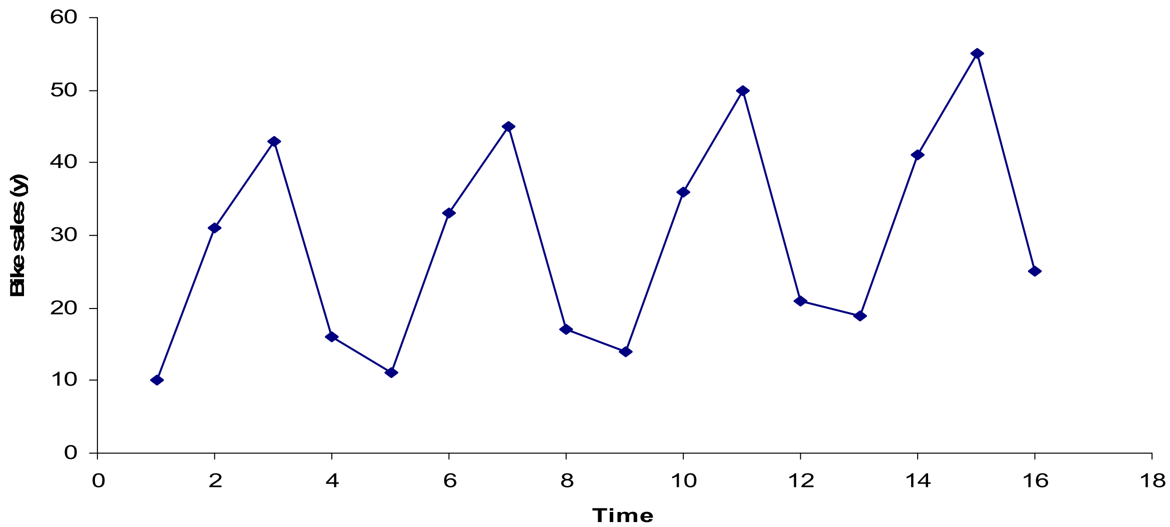
$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-3}, \qquad s = \sqrt{MSE}$$

Procedures of Additive Holt-Winters Method

- Consider the Mountain Bike example,

Quarterly sales of the TRK-50 Mountain Bike				
	Year			
Quarter	1	2	3	4
1	10	11	14	19
2	31	33	36	41
3	43	45	50	55
4	16	17	21	25



Procedures of Additive Holt-Winters Method

- Observations:
  - Linear upward trend over the 4-year period
  - Magnitude of seasonal span is almost constant as the level of the time series increases

  $\Rightarrow$ Additive Holt-Winters method can be applied to forecast future sales

Procedures of Additive Holt-Winters Method

- **Step 1**: Obtain initial values for the level ℓ_0, the growth rate b_0, and the seasonal factors sn_{-3}, sn_{-2}, sn_{-1}, and sn_0, by fitting a least squares trend line to <u>at least four or five years</u> of the historical data.
  - y-intercept = ℓ_0; slope = b_0

Procedures of Additive Holt-Winters Method

- Example
  - Fit a least squares trend line to all 16 observations
  - Trend line

    $$\hat{y}_t = 20.85 + 0.980882\, t$$

  - $\ell_0 = 20.85$; $b_0 = 0.9809$

SUMMARY OUTPUT	
Regression Statistics	
Multiple R	0.320508842
R Square	0.102725918
Adjusted R Square	0.038634912
Standard Error	14.28614022
Observations	16
ANOVA	
	df
Regression	1
Residual	14
Total	15
	Coefficients
Intercept	20.85
Time	0.980882353

Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  1. Compute $\hat{y}_t$ for each time period that is used in finding the least squares regression equation. In this example, *t* = 1, 2, …, 16.

$$\hat{y}_1 = 20.85 + 0.980882(1) = 21.8309$$

$$\hat{y}_2 = 20.85 + 0.980882(2) = 22.8118$$

$$\ldots\ldots$$

$$\hat{y}_{16} = 20.85 + 0.980882(16) = 36.5441$$

Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  2. Detrend the data by computing $S_t = y_t - \hat{y}_t$ for each observation used in the least squares fit. In this example, *t* = 1, 2, ..., 16.

$$S_1 = y_1 - \hat{y}_1 = 10 - 21.8309 = -11.8309$$

$$S_2 = y_2 - \hat{y}_2 = 31 - 22.8112 = 8.1882$$

......

$$S_{16} = y_{16} - \hat{y}_{16} = 25 - 36.5441 = -11.5441$$

Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  3. Compute the average seasonal values for each of the *L* seasons. The *L* averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

$$\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4}$$

$$= \frac{(-11.8309) + (-14.7544) + (-15.6779) + (-14.6015)}{4} = -14.2162$$

Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Compute the average of the *L* seasonal factors. The average should be 0.

Procedures of Additive Holt-Winters Method

- **Step 3**: Calculate a point forecast of y_1 from time 0 using the initial values

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (T = 0, p = 1)$$

$$\hat{y}_1(0) = \ell_0 + b_0 + sn_{1-4} = \ell_0 + b_0 + sn_{-3}$$

$$= 20.85 + 0.9809 + (-14.2162) = 7.6147$$

Procedures of Additive Holt-Winters Method

- **Step 4**: Update the estimates ℓ_T, b_T, and sn_T by using some predetermined values of smoothing constants.
- Example: let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

$$\ell_1 = \alpha(y_1 - sn_{1-4}) + (1-\alpha)(\ell_0 + b_0)$$
$$= 0.2(10-(-14.2162)) + 0.8(20.85 + 0.9808) = 22.3079$$

$$b_1 = \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0$$
$$= 0.1(22.3079 - 20.85) + 0.9(0.9809) = 1.0286$$

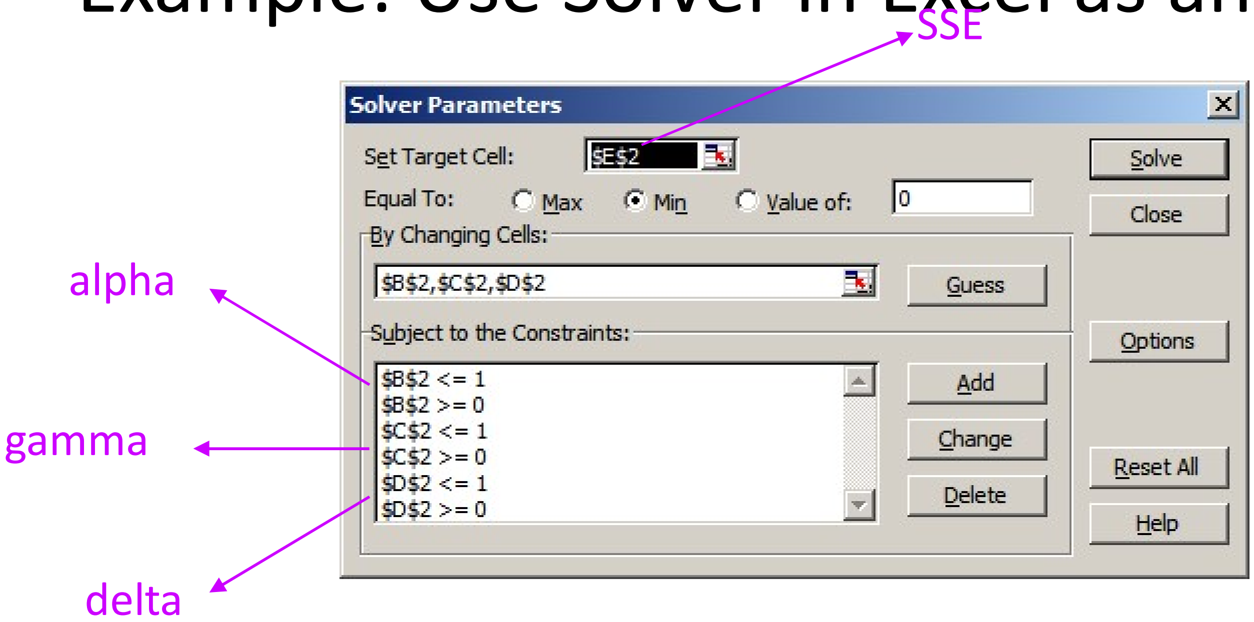
$$sn_1 = \delta(y_1 - \ell_1) + (1-\delta)sn_{1-4}$$
$$= 0.1(10 - 22.3079) + 0.9(-14.2162) = -14.0254$$

$$\hat{y}_2(1) = \ell_1 + b_1 + sn_{2-4} = \ell_1 + b_1 + sn_{-2}$$
$$= 22.3079 + 1.0286 + 6.5529 = 29.8895$$

1	n	alpha	gamma	delta	SSE	MSE	s	
2	16	0.2000	0.1000	0.1000	25.2166	1.9397	1.3927	
3								
4								
5						Forecast		Squared
6				Growth	Seasonal	Made Last	Forecast	Forecast
7	Time	y	Level	Rate	Factor	Period	Error	Error
8	-3				-14.2162			
9	-2				6.5529			
10	-1				18.5721			
11	0		20.85	0.9809	-10.9088			
12	1	10	22.30794	1.0286	-14.0254	7.6147	2.3853	5.6896
13	2	31	23.55864	1.0508	6.6418	29.8895	1.1105	1.2333
14	3	43	24.57314	1.0472	18.5575	43.1815	-0.1815	0.0329
15	4	16	25.87801	1.0729	-10.8057	14.7115	1.2885	1.6603
16	5	11	26.56583	1.0344	-14.1794	12.9256	-1.9256	3.7079
17	6	33	27.35185	1.0096	6.5424	34.2420	-1.2420	1.5427
18	7	45	27.97764	0.9712	18.4040	46.9190	-1.9190	3.6825
19	8	17	28.72023	0.9483	-10.8972	18.1431	-1.1431	1.3067
20	9	14	29.37074	0.9186	-14.2985	15.4892	-1.4892	2.2176
21	10	36	30.12295	0.9019	6.4759	36.8317	-0.8317	0.6918
22	11	50	31.1391	0.9133	18.4497	49.4289	0.5711	0.3262
23	12	21	32.0214	0.9102	-10.9096	21.1553	-0.1553	0.0241
24	13	19	33.00502	0.9176	-14.2692	18.6331	0.3669	0.1346
25	14	41	34.04291	0.9296	6.5240	40.3985	0.6015	0.3618
26	15	55	35.28807	0.9612	18.5759	53.4222	1.5778	2.4894
27	16	25	36.18131	0.9544	-10.9368	25.3396	-0.3396	0.1153

Procedures of Additive Holt-Winters Method

- **Step 5**: Find the most suitable combination of α, γ, and δ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration



1	n	alpha	gamma	delta	SSE	MSE	s	
2	16	0.5606	0.0000	0.0000	18.7975	1.4460	1.2025	
3								
4								
5						Forecast		Squared
6				Growth	Seasonal	Made Last	Forecast	Forecast
7	Time	y	Level	Rate	Factor	Period	Error	Error
8	-3				-14.2162			
9	-2				6.5529			
10	-1				18.5721			
11	0		20.85	0.9809	-10.9088			
12	1	10	23.16818	0.9809	-14.2162	7.6147	2.3853	5.6896
13	2	31	24.31613	0.9809	6.5529	30.7020	0.2980	0.0888
14	3	43	24.80977	0.9809	18.5721	43.8691	-0.8691	0.7553
15	4	16	26.41755	0.9809	-10.9088	14.8818	1.1182	1.2503
16	5	11	26.17496	0.9809	-14.2162	13.1823	-2.1823	4.7622
17	6	33	26.75847	0.9809	6.5529	33.7088	-0.7088	0.5024
18	7	45	27.00412	0.9809	18.5721	46.3114	-1.3114	1.7198
19	8	17	27.94229	0.9809	-10.9088	17.0762	-0.0762	0.0058
20	9	14	28.5268	0.9809	-14.2162	14.7070	-0.7070	0.4998
21	10	36	29.47369	0.9809	6.5529	36.0606	-0.0606	0.0037
22	11	50	31.00029	0.9809	18.5721	49.0266	0.9734	0.9474
23	12	21	31.94061	0.9809	-10.9088	21.0723	-0.0723	0.0052
24	13	19	33.0867	0.9809	-14.2162	18.7053	0.2947	0.0868
25	14	41	34.28034	0.9809	6.5529	40.6205	0.3795	0.1440
26	15	55	35.91533	0.9809	18.5721	53.8333	1.1667	1.3612
27	16	25	36.34264	0.9809	-10.9088	25.9874	-0.9874	0.9749

Additive Holt-Winters Method

- *p*-step-ahead forecast made at time *T*

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \qquad (p = 1, 2, 3, \ldots)$$

- Example

$$\hat{y}_{17}(16) = \ell_{16} + b_{16} + sn_{17-4} = 36.3426 + 0.9809 - 14.2162 = 23.1073$$

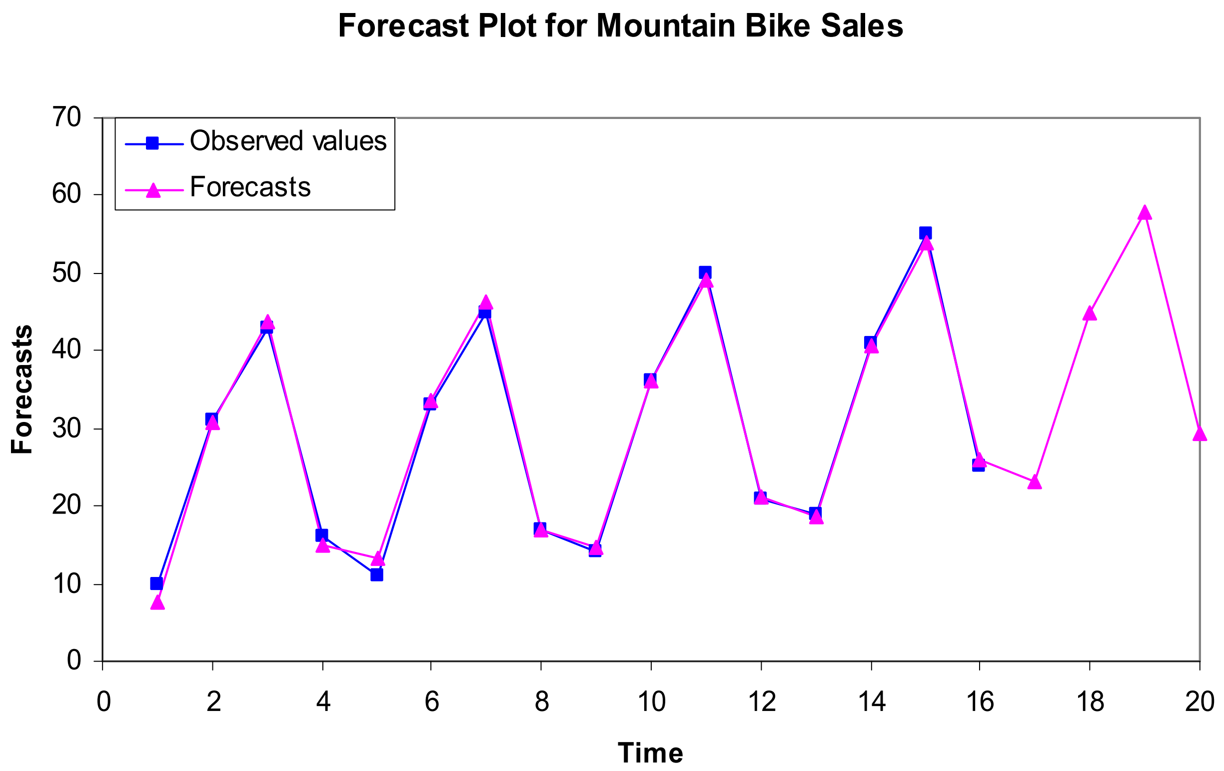
$$\hat{y}_{18}(16) = \ell_{16} + 2b_{16} + sn_{18-4} = 36.3426 + 2(0.9809) + 6.5529 = 44.8573$$

$$\hat{y}_{19}(16) = \ell_{16} + 3b_{16} + sn_{19-4} = 36.3426 + 3(0.9809) + 18.5721 = 57.8573$$

$$\hat{y}_{20}(16) = \ell_{16} + 4b_{16} + sn_{20-4} = 36.3426 + 4(0.9809) - 10.9088 = 29.3573$$

Additive Holt-Winters Method

- Example



Chapter Summary

- Simple Exponential Smoothing
  - No trend, no seasonal pattern
- Holt’s Trend Corrected Exponential Smoothing
  - Trend, no seasonal pattern
- Holt-Winters Methods
  - Both trend and seasonal pattern
    - Multiplicative Holt-Winters method
    - Additive Holt-Winters Method