1

a) 2 partículas → 3·2 coordenadas

Coplanarias → −1·2 " (plano 2D)

2 cuerdas → −2 "

⇒ 6−2−2 = 2 coordenadas generalizadas.

Otra manera de verlo es que si sujetas m_1, m_2 describe un arco (varied lineal 1D), y idem si sujetas m_2 (otro arco 1D) → en total problema 2D. → 2 q_i.

b)

(θ, ϕ) → son válidas, describen unívocamente todo el espacio de posiciones. Si fijas θ, ϕ lo puedes variar entre 0 y 2π de manera independiente.

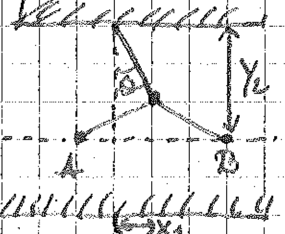
Idem para (θ, ψ), (ϕ, ψ), son independientes y no hay degeneración. Otra manera de verlo a partir de (θ, ϕ) es que:

$$\theta + \psi = \phi$$

$$(\theta, \phi) \rightarrow (\theta, \psi + \theta) \rightarrow (\theta, \psi) \rightarrow (\phi - \psi, \psi) \rightarrow (\phi, \psi)$$

Al ser un problema con simetría polar no valdrán las cartesianas como coordenadas generalizadas. Aunque las combinaciones sean independientes (θ, y_2), habrá degeneración, con lo que dos coordenadas definen más de un punto del espacio.

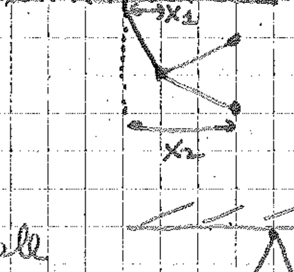
(θ, y_2) → degeneración doble



A y B mismas y_2 y θ por simetría

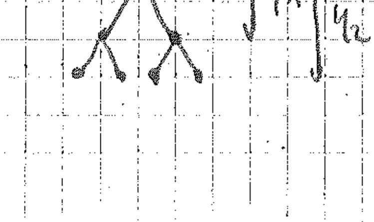
(x_1, x_2) → " cuádruple

↳ 2 si quitamos el techo
2 + (m₂ arriba)



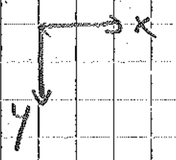
(y_1, y_2) → degenerad cuádruple

↳ idem pero rotado



c) $\delta W = \sum_\alpha \vec{f}_\alpha \cdot \delta \vec{r}_\alpha = \sum_i \sum_\alpha \overbrace{\vec{f}_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}}^{Q_i} \delta q_i$ $\qquad \alpha = 1,2 \quad i = 1,2$

$\vec{r}_1 = l_1 (\sin\theta, \cos\theta)$ Stma. refere:



$\vec{r}_2 = \vec{r}_1 + l_2 (\sin\phi, \cos\phi)$

$$\left\{ \begin{array}{l} \frac{\partial \vec{r}_1}{\partial \theta} = l_1 (\cos\theta, -\sin\theta) \\ \frac{\partial \vec{r}_2}{\partial \theta} = \frac{\partial \vec{r}_1}{\partial \theta} = l_1 (\cos\theta, -\sin\theta) \\ \frac{\partial \vec{r}_1}{\partial \phi} = 0 \\ \frac{\partial \vec{r}_2}{\partial \phi} = l_2 (\cos\phi, -\sin\phi) \end{array} \right.$$

Fuerzas netas: (tensiones se cancelan en sólido rígido)

$$\left\{ \begin{array}{l} \vec{f}_1 = (0, m_1 g) \\ \vec{f}_2 = (F, m_2 g) \end{array} \right.$$

$q_1 = \theta$
$q_2 = \phi$

$$\delta W = \left(-m_1 g \sin\theta\, l_1 + F l_1 \cos\theta - m_2 g \sin\theta\, l_1\right) d\theta + \left(F l_2 \cos\phi - m_2 g \sin\phi\, l_2\right) d\phi$$

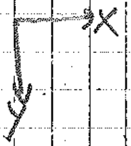
Posición de equilibrio: $\delta W = 0$

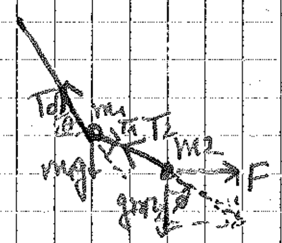
Como θ y ϕ son independientes, cada factor debe anularse por separado:

- $(m_1 + m_2) g l_1 \sin\theta_{eq} = F l_1 \cos\theta_{eq}$
  $\hookrightarrow \tan\theta_{eq} = \frac{F}{(m_1+m_2) g}$ //

- $F l_2 \cos\phi_{eq} = m_2 g \sin\phi_{eq}\, l_2$
  $\hookrightarrow \tan\phi_{eq} = \frac{F}{m_2 g}$ //

- 2ª manera: diagrama de fuerzas de Newton





$T_2 = (-F, -m_2 g) \quad (\sum F_i = 0)$
$T_1 = -T_2$

$\tan\phi_{eq} = \frac{F}{m_2 g}$ // ✓

$(0, m_1 g) + (F, m_2 g) = -T_0 (\sin\theta_{eq}, \cos\theta_{eq})$

$\hookrightarrow \frac{-T_0 \sin\theta_{eq}}{-T_0 \cos\theta_{eq}} = \tan\theta_{eq} = \frac{F}{(m_1+m_2) g}$ // ✓

d) Fuerzas generalizadas (ver c)) → dimensiones de momento de una fuerza

$Q_\theta = F l_1 \cos\theta - (m_1 + m_2) g l_1 \sin\theta$

$Q_\phi = F l_2 \cos\phi - m_2 g l_2 \sin\phi$

Si añadimos las tensiones:

$\tilde{\vec{f}_2} = \vec{f_2} + T_2(-\sin\phi, -\cos\phi)$

$\tilde{\vec{f}_1} = \vec{f_1} + T_2(\sin\phi, \cos\phi) + T_0(-\sin\theta, -\cos\theta)$

$\tilde{Q}_\theta = Q_\theta - T_0 \sin\theta\cos\theta + T_2 \sin\phi\cos\theta - T_2\cos\phi\sin\theta + T_0\cos\theta\sin\theta$
$\quad - T_2 \sin\phi\cos\theta \qquad + T_2\cos\phi\sin\theta$

$= Q_\theta + T_2\sin(\phi-\theta) - T_2\sin(\phi-\theta) = Q_\theta$

$\tilde{Q}_\phi = Q_\phi - T_2\sin\phi\cos\phi + T_2\cos\phi\sin\phi = Q_\phi$

↳ las tensiones se cancelan → podemos prescindir de ellas

e) $\mathcal{L} = T - V$

$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} = \frac{1}{2} m_1 l_1^2 \dot\theta^2 + \frac{1}{2} m_2 \Big(\dot{\vec{r}}_1^{\,2} + 2\dot{\vec{r}}_1 \cdot$

$\cdot \frac{d}{dt}\left[l_2(\sin\phi, \cos\phi)\right] + l_2^2 \dot\phi^2 \Big)$

$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot\theta^2 + \frac{1}{2} m_2 l_2^2 \dot\phi^2 + m_2 l_1 l_2 (\cos\theta\cos\phi + \sin\theta\sin\phi) \cdot \dot\theta\dot\phi$

$V = -m_1 g y_1 - m_2 g y_2 = -m_1 g l_1 \cos\theta - m_2 g (l_1\cos\theta + l_2\cos\phi)$

$= -(m_1 + m_2) g l_1 \cos\theta - m_2 g l_2 \cos\phi$

$\mathcal{L} = \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta^2 + m_2 l_1 l_2 \cos(\theta-\phi) \cdot \dot\theta \cdot \dot\phi + \frac{1}{2} m_2 l_2^2 \dot\phi^2$
$\quad + (m_1+m_2) g l_1 \cos\theta + m_2 g l_2 \cos\phi$

→ $F = 0$

$\frac{\partial \mathcal{L}}{\partial \dot\theta} = (m_1 + m_2) l_1^2 \dot\theta + m_2 l_1 l_2 \dot\phi \cos(\theta - \phi)$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = (m_1+m_2) l_1^2 \ddot\theta + m_2 l_1 l_2 \ddot\phi \cos(\theta-\phi) - m_2 l_1 l_2 \dot\phi \sin(\theta-\phi) \cdot (\dot\theta - \dot\phi)$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_2 l_1 l_2 \dot\theta \dot\phi \sin(\theta-\phi) - (m_1+m_2) g\, l_1 \sin\theta$$

Ec. Euler-Lagrange:

$$(m_1+m_2) l_1^2 \ddot\theta + m_2 l_1 l_2 \ddot\phi \cos(\theta-\phi) - m_2 l_1 l_2 \dot\phi \sin(\theta-\phi)\cdot(\dot\theta-\dot\phi)$$
$$+ m_2 l_1 l_2 \dot\phi \dot\theta \sin(\theta-\phi) + (m_1+m_2) g\, l_1 \sin\theta = 0$$

$$\hookrightarrow (m_1+m_2) l_1^2 \ddot\theta + m_2 l_1 l_2 \ddot\phi \cos(\theta-\phi) + m_2 l_1 l_2 \dot\phi^2 \sin(\theta-\phi)$$
$$+ (m_1+m_2) l_1 g \sin\theta = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = m_2 l_1 l_2 \dot\theta \cos(\theta-\phi) + m_2 l_2^2 \dot\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = m_2 l_1 l_2 \ddot\theta \cos(\theta-\phi) - m_2 l_1 l_2 \dot\theta \sin(\theta-\phi)\cdot(\dot\theta-\dot\phi)$$
$$+ m_2 l_2^2 \ddot\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_2 l_1 l_2 \dot\theta \dot\phi \cdot \sin(\theta-\phi)\cdot(-1) - m_2 g\, l_2 \sin\phi$$

$$\hookrightarrow m_2 l_1 l_2 \ddot\theta \cos(\theta-\phi) - m_2 l_1 l_2 \dot\theta^2 \sin(\theta-\phi) + m_2 g\, l_2 \sin\phi = 0$$
$$+ m_2 l_2^2 \ddot\phi$$

→ es un problema de caos clásico:
depende fuerte en condiciones iniciales

→ comprobar límite péndulo simple

(2)

a) 2 grados de libertad : $\begin{cases} \phi = \omega t & \text{(controlado)} \\ \ell & \to \text{coordenadas generalizadas} \end{cases}$

b) $\vec{r} = \ell(t)\,\hat{u_r}$

con $\hat{u_r} = (\sin\alpha\cos\phi,\ \sin\alpha\sin\phi,\ \cos\alpha)$; $\alpha = cte$

$$\dot{\vec{r}} = \dot{\ell}\,\hat{u_r} + \ell\cdot(-\sin\alpha\sin\phi,\ \sin\alpha\cos\phi,\ 0)\cdot\dot{\phi}$$

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\Big[\dot{\ell}^2 + \ell^2\sin^2\alpha\,\dot{\phi}^2 + 2\ell\dot{\ell}\big(\cancel{-\sin^2\alpha\sin\phi\cos\phi + \sin^2\alpha\sin\phi\cos\phi}\big)\Big]$$

$$V = mg\ell\cos\alpha$$

$$\mathcal{L} = \frac{m}{2}\big(\dot{\ell}^2 + \ell^2\sin^2\alpha\,\underbrace{\dot{\phi}^2}_{\omega^2}\big) - mg\ell\cos\alpha$$

ϕ: coordenada cíclica, pero ya fijada externamente $\to \phi = \omega t$, es un parámetro, no una coordenada generalizada.

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\ell^2\sin^2\alpha\,\omega = P_\phi = cte \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

c) $\dfrac{\partial\mathcal{L}}{\partial\dot{\ell}} = m\dot{\ell} \to \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot{\ell}}\right) = m\ddot{\ell}$

$$\frac{\partial\mathcal{L}}{\partial\ell} = m\ell\sin^2\alpha\cdot\omega^2 - mg\cos\alpha$$

Como no piden fuerza de ligadura, no hace falta usar los multiplicadores de Lagrange, directamente sustituyes ω

$$\hookrightarrow\ m\ddot{\ell} - m\ell\sin^2\alpha\,\omega^2 + mg\cos\alpha = 0$$

$$\ddot{\ell} - \omega^2\ell\sin^2\alpha = -mg\cos\alpha$$

Ensayo: $\ell = A\cdot e^{\tilde{\alpha}t} + B\cdot e^{-\tilde{\alpha}t} + C$

$$\tilde{\alpha}^2Ae^{\tilde{\alpha}t} + \tilde{\alpha}^2Be^{-\tilde{\alpha}t} - (\omega\sin\alpha)^2(Ae^{\tilde{\alpha}t} + Be^{-\tilde{\alpha}t}) - \omega^2\sin^2\alpha\,C = -mg\cos\alpha$$

$$\hookrightarrow\ C = \frac{mg\cos\alpha}{\omega^2\sin^2\alpha} = \frac{mg}{\omega^2\sin\alpha\tan\alpha}$$

A: $\tilde{\alpha}^2 - (\omega \sin\alpha)^2 = 0$
B: idem

$\Big\}\ \tilde{\alpha} = \pm\ \omega \sin\alpha$

A y B: a determinar según condiciones iniciales

Por ejemplo:

$\ell(0) = A + B + C = \ell_0$

$\dot{\ell}(0) = A - B = 0 \rightarrow A = B \rightarrow 2A = -C + \ell_0 \Rightarrow A = -\dfrac{mg}{2\omega^2 \sin\alpha \tan\alpha} + \dfrac{\ell_0}{2}$

$$\ell(t) = \frac{mg}{2\omega^2 \sin\alpha \tan\alpha}\left(2 - e^{\omega \sin\alpha t} - e^{-\omega \sin\alpha t}\right) + \frac{\ell_0}{2}\left(e^{\omega \sin\alpha t} + e^{-\omega \sin\alpha t}\right)$$

$$= \frac{mg}{\omega^2 \sin\alpha \tan\alpha}\left(1 - \cosh(\omega \sin\alpha\, t)\right) + \ell_0 \cosh(\omega \sin\alpha\, t)$$

se escapa

$\phi = \omega t$

Se estabiliza en $\ell(t) = \dfrac{mg}{\omega^2 \sin\alpha \tan\alpha} = cte$

si $\ell_0 = \dfrac{mg \cos\alpha_{eq}}{\omega^2 \sin^2\alpha_{eq}} \rightarrow \dfrac{1-\cos^2\alpha_{eq}}{\cos\alpha_{eq}} = \dfrac{mg}{\ell_0 \omega^2}$

$\hookrightarrow \cos^2\alpha_{eq} + \dfrac{mg}{\ell_0\omega^2}\cos\alpha_{eq} - 1$

$$\alpha_{eq} = \frac{-\dfrac{mg}{\ell_0\omega^2} + \sqrt{\left(\dfrac{mg}{\ell_0\omega^2}\right)^2 + 4}}{2} \qquad \alpha \in (0, \pi/2)$$

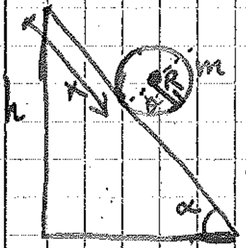
Si $\alpha > \alpha_{eq} \rightarrow$ cae a $\ell = 0$ $(\ell \rightarrow -\infty)$ para $t \rightarrow \infty$

Si $\alpha < \alpha_{eq} \rightarrow$ se escapa $(\ell \rightarrow \infty)$ " "

$\alpha = \alpha_{eq} \rightarrow$ equilibrio inestable (no es solución física relevante)

(3)

a)



2 coordenadas generalizadas: $\left\{ \begin{array}{l} \theta \\ x \end{array} \right.$

Si supones condición de rodadura

$\equiv$ 1 ligadura no holónoma

↳ "2-1" 1 grado de libertad

b)

θ, x

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

I: momento de inercia aro muy fino $= MR^2$

$$\hookrightarrow T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m R^2 \dot{\theta}^2$$

$$V = -mg z_{cm}$$

$$\left. \begin{array}{l} \cos\alpha = \dfrac{d}{R} \\ \sin\alpha = \dfrac{e}{x} \end{array} \right\} z_{cm} = e - \frac{d}{2}$$

$$= x \sin\alpha - \frac{R}{2}\cos\alpha$$

$$V = -mgx \sin\alpha + \frac{mgR\cos\alpha}{2} = -mgx\sin\alpha + cte$$

... origen, sin importancia

$$\mathcal{L} = T - V = \frac{m}{2}\left(\dot{x}^2 + R^2\dot{\theta}^2\right) + mgx\sin\alpha + cte$$

c)

Condición de rodadura: $R\,d\theta = dx$

↳ ligadura no holónoma

$$R\,d\theta - dx = 0 = \vec{\Lambda}\, d\vec{q} \; ; \; \Lambda = (R, -1) \; ; \; \vec{q} = (\theta, x)$$

Ec. Euler-Lagrange

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \lambda \cdot \Lambda_i$$

$m\ddot{x} - mg\sin\alpha = -\lambda$

$mR^2\ddot{\theta} = \lambda R$

$R\,d\theta = dx \rightarrow R\dot{\theta} = \dot{x} \rightarrow R\ddot{\theta} = \ddot{x}$

d) ↳ $\lambda = \frac{mR^2\ddot{\theta}}{R} = mR\ddot{\theta} = m\ddot{x}$

↳ $m\ddot{x} + \lambda - mg\sin\alpha = 2m\ddot{x} - mg\sin\alpha = 0$

↳ $\ddot{x} = \frac{g\sin\alpha}{2}$ → Aceleración

↳ $\ddot{\theta} = \frac{g}{2R}\sin\alpha$

Fuerza de rozamiento:

$\lambda = m\ddot{x} = \frac{mg\sin\alpha}{2}$

$F_x = -\frac{mg\sin\alpha}{2}$
(fuerza de contacto)

$F_\theta = \frac{mgR\sin\alpha}{2}$
(fuerza momento)

$\dot{x} = \int \ddot{x}\,dt = \frac{g\sin\alpha \cdot t}{2} + v_0$

$x = \frac{g\sin\alpha}{4} t^2 + v_0 \cdot t + x_0$

$\sin\alpha = \frac{h}{L} \rightarrow L = \frac{h}{\sin\alpha}$

Suponemos $x_0 = 0$, $v_0 = 0$

$L = \frac{g\sin\alpha}{4} t_f^2 \rightarrow t_f = \sqrt{\frac{4L}{g\sin\alpha}} = \sqrt{\frac{4h}{g\sin^2\alpha}}$

$v_f = \frac{g\sin\alpha}{2} t_f = \frac{g\sin\alpha}{2} \cdot 2\sqrt{\frac{h}{g\sin^2\alpha}} = \sqrt{gh}$

O por conservación de energía

Rodadura

$mgh = \frac{1}{2} m v_f^2 + \frac{1}{2} mR^2 \dot{\theta}_f^2 = \frac{1}{2} m v_f^2 + \frac{1}{2} mR^2 \frac{v_f^2}{R^2} = m v_f^2$

↳ $v_f = \sqrt{gh}$

Fuerza de contacto evita que deslice, pero es sobre 1 punto y no hace trabajo; convierte en potencial en rotación rueda de forma limpia.

$\ddot{x} = g\sin\alpha - 0$

Si no ligadura ($\lambda = 0$), aceleración sería el doble, cae más rápido, pero perderías realmente E por rozamiento si la superficie no es como un bloque de hielo.

↳ ($\mu \approx 0$)

(4)

$\vec{r} = \rho \cdot \hat{u}_r$

$\vec{v} = \dot{\rho} \cdot \hat{u}_r + \rho \cdot \dot{\hat{u}}_r = \dot{\rho}\, \hat{u}_r + \rho \dot{\theta} \cdot u_\theta$

$T = \frac{1}{2} m \vec{v}^2 = \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\theta}^2)$

$V = mg\cos\theta$ (origen en $\theta = \pi/2$)

$\mathcal{L} = T - V = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2) - mg\rho\cos\theta$

↳ Se desliza sobre alambre: Ligadura $\rho = R$

↳ $\rho - R = 0 \rightarrow \Lambda_i\, dq_i = 0$

↳ $(1, 0) \cdot (d\rho, d\theta) = 0$

↓ Ec. Euler-Lagrange con multiplicadores λ:

$$\begin{cases} m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda \\ m\rho^2\ddot{\theta} - mg\rho\sin\theta = 0 \end{cases}$$

$\rho = R$

$$\begin{cases} \lambda = mg\cos\theta - mR\dot{\theta}^2 \equiv \text{reacción del aro} \\ R\ddot{\theta} = g\sin\theta \rightarrow \ddot{\theta} = \frac{g}{R}\sin\theta \end{cases}$$

En el momento del despegue, $\lambda = 0$ (fuerza normal al plano se anula)

↳ $mg\cos\theta_{max} = mR\dot{\theta}_{max}^2$

↳ $\cos\theta_{max} = \frac{R}{g}\dot{\theta}_{max}^2$

$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{g}{R}\sin\theta \Rightarrow d\dot{\theta} = \frac{g}{R}\sin\theta\, dt = \frac{g}{R}\sin\theta \frac{d\theta}{\frac{d\theta}{dt}}$

↳ $\dot{\theta}\, d\dot{\theta} = \frac{g}{R}\sin\theta \cdot d\theta$ $\quad \int_0^{max}$

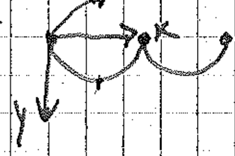
$\frac{\dot{\theta}_{max}^2}{2} = -\frac{g}{R}(\cos\theta_{max} - 1)$

$\cos\theta_{max} = \frac{2R}{g} \cdot \frac{g}{R}(1 - \cos\theta_{max})$

↳ $\cos\theta_{max} = \frac{2}{3}$

↳ $\theta_{max} = a\cos(2/3)$

⑤ $\begin{cases} x(\theta) = R(\theta - \sin\theta) \to \dot{x} = R\dot{\theta}(1-\cos\theta) \\ y(\theta) = R(1-\cos\theta) \to \dot{y} = R\dot{\theta}\sin\theta \end{cases}$



a) $T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}R^2\dot{\theta}^2(1+\cos^2\theta - 2\cos\theta + \sin^2\theta)$

$= \frac{m}{2}R^2\, 2(1-\cos\theta)\dot{\theta}^2 = 2mR^2\sin^2\frac{\theta}{2}\cdot\dot{\theta}^2$

$V = -mgy = -mgR(1-\cos\theta) = -2mgR\sin^2\frac{\theta}{2}$

$\mathcal{L} = T - V = 2mR^2\dot{\theta}^2\sin^2\frac{\theta}{2} + 2mgR\sin^2\frac{\theta}{2}$

$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 4mR^2\dot{\theta}\sin^2\frac{\theta}{2} \to \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = 4mR^2\ddot{\theta}\sin^2\frac{\theta}{2} + 4mR^2\dot{\theta}^2\cdot\underbrace{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{1}{2}}_{=\sin\theta}$

$\frac{\partial\mathcal{L}}{\partial\theta} = 2mR(R\dot{\theta}^2+g)\frac{\sin\theta}{2}$

$\hookrightarrow$ E-L: $4mR^2\ddot{\theta}\sin^2\frac{\theta}{2} + 2mR^2\dot{\theta}^2\sin\theta - mR^2\dot{\theta}^2\sin\theta - gmR\sin\theta = 0$

$\hookrightarrow 4\ddot{\theta}\sin^2\frac{\theta}{2} + \dot{\theta}^2\sin\theta - \frac{g}{R}\sin\theta = 0$

$\ddot{\theta}\sin^2\frac{\theta}{2} + \left(\dot{\theta}^2 - \frac{g}{R}\right)\frac{\sin\theta}{4} = 0$

Pasamos a coordenada generalizada s (arco): $ds^2 = dx^2 + dy^2$

Vía 1: $dx = R\,d\theta(1-\cos\theta)$; $dy = R\,d\theta\sin\theta$

$ds^2 = R^2 d\theta^2(\sin^2\theta + 1 + \cos^2\theta - 2\cos\theta) = 4R^2\sin^2\theta/2\; d\theta^2$

Vía 2: $\ddot{\theta}\sin^2\frac{\theta}{2} + \left(\dot{\theta}^2 - \frac{g}{R}\right)\frac{2\sin\theta/2\cos\theta/2}{4} = 0$

$\hookrightarrow \ddot{\theta}\sin\frac{\theta}{2} + \left(\dot{\theta}^2 - \frac{g}{R}\right)\frac{1}{2}\cos\frac{\theta}{2}$

Por vía 2, cambio de variable a: $q = \cos\theta/2 \propto s = 4R\cos\theta/2$

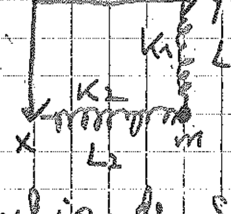
$\dot{q} = -\frac{1}{2}\sin\frac{\theta}{2}\dot{\theta} \to \ddot{q} = -\frac{1}{4}\cos\frac{\theta}{2}\cdot\dot{\theta}^2 - \frac{1}{2}\sin\frac{\theta}{2}\cdot\ddot{\theta}$

$\ddot{\theta}\sin\theta/2 = -2\ddot{q} - \frac{1}{2}\dot{\theta}^2\cos\frac{\theta}{2}$ → sustituir en ec. de movimiento

$-2\ddot{q} - \cancel{\frac{1}{2}\dot{\theta}^2\cos\frac{\theta}{2}} + \cancel{\dot{\theta}^2\cdot\frac{1}{2}\cos\theta/2} - \frac{g}{2R}q = 0$

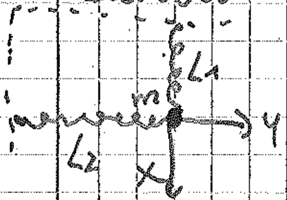
$\hookrightarrow \ddot{q} + \frac{g}{4R}q = 0$ → ec. oscilador armónico → $q \propto s$, la variable de arco oscila alrededor del equilibrio ($\theta = \pi/2$ $\hookrightarrow s = 4R/\sqrt{2}$)

⑥

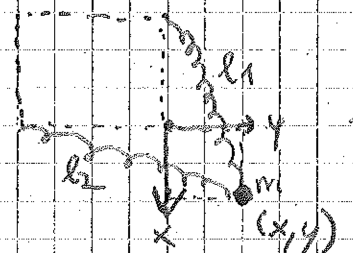


L_1, L_2 : longitudes en reposo de los muelles. En la posición (L_1, L_2), la m está en equilibrio.

Cambio de s.ref, lo centro en (L_1, L_2).



→ Fuera del equilibrio:



$$l_1^2 = (L_1+x)^2 + y^2$$
$$l_2^2 = (L_2+y)^2 + x^2$$
(por Pitágoras)

2 grados de libertad: $\{x, y\}$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$V = \frac{1}{2}k_1(l_1 - L_1)^2 + \frac{1}{2}k_2(l_2 - L_2)^2 = \frac{1}{2}k_1 l_1^2 + \frac{1}{2}k_1 L_1^2 + \frac{1}{2}k_2 l_2^2 + \frac{1}{2}k_2 L_2^2 - k_1 l_1 L_1 - k_2 l_2 L_2$$

$$= \frac{1}{2}(k_1 l_1^2 + k_2 l_2^2) + \text{ctes} - k_1 l_1 L_1 - k_2 l_2 L_2$$

$$= \frac{1}{2}\left[k_1((L_1 + x)^2 + y^2) + k_2((L_2 + y)^2 + x^2)\right]$$
$$- \left[k_1 L_1 \underbrace{\sqrt{(L_1+x)^2 + y^2}}_{\sqrt{}_1} + k_2 L_2 \underbrace{\sqrt{(L_2+y)^2 + x^2}}_{\sqrt{}_2}\right] + \text{ctes}$$

$$\mathcal{L} = T - V = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + k_1 L_1 \sqrt{(L_1+x)^2 + y^2} + k_2 L_2 \sqrt{(L_2+y)^2 + x^2}$$
$$- \frac{k_1}{2}\left[(L_1+x)^2 + y^2\right] - \frac{k_2}{2}\left[(L_2+y)^2 + x^2\right] + \text{ctes}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = k_1 L_1 \frac{1}{2\sqrt{}_1} \cdot 2(L_1 + x) + k_2 L_2 \frac{1}{2\sqrt{}_2} \cdot 2x$$
$$- 2\frac{k_1}{2}(L_1 + x) - 2\frac{k_2}{2}x$$

$$m\ddot{x} = k_1\left(\frac{L_1}{\sqrt{}_1} - 1\right)(L_1 + x) + k_2\left(\frac{L_2}{\sqrt{}_2} - 1\right) \cdot x$$

Idem para y, por simetría: $x \leftrightarrow y$, $1 \leftrightarrow 2$

$$m\ddot{y} = k_2\left(\frac{L_2}{\sqrt{}_2} - 1\right)(L_2 + y) + k_1\left(\frac{L_1}{\sqrt{}_1} - 1\right) y$$

Si $k_1 = k_2 = k$, $L_1 = L_2 = L$

$$m\ddot{x} = k\left[\left(\frac{L}{\sqrt{(L+x)^2+y^2}} - 1\right)(L+x) + \left(\frac{L}{\sqrt{(L+y)^2+x^2}} - 1\right) x\right]$$

$m\ddot{y}$ = Idem intercambiando $x \leftrightarrow y$

↳ No tiene solución analítica, no es oscilador armónico.

En el límite $x, y \gg L$ (oscilaciones grandes): ↳ movimiento raro, ¿caótico?

$(x_0, y_0) = (A, A)$ ↳ $\frac{L}{\sqrt{(L+x)^2+y^2}} \approx \frac{L}{\sqrt{x^2+y^2}} \ll 1$

↳ $m\ddot{x} = -k(L+x) - kx \approx -2kx$

$$\left.\begin{aligned} \ddot{x} + \frac{2k}{m}x &= 0 \\ \ddot{y} + \frac{2k}{m}y &= 0 \end{aligned}\right\} \text{Oscilador armónico}$$

↳ $(x, y) = A(\cos\omega t, \sin\omega t)$

con $\omega = \sqrt{\frac{2k}{m}}$

Si $k_1 \neq k_2 \to k = k_1 + k_2$, no cambia ↑.
$L_1 \neq L_2$ → no influye

Es decir, la anisotropía del sistema no es relevante para oscilaciones grandes. A grandes distancias se ve como un punto, no importa lo que pase localmente cerca del equilibrio (hace cosas raras).
↳ y los muelles ≈ paralelos

Macroscópicamente domina la oscilación armónica.

Este procedimiento es similar a realizar una expansión multipolar a 1er orden.

(7) $\alpha(\vec{q}, \dot{\vec{q}}, \ddot{\vec{q}}, t)$

$$S = \int \alpha \cdot dt$$

a) siempre se hacen variaciones a t fijo (ppo. D'Alembert)

$$\delta S = 0 = \int_{t_1}^{t_2} \delta\alpha \cdot dt$$

$$\delta\alpha = \frac{\partial\alpha}{\partial q_i}\delta q_i + \frac{\partial\alpha}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial\alpha}{\partial \ddot{q}_i}\delta \ddot{q}_i + \overset{t \text{ fijo}}{0}$$

$$= \frac{\partial\alpha}{\partial q_i}\delta q_i + \frac{\partial\alpha}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) + \frac{\partial\alpha}{\partial \ddot{q}_i}\frac{d}{dt}\left(\frac{d}{dt}\delta q_i\right)$$

$$\delta S = \int_{t_1}^{t_2} \delta\alpha \cdot dt$$

$$\to \int_{t_1}^{t_2} \frac{\partial\alpha}{\partial \dot{q}_i} \cdot \frac{d}{dt}(\delta q_i)\,dt \overset{\text{por partes}}{=} \frac{\partial\alpha}{\partial \dot{q}_i} \cdot \cancel{\delta q_i}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial\alpha}{\partial \dot{q}_i}\right)\delta q_i \cdot dt$$

0 porque $\delta q_i(t_1) = \delta q_i(t_2)$, $\frac{\partial\alpha}{\partial \dot{q}_i}(t_1) = \frac{\partial\alpha}{\partial \dot{q}_i}(t_2)$ } extremos fijos (enunciado)

$$\to \int_{t_1}^{t_2} \frac{\partial\alpha}{\partial \ddot{q}_i}\frac{d}{dt}\left(\frac{d}{dt}\delta q_i\right)dt \overset{\text{por partes}}{=} \cancel{\frac{\partial\alpha}{\partial \ddot{q}_i} \cdot \frac{d}{dt}(\delta q_i)\Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d}{dt}(\delta q_i) \cdot \frac{d}{dt}\left(\frac{\partial\alpha}{\partial \ddot{q}_i}\right) \cdot dt$$

0 por idem

$$\overset{\text{p. partes}}{=} - \cancel{\delta(q_i) \cdot \frac{d}{dt}\left(\frac{\partial\alpha}{\partial \ddot{q}_i}\right)\Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2} \frac{d^2}{dt^2}\left(\frac{\partial\alpha}{\partial \ddot{q}_i}\right) \cdot \delta(q_i)$$

$$\to \delta S = \int_{t_1}^{t_2}\left(\frac{d^2}{dt^2}\left(\frac{\partial\alpha}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial\alpha}{\partial \dot{q}_i}\right) + \frac{\partial\alpha}{\partial q_i}\right) \cdot \delta q_i \cdot dt = 0$$

$\to \delta q_i$ a t cte $\to$ integrando = 0

$$\frac{d^2}{dt^2}\left(\frac{\partial\alpha}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial\alpha}{\partial \dot{q}_i}\right) + \frac{\partial\alpha}{\partial q_i} = 0$$

b)

$$\mathcal{L} = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2$$

$$\frac{\partial \mathcal{L}}{\partial \ddot{q}} = -\frac{m}{2} q \rightarrow \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\right) = -\frac{m}{2}\ddot{q}$$

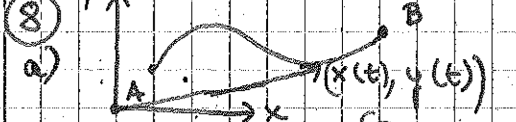
$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial q} = -\frac{m}{2}\ddot{q} - Kq$$

$$\Rightarrow -\frac{m}{2}\ddot{q} - \frac{m}{2}\ddot{q} - Kq = 0$$

↳ $\ddot{q} + \frac{k}{m} q = 0$ → oscilador armónico

(8) a)



$\vec{r} = (x(t), y(t))$; $\vec{r}_{A,B} = (x_{A,B}, y_{A,B})$ fijos $\forall t$

Longitud $S_{AB} = \int_{S_A}^{S_B} ds = \int_{t_A}^{t_B} \frac{ds}{dt}\cdot dt$; $ds^2 = dx^2 + dy^2$

$$S_{AB} = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t_A}^{t_B} \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} dt = \int_{t_A} L(\dot{\vec{r}})\cdot dt$$

$$\delta S_{AB} = \int_{t_A}^{t_B} \delta L(\dot{\vec{r}})\, dt = \int_{t_A}^{t_B} \left[L(\dot{\vec{r}} + \delta\dot{\vec{r}}) - L(\dot{\vec{r}})\right] dt =$$

$$= \int_{t_A}^{t_B} \Big[L(\dot{\vec{r}}) + \underbrace{\frac{\partial L}{\partial \dot{r}_i}\delta \dot{r}_i}_{\text{Taylor}} - L(\dot{\vec{r}})\Big] dt = \int_{t_A}^{t_B} \frac{\partial L}{\partial \dot{r}_i}\frac{d}{dt}(\delta r_i)\, dt =$$

$$= \int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\delta r_i\right) dt - \int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right)\cdot \delta r_i\, dt = \cancel{\frac{\partial L}{\partial \dot{r}_i}\delta r_i \Big|_{t_A}^{t_B}} = 0$$

porque $\vec{r}_{A,B}$ fijos $\forall t$

$$-\int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right)\cdot \delta r_i \cdot dt = \delta S_{AB} = 0$$

↳ extremal (longitud mínima)

→ como δr_i son independientes, → $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = 0$

$\Rightarrow \frac{\partial L}{\partial \dot{r}_i} = \text{cte} \rightarrow \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}\cdot\frac{2\dot{x}}{L} = \frac{\dot{x}}{L} = \text{cte}_1$

No demuestro que sea única ni que sea el mínimo, por simplicidad.

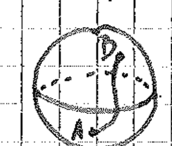
↳ $\frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{L} = \text{cte}_2$ ⟹ $\frac{\dot{x}}{\dot{y}} = \text{cte}_3$ → Solución trivial (la + sencilla):

$\left.\begin{array}{l}\dot{x} = \text{cte}_4 = \lambda \\ \dot{y} = \text{cte}_5 = \beta\end{array}\right\}$ $\vec{r} = \begin{pmatrix}\lambda \\ \beta\end{pmatrix}\cdot t$ (ecuación paramétrica de la recta)

↳ $x = \lambda t$; $y = \beta t$

↳ $y = \beta x/\lambda = \beta/\lambda \cdot x$

b) Esfera $\to$ idem pero con 3 coordenadas y 1 ligadura



$$S_{AB} = \int_{t_A}^{t_B} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\cdot dt \to L = L(\dot{x},\dot{y},\dot{z}) = L(\dot{\vec{r}})$$

$x^2+y^2+z^2 = R^2$ $\to$ mult. Lagrange

$$L' = \underbrace{\sqrt{x'^2+y'^2+z'^2}}_{L} + \lambda(x^2+y^2+z^2)$$

$$S_{AB} = \int L(\vec{r},\dot{\vec{r}})\,dt = \int \left(L' - \lambda(x^2+y^2+z^2)\right)dt$$

$\hookrightarrow \delta S_{AB} = \int \delta L\cdot dt = \int dL'\,dt - \int \lambda\,\delta(\underbrace{x^2+y^2+z^2}_{R^2})\,dt$ $\to 0$

$\hookrightarrow \delta S_{AB} = \int \delta L'(\vec{r},\dot{\vec{r}})\,dt = \int \left[L'(\vec{r}+\delta\vec{r},\dot{\vec{r}}+\delta\dot{\vec{r}}) - L'(\vec{r},\dot{\vec{r}})\right]dt$

$$= \int\left[L'(\vec{r},\dot{\vec{r}}) + \frac{\partial L'}{\partial r_i}\delta r_i + \frac{\partial L'}{\partial \dot{r}_i}\delta \dot{r}_i - L'(\vec{r},\dot{\vec{r}})\right]dt$$

$$= \int\left(\frac{\partial L'}{\partial r_i}\delta r_i + \frac{\partial L'}{\partial \dot{r}_i}\frac{d}{dt}(\delta r_i)\right)dt \overset{\text{idem a a)}}{=} \int\left[\frac{\partial L'}{\partial r_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}_i}\right)\right]dt\,\delta r_i = 0$$

$\hookrightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}_i}\right) - \frac{\partial L'}{\partial r_i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right) - 2\lambda x = 0$

$\hookrightarrow$ idem para y, z

$\hookrightarrow \frac{d}{dt}\underbrace{\left\{\frac{(\dot{x},\dot{y},\dot{z})}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right\}}_{\equiv\hat{\tau}} - 2\lambda(x,y,z) = 0$ $\qquad \hat{\tau} = \hat{\dot{\vec{r}}} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$

$\hookrightarrow$ Vector tangente

$\hookrightarrow \frac{d}{dt}\hat{\tau} = 2\lambda\vec{r} \Rightarrow \vec{r}\times\left(\frac{d}{dt}\hat{\tau}\right) = -2\lambda\,\vec{r}\times\vec{r} = 0$

$\hookrightarrow \frac{d}{dt}(\vec{r}\times\hat{\tau}) = \dot{\vec{r}}\times\hat{\tau} + \vec{r}\times\left(\frac{d\hat{\tau}}{dt}\right) = \underbrace{|\dot{\vec{r}}|\,\hat{\tau}\times\hat{\tau}}_{0} + \underbrace{\vec{r}\times\left(\frac{d\hat{\tau}}{dt}\right)}_{0} = 0$

$\Rightarrow \vec{r}\times\hat{\tau} = \vec{C}$ $\to$ Implica circunferencia de radio máximo

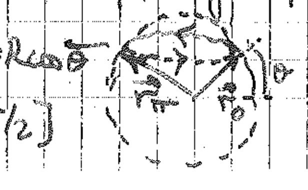
$\hookrightarrow$ Implica que trayectoria está contenida en un plano

$\hookrightarrow$ circunferencia

$\hookrightarrow$ Implica que su radio es máximo, porque si no $\vec{C}$ cambiaría por ejemplo en 0 y π

Radio $\rho = R\cos\theta$

$\theta \in [0, \pi/2]$

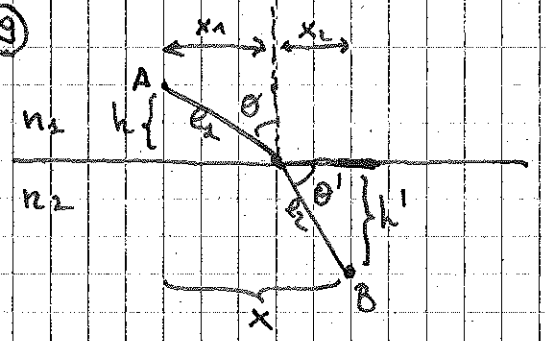


$\vec{r}_0 = R(\cos\theta, \sin\theta, 0)$ $\quad \vec{\tau}_0 = (0,0,-1) \to \vec{C}_0 = \pm(-\sin\theta, \cos\theta, 0)$

$\vec{r}_\pi = R(-\cos\theta, \sin\theta, 0)$ $\quad \vec{\tau}_\pi = (0,0,1) \to \vec{C}_\pi = (\sin\theta, \cos\theta, 0)$

$\vec{C}_0 = \vec{C}_\pi = \vec{C}$ sii $\theta = 0$ $\to$ Radio máximo $\rho = R$

⑨



Dos puntos arbitrarios A y B, fijos.

Principio de Fermat: El rayo de luz sigue la trayectoria que minimiza el camino óptico.

Camino óptico $L = \int n \cdot ds$

En este caso, $L = L_1 + L_2 = n_1 \ell_1 + n_2 \ell_2$

A y B fijos $\to x, h, h'$ ctes

$$\left.\begin{aligned} \tan\theta &= \frac{x_1}{h} \\ \tan\theta' &= \frac{x_2}{h'} \end{aligned}\right\} \quad x = x_1 + x_2 = \text{cte} = h\tan\theta + h'\tan\theta'$$

$$dx = 0 = -\frac{h}{\cos^2\theta}\,d\theta - \frac{h'}{\cos^2\theta'}\,d\theta'$$

$$\hookrightarrow \frac{h'}{\cos^2\theta'}\,d\theta' = -\frac{h}{\cos^2\theta}\,d\theta$$

$$\left.\begin{aligned} \cos\theta &= \frac{h}{\ell_1} \\ \cos\theta' &= \frac{h'}{\ell_2} \end{aligned}\right\} \quad L = n_1\frac{h}{\cos\theta} + n_2\frac{h'}{\cos\theta'}$$

Ppo. de Fermat: $\delta L = 0$

$$\delta L = -\frac{n_1 h}{\cos^2\theta}\sin\theta\,d\theta + (-)n_2\frac{h'\sin\theta'}{\cos^2\theta'}\,d\theta' =$$

$$= -\frac{n_1 h}{\cos^2\theta}\sin\theta\,d\theta + n_2\sin\theta'\frac{h}{\cos^2\theta}\,d\theta = 0 \qquad \theta \in [0, \tfrac{\pi}{2})$$

$$\hookrightarrow \frac{h}{\cos^2\theta}\left(n_2\sin\theta' - n_1\sin\theta\right)d\theta = 0 \qquad \forall\theta$$

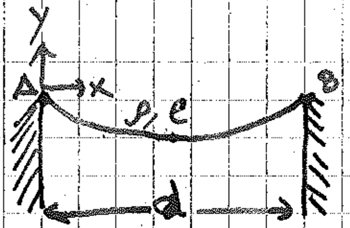
↳ Implica ley de Snell:

$$n_1\sin\theta = n_2\sin\theta'$$

Idem para la reflexión, pero con $n_1 = n_2$, y θ' hacia arriba $\theta \, \theta'$

$\hookrightarrow \sin\theta = \sin\theta' \to \theta = \theta'$

(10)



$\ell = cte$ (no es elástico)

$$\ell = \int_A^B ds = \int_A^B \sqrt{(dx)^2+(dy)^2} = \int_{x=0\equiv x_A}^{d\equiv x_B} \sqrt{1+(y')^2}\cdot dx \quad ; \quad y' \equiv y'(x) \equiv \frac{dy}{dx}$$

... x_i fijo

↳ ℓ es análogo a la acción, con un lagrangiano efectivo $L(y') \equiv \sqrt{1+(y')^2}$, donde x hace el papel del parámetro t:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = A \implies \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = A = \frac{y''}{\sqrt{1+y'^2}} - \frac{(y')^2 y''}{(1+y'^2)^{3/2}}$$

$$\delta\ell = \int_0^d (A\cdot\delta y)\cdot dx = 0 \qquad A = \frac{y''+y''y'^2-y'^2y''}{(1+y'^2)^{3/2}} = y''/(1+y'^2)^{3/2}$$

Por otro lado, la cuerda colgará de manera que minimice el potencial

densidad lineal ρ

$$V = \int_A^B dm\cdot g\cdot y = g\int_A^B \rho\cdot ds\cdot y = \rho g \int_0^d \sqrt{1+y'^2}\cdot y\, dx$$

$$L'(y,y') = \rho g\, y\sqrt{1+y'^2} \;\rightarrow\; \delta V = \int_0^d B\,\delta y\, dx$$

$$B = \frac{d}{dx}\left(\frac{\partial L'}{\partial y'}\right) - \frac{\partial L}{\partial y} = \rho g\left\{\frac{d}{dx}\left(\frac{y\cdot y'}{\sqrt{1+y'^2}}\right) - \sqrt{1+y'^2}\right\}$$

$$= \rho g\left\{\frac{(y'^2+yy'')\cdot(1+y'^2) - yy'^2y'' - (1+y'^2)^2}{(1+y'^2)^{3/2}}\right\}$$

$$= \rho g\left\{y'^2 + \cancel{y'^4} + yy'' + \cancel{yy'^2y''} - \cancel{yy'^2y''} - 1 - \cancel{y'^4} - 2y'^2\right\}/(1+y'^2)^{3/2}$$

$$= \rho g\{yy'' - y'^2 - 1\}/\{1+y'^2\}^{3/2}$$

Como tenemos dos acciones, se deben cumplir simultáneamente: Basta con que A y B sean proporcionales (Comparten subespacio)

↳ $A - \lambda B = 0$

Equivale a escribir $\tilde{L} = L' - \lambda L$

$$A - \lambda B = 0$$

$$\to \frac{y'' - \lambda \rho g (y y'' - y'^2 - 1)}{(1+y'^2)^{3/2}} = 0 = \frac{y'' - \lambda \rho g\, y y''}{(1+y'^2)^{3/2}} + \rho g \lambda (1+y'^2)^{-1/2} = 0$$

$$\to y''(1 - \lambda \rho g y) + \rho g \lambda (1 + y'^2) = 0$$

$$y'' \left(\frac{1}{\rho g \lambda} - y \right) + (1+y'^2) = 0$$

Es inmediato comprobar que las soluciones tipo $\cosh(ax) + B$ son solución de la ecuación. Las constantes se determinan con ℓ y d.

(11)

superficie esfera

$$F_r \propto \dot z,\ \eta,\ S_{esf},\ \frac{1}{m}$$

z $\uparrow$ η (m) $\updownarrow 2R$ $\downarrow g$

$F_r = 6\pi\eta\,\dot z\cdot R$ (fórmula de Stokes)
↳ caso 1D

$$T = \frac{m}{2}\dot z^2\ ;\ V = +mgz \rightarrow \mathcal{L} = T - V = \frac{m}{2}\dot z^2 - mgz$$

Función de disipación $\mathcal{F} = ?$ $\rightarrow$ $F_r = \frac{\partial}{\partial \dot z}\mathcal{F} \rightarrow \mathcal{F} = \int F_r\cdot d\dot z =$

$$\mathcal{F} = 3\pi\eta\,\dot z^2 R$$

Ec. de Euler - Lagrange con disipación

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right) - \frac{\partial L}{\partial z} + \frac{\partial \mathcal{F}}{\partial \dot z} = 0$$

$$\hookrightarrow\ m\ddot z + mg + 6\pi\eta R\dot z = 0$$

$$\ddot z + \frac{6\pi\eta R}{m}\dot z + g = 0 \qquad \text{Ec. de movim.}$$

Cambio $v = \dot z$

$$\dot v + \frac{6\pi\eta R}{m}v + g = 0$$

$$\frac{dv}{dt} = -g - \frac{6\pi\eta R}{m}v \rightarrow -dt = \frac{dv}{g + \frac{6\pi\eta R}{m}v}$$

$$-\int_{t_0}^{t} dt = -(t - t_0) = \int_{v(t_0)}^{v(t)} \dots = \frac{m\ln}{6\pi\eta R}\left(g + \frac{6\pi\eta R}{m}v\right)\Bigg|_{t_0}^{t}$$

$$\hookrightarrow \frac{g + \frac{6\pi\eta R}{m}v(t)}{g + \frac{6\pi\eta R}{m}v(0)} = e^{-\frac{6\pi\eta R}{m}(t-t_0)} \qquad \gamma = \frac{6\pi\eta R}{m}$$

$$\hookrightarrow g + \gamma v(t) = [g + \gamma v(0)]\cdot e^{-\gamma(t-t_0)}$$

$$v(t) = -\frac{g}{\gamma}\left(1 - e^{-\gamma(t-t_0)}\right) + v(0)\cdot e^{-\gamma(t-t_0)}$$

$$z(t) - z(t_0) = \int v(t)\cdot dt = -\frac{g}{\gamma}\left(t + \frac{1}{\gamma}e^{-\gamma(t-t_0)}\right) - \frac{v(0)}{\gamma}e^{-\gamma(t-t_0)}\Bigg|_{t_0}^{t}$$

$$= -\frac{g}{\gamma}\left((t-t_0) + \frac{1}{\gamma}e^{-\gamma(t-t_0)} - \frac{1}{\gamma}\right) - \frac{v(0)}{\gamma}e^{-\gamma(t-t_0)} + \frac{v(0)}{\gamma}$$

$$v(t\to\infty) = -\frac{g}{\gamma} = -\frac{mg}{6\pi\eta R} \quad \text{(velocidad terminal)}$$

(12)

$$L = \frac{m}{2}(\dot q_1^2 + \dot q_2^2) - mg\,q_2 - q_3\left(\sqrt{q_1^2 + q_2^2} - \ell\right)$$

$$\frac{\partial L}{\partial \dot q_1} = m\dot q_1 \quad ; \quad \frac{\partial L}{\partial \dot q_2} = m\dot q_2 \quad ; \quad \frac{\partial L}{\partial \dot q_3} = 0$$

$$\frac{\partial L}{\partial q_1} = -q_3 q_1\left(\sqrt{q_1^2 + q_2^2}\right)^{-1/2}$$

$$\frac{\partial L}{\partial q_2} = -mg - q_3 q_2 (q_1^2 + q_2^2)^{-1/2}$$

$$\frac{\partial L}{\partial q_3} = -\left(\sqrt{q_1^2 + q_2^2} - \ell\right)$$

$\hookrightarrow$ E.L$_1$: $\quad m\ddot q_1 + q_3 q_1 (q_1^2 + q_2^2)^{-1/2} = 0$

E.L$_2$: $\quad m\ddot q_2 + mg + q_3 q_2 (q_1^2 + q_2^2)^{-1/2} = 0$

E.L$_3$: $\quad \sqrt{q_1^2 + q_2^2} - \ell = 0 \qquad (\ell = \text{cte})$

$$\hookrightarrow \begin{cases} m\ddot q_1 + q_3 q_1/\ell = 0 \\ m\ddot q_2 + q_3 q_2/\ell + mg = 0 \end{cases}$$

$\to$ Ecus de movimiento (q$_3$ es conde una ligadura, es un multiplicador) $\searrow$ $\nexists\ \dot q_3$ en L
variable auxiliar que mejora aspecto del Lagrangiano.

$$p_1 = \frac{\partial L}{\partial \dot q_1} = m\dot q_1 \quad ; \quad p_2 = \frac{\partial L}{\partial \dot q_2} = m\dot q_2 \quad ; \quad p_3 = 0$$

$$H = m\dot q_1^2 + m\dot q_2^2 - L = \frac{m}{2}(\dot q_1^2 + \dot q_2) + mg\,q_2 + q_3(\underbrace{\sqrt{q_1^2 + q_2^2} - \ell}_{=0}) = E$$

$$H = E = \frac{m}{2}(\dot q_1^2 + \dot q_2^2) + mg\,q_2 \quad \to E \text{ no depende explícitamente de } q_3$$

Truco: derivar ligadura respecto a t para relacionar con Euler-Lagrange:

$$\ell^2 = q_1^2 + q_2^2 \quad \to \quad 0 = 2q_1\dot q_1 + 2q_2\dot q_2 = \frac{d}{dt}\ell^2$$

$$\hookrightarrow \frac{d^2}{dt^2}\ell^2 = 2\ddot q_1 q_1 + 2\dot q_1^2 + 2\ddot q_2 q_2 + 2\dot q_2^2 = 0$$

$$\hookrightarrow \dot q_1^2 + \dot q_2^2 = -\ddot q_1 q_1 - \ddot q_2 q_2 = q_1\frac{q_3 q_1}{m\ell} + q_2\left(\frac{q_3 q_2}{m\ell} + g\right)$$

$$= \frac{q_3}{m\ell}(\underbrace{q_1^2 + q_2^2}_{\ell^2}) + q_2 g = q_3\frac{\ell}{m} + q_2 g$$

$$E = \frac{m}{2}(\dot q_1^2 + \dot q_2^2) + mg q_2 = q_3\frac{\ell}{2} + \frac{m}{2}q_2 g + mg q_2 = q_3\frac{\ell}{2} + \frac{3}{2}m q_2 g$$

$$\hookrightarrow q_3 = \underline{\frac{2E - 3mq_2 g}{\ell}}$$

(13)

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - q\left(\phi - \frac{1}{c}\dot{\vec{x}} \cdot \vec{A}\right) = \frac{m}{2}\dot{x}_i\dot{x}_i - q\left(\phi - \frac{1}{c}\epsilon_{ijk}\dot{x}_j A_k\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = 0$$

↳ $\frac{\partial \mathcal{L}}{\partial \dot{x}_m} = p_m$ se conserva

$$p_m = m\dot{x}_m + \frac{q}{c}\epsilon_{ijk}A_k \cdot \delta_{jm}$$

$\phi(y,z)$; $\vec{A}(y,z)$

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + \frac{q}{c}(v_x A_x + v_y A_y + v_z A_z)$$

x cíclica: $\frac{\partial \mathcal{L}}{\partial x} = 0$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x = cte$$

↳ campo y partícula se intercambian momento

(14) $(\phi, A) \rightarrow (\phi', A')$: $\phi' = \phi - \frac{1}{c}\frac{\partial \chi(x,t)}{\partial t}$; $\vec{A}' = \vec{A} + \vec{\nabla}\chi(x,t)$

$$\mathcal{L} = T - q\phi + \frac{q}{c}\vec{v} \cdot \vec{A} = \frac{m}{2}\dot{x}_k\dot{x}_k - q\phi(x) + \frac{q}{c}\dot{x}_k A_k$$

$$\mathcal{L}' = \frac{m}{2}\dot{x}_k\dot{x}_k - q\phi + \frac{q}{c}\frac{\partial \chi(x,t)}{\partial t} + \frac{q}{c}\dot{x}_k A_k + \frac{q}{c}\dot{x}_k\frac{\partial}{\partial x_k}\chi(x,t)$$

$$= \mathcal{L} + \frac{q}{c}\left(\frac{\partial}{\partial t} + \dot{x}_k\partial_k\right)\chi(x,t)$$

Aparte $\frac{d}{dt}[\chi(x,t)] = \frac{\partial \chi}{\partial x_i}\cdot\frac{\partial x_i}{\partial t} + \frac{\partial \chi}{\partial t} = \dot{x}_i\partial_i\chi + \frac{\partial}{\partial t}\chi = \left(\frac{\partial}{\partial t} + \dot{x}_k\partial_k\right)\chi$

↳ $\mathcal{L}' = \mathcal{L} + \frac{q}{c}\frac{d}{dt}\chi(x,t)$

→ La nueva lagrangiana difiere en una derivada total, con lo que no afecta a la física, porque al integrar $S' = \int L\,dt$ y estar los extremos fijos, se cancela.
$= S + cte$ ↳ al hacer variaciones $\delta S' = \delta S$

Esto también puede verse en las ecuaciones de Lagrange:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}'}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\frac{dF}{dt}\right) - \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) \quad F(q,t)$$

$$= \ldots + \underbrace{\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}\right)\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)}_{=0}$$

porque hay igualdad de derivadas cruzadas y $\frac{\partial F}{\partial \dot{q}} = 0$.

Por otro lado, los campos E y B tampoco cambian, pues:

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \underbrace{\vec{\nabla}\left(\frac{1}{c}\frac{\partial \chi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\chi}_{=0}$$

$$= \vec{E} \checkmark$$

derivadas cruzadas

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times(\vec{\nabla}\chi)}_{=0} = \vec{B} \checkmark$$

Las transformaciones de Gauge no cambian la física, a nivel de campos y ecuaciones de movimiento (lo que mides). Por tanto hay grados de libertad en los potenciales que puedes fijar sin alterar la física (por ejemplo el origen de potencial, etc).

(15)

$$\vec{B} = (0, 0, B) = B\hat{z}$$

a) $\vec{A} = -yB\hat{x}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow B_i = \epsilon_{ijk}\partial_j A_k = \epsilon_{ijx}\partial_j \underset{-yB}{\underset{\parallel}{A_x}} = \epsilon_{iyx}\partial_y A_x \Rightarrow B_z = \epsilon_{zyx}(-B) = +B$$

o bien

$$\vec{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yB & 0 & 0 \end{vmatrix} = B\hat{k} = B\hat{z} \checkmark$$

b) $\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{q}{c}\dot{x}\cdot By$

$x, z \to$ cíclicas $\left(\frac{\partial\mathcal{L}}{\partial x} = \frac{\partial\mathcal{L}}{\partial z} = 0\right)$

c) $p_x = cte = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} - \frac{qBy}{c} \to \dot{x} = \frac{p_x + qBy/c}{m}$

$p_z = cte = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z} \to z = \frac{p_z}{m}t + z_0 = v_z t + z_0$

$\to \dot{z} = p_z/m$

$\frac{\partial\mathcal{L}}{\partial\dot{y}} = m\dot{y}$; $\frac{\partial\mathcal{L}}{\partial y} = -\frac{q}{c}\dot{x}B$

$\to m\ddot{y} + \frac{q}{c}\dot{x}B = 0 = m\ddot{y} + \frac{q}{c}B\cdot\left(\frac{p_x + qBy/c}{m}\right)$

$\ddot{y} + \frac{qB}{mc}p_x + \frac{q^2B^2}{m^2c^2}y = 0$ → tipo osc. armónico desplazado

Ensayo $y(t) = A\cos(\alpha t + \delta) + C$

$$-\alpha^2 + \frac{q^2B^2}{m^2c^2} = 0 \rightarrow \alpha = \frac{qB}{mc} \rightarrow \text{frecuencia de ciclotrón}$$

$$\frac{qB}{m^2c} p_x + \frac{q^2B^2}{m^2c^2} G = 0 \rightarrow G = -\frac{p_x c}{qB}$$

$$\Rightarrow y(t) = A\cdot\cos(\alpha t + \delta) - \frac{p_x c}{qB}$$

$$y(0) = A\cos\delta - \frac{p_x c}{qB}$$

$$\text{Si } \delta = \frac{\pi}{2} \rightarrow y(t) = A\sin(\alpha t) + y(0)$$

$$A = \frac{\dot y(0)}{\alpha}$$

$$\begin{cases} \dot x = \dfrac{p_x + qBy/c}{m} \rightarrow x = \dfrac{p_x}{m}\cdot t + \dfrac{qB}{c}\cdot\left(-\dfrac{A}{\alpha}\cos\alpha t + y(0)\cdot t\right) \\ \dot y = A\alpha\cos\alpha t = \dot y(0)\cos\alpha t \rightarrow y = A\sin(\alpha t) + y(0) \end{cases}$$

$$\dot z = p_z/m \rightarrow z = \frac{p_z}{m}\cdot t + z_0$$

$$H = p_x\cdot\dot x + p_z\cdot\overset{p_z/m}{\dot z} + p_y\cdot\dot y - p_y^2/2m - p_z^2/2m - \frac{m}{2}\dot x^2 + \frac{q}{c}\dot x B y$$

$$= m\dot x^2 - \cancel{\frac{qBy\dot x}{c}} + p_y^2/2m + p_z^2/2m - \frac{m}{2}\dot x^2 + \cancel{\frac{q}{c}\dot x By}$$

$$= p_y^2/2m + \frac{p_z^2}{2m} + \frac{m}{2}\left(\frac{p_x + qBy/c}{m}\right)^2 =$$

$$= \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{2p_x qBy}{2cm} + \frac{q^2B^2y^2}{2c^2m}$$

$$= \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{qB}{cm}p_x y + \frac{q^2B^2}{c^2m}y^2 = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \alpha p_x y + m\alpha^2 y^2$$

$$= H(y, p_y) = E$$

$$\frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{m}{2}\dot y(0)^2 + \alpha p_x y(0) + m\alpha^2 y(0)^2 = E$$

Si lo hacemos por cuadraturas:

$$\left.\begin{array}{l}\xi = x+iy \\ \xi^* = x-iy\end{array}\right\}\left\{\begin{array}{l} x = (\xi+\xi^*)/2 \\ y = (\xi-\xi^*)/2i = -i(\xi-\xi^*)/2\end{array}\right.$$

$$\mathcal{L} = \frac{m}{2}\dot{z}^2 + \frac{m}{2\cdot 4}\left((\dot\xi+\dot\xi^*)^2 + i^2(\dot\xi-\dot\xi^*)^2\right) - \frac{q}{2c\cdot 2}(\dot\xi+\dot\xi^*)\cdot B\cdot(-i)(\dot\xi-\dot\xi^*)$$

$$= \frac{m}{2}\dot z^2 + \frac{m}{8}\left(\dot\xi^2+\dot\xi^{*2}+2\dot\xi\dot\xi^* - \dot\xi^2 - \dot\xi^{*2} + 2\dot\xi\dot\xi^*\right) + i\frac{q}{4c}(\dot\xi^2-\dot\xi^{*2})B$$

$$= \frac{m}{2}\dot z^2 + \frac{m}{4}\dot\xi\dot\xi^* + \frac{iqB}{4c}(\dot\xi^2-\dot\xi^{*2})$$

$$\frac{\partial L}{\partial \dot z} = m\dot z = p_z = cte \rightarrow z = z_0 + \frac{p_z}{m}\cdot t$$

$$\frac{\partial L}{\partial \dot\xi} = cte = \frac{m}{4}\dot\xi^* + \frac{iqB}{2c}\dot\xi \equiv p_+ \rightarrow \frac{m}{4}\xi^* + \frac{iqB}{2c}\xi = p_+ t + \overset{cte}{K_+}$$

$$\frac{\partial L}{\partial \dot\xi^*} = \frac{m}{4}\dot\xi - \frac{iqB}{2c}\dot\xi^* = cte \equiv p_- \rightarrow \frac{m}{4}\xi - \frac{iqB}{2c}\xi^* = p_- t + K_-$$

$$H = \frac{m}{4}\cancel{\dot\xi^*}\dot\xi + \frac{iqB}{2c}\dot\xi^2 + \frac{m}{4}\dot\xi\dot\xi^* - \frac{iqB}{2c}\dot\xi^{*2} + m\dot z^2 - \frac{m}{2}\dot z^2$$

$$- \cancel{\frac{m}{4}\dot\xi\dot\xi^*} - \frac{iqB}{4c}\dot\xi^2 + \frac{iqB}{4c}\dot\xi^{*2}$$

$$= \frac{m}{4}\dot\xi\dot\xi^* + \frac{m}{2}\dot z^2 + \frac{iqB}{4c}(\dot\xi^2-\dot\xi^{*2}) = E$$

??

$$\vec\nabla\times\vec A = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & xB & 0\end{vmatrix} = -B(-\hat k) = B\hat k = \vec B \checkmark$$

$$\mathcal{L}' = \frac{m}{2}(\dot x^2+\dot y^2+\dot z^2) + \frac{q}{c}\dot y\cdot xB = \alpha + \frac{qB}{c}\dot y x + \frac{qB}{c}\dot x y$$

$$= \alpha + \frac{d}{dt}\left(\frac{qB}{c}xy\right)$$ → En efecto, difiere de la anterior en una derivada total.

(16) μ^-

$$\begin{cases} h = 9000\,m \\ N(h) = 10^8 = N_0 \\ \tau_0 = 2{,}197\,\mu s \\ v = 0{,}9978\,c \end{cases}$$

$$N(t) = N_0 \cdot e^{-t/\tau}$$

$\underbrace{\text{Desde Tierra:}}$

$$t = h/v$$

$$\tau = \overset{\text{dilatac. temporal}}{\gamma \tau_0}$$

$$N(t) = N_0\, e^{-\frac{h}{v\gamma\tau_0}} = N_{\text{nivel del mar}}$$

$$= 0{,}40 \cdot 10^8$$

Desde el muón

$$\tau' = \tau_0$$

$$t' = \frac{h'}{v} = \overset{\text{contracción de longitudes}}{\frac{h/\gamma}{v}}$$

$$N(t) = N_0 \cdot e^{-\frac{t'}{\tau'}} = N_0 \cdot e^{-\frac{h}{\gamma v \tau_0}}$$ coincide ✓

(17)

Suponemos sistema inercial.

$$\left.\begin{aligned} v &= 1\,km/s \\ h &= 20\,Km \\ x &= 2\pi(R_T + h) \end{aligned}\right\}$$

$R_T = 6370\,Km$

Desde Tierra:

$$\Delta t = \frac{x}{v}$$

$$\Delta t' = \Delta t/\gamma\ ;$$

$$\Delta t = \gamma \Delta t'$$

$\vdots$ reloj en reposo en casa

$$\rightarrow \Delta t - \Delta t' = \frac{x}{v}\left(1 - \frac{1}{\gamma}\right) = \frac{x}{v} \cdot \frac{\gamma - 1}{\gamma}$$

Desde avión no es conveniente hacerlo por ser no inercial (paradoja de los gemelos)

$$x' = \frac{x}{\gamma}$$

$$\Delta t' = \frac{x'}{v} = \frac{x}{\gamma v}$$

$$\Delta t = \gamma \Delta t' \quad \text{y no} \quad \Delta t = \Delta t'/\gamma$$

$$\Delta t - \Delta t' = \frac{x}{v}\left(1 - \frac{1}{\gamma}\right) = \frac{x}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \simeq \frac{x}{v}\left(1 - 1 + \frac{v^2}{2c^2}\right)$$

calculadora no puede

$$= \frac{x \cdot v}{2c^2} = \pi (R_T + h)\frac{v}{c^2} = 223\,ns$$

(18) $\beta = 0{,}8$

$\Delta\tau = 1h = 3600s$

Desde Tierra (base):

$d(t) = \beta c t$

Nave emite en $t_e = \overset{\text{dilatac.}}{\gamma} \cdot \Delta\tau$

↳ llega a Tierra en $d(t_e)/c \rightarrow t_{1_r} = \gamma\Delta\tau + d(t_e)/c$ recepc.

$$t_{1_r} = \gamma\Delta\tau + \beta\gamma\Delta\tau = \gamma\Delta\tau(1+\beta) = \frac{\Delta\tau(1+\beta)}{\sqrt{1-\beta^2}} = \Delta\tau\sqrt{\frac{1+\beta}{1-\beta}}$$

Para el fotón enésimo:

$$t_{e_n} = \gamma n \Delta\tau \quad , \quad t_{n_r} = n\gamma\Delta\tau + \frac{\beta c(n\gamma\Delta\tau)}{c}$$

$$= n\gamma\Delta\tau(1+\beta) = n\Delta\tau\sqrt{\frac{1+\beta}{1-\beta}}$$

Podemos formularlo con cuadrivectores para verlo más claro

S: Tierra, S': nave

↳ $x' = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}}_{\Lambda} x$; $x = \underbrace{\begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}}_{\Lambda^{-1}} x'$ Tomo $c = 1$

Nave emite en S' cada $\tau = \Delta\tau \rightarrow$ suceso de emisión x'_N

$$x'_N = \begin{pmatrix} n\tau \\ 0 \end{pmatrix} \quad n = 0, 1, \ldots \in \mathbb{N}$$

$$x_N = \Lambda^{-1} x'_N = \begin{pmatrix} \gamma n\tau \\ \gamma\beta n\tau \end{pmatrix} = \underbrace{\gamma n\tau}_{\text{dilat. tiempo}} \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

Suceso de recepción en S: tarda en llegar $(x_N)^1/c$ al origen de S

$$y_T = \begin{pmatrix} (x_N)^0 + (x_N)^1/c \\ 0 \end{pmatrix} = \gamma n\tau \begin{pmatrix} 1+\beta \\ 0 \end{pmatrix} = n\tau\sqrt{\frac{1+\beta}{1-\beta}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ → medido en S

$$y'_T = \Lambda y_T = \gamma n\tau(1+\beta) \cdot \begin{pmatrix} \gamma \\ -\gamma\beta \end{pmatrix} = \gamma^2(1+\beta)n\tau \begin{pmatrix} 1 \\ -\beta \end{pmatrix}$$

$$= \frac{n\tau}{1-\beta}\begin{pmatrix} 1 \\ -\beta \end{pmatrix}$$ → medido en S'

Análogo para los fotones emitidos por T. Por simetría, basta con cambiar N por T, β por −β y prima por sin prima.

$$X_T = \begin{pmatrix} n\tau \\ 0 \end{pmatrix} \hat{=} X'_N$$

sólo en las componentes espaciales que estén aisladas

$T \equiv S$, $N \equiv S'$

$$X'_T = n\tau\begin{pmatrix} \gamma \\ -\gamma\beta \end{pmatrix} = n\tau\gamma\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \hat{=} X_N$$

→ S' ve que tiempo en S se dilata, tarda γτ en emitir.

$$Y'_N = n\tau\sqrt{\frac{1+\beta}{1+\beta}}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} n\tau\gamma + (n\tau\gamma\beta)/1-\beta \\ 0 \end{pmatrix} = \frac{n\tau\gamma}{1-\beta}\begin{pmatrix} 1 \\ 0 \end{pmatrix} ✓$$

$$Y_N = \begin{pmatrix} n\tau + \frac{n\beta\tau}{1-\beta} \\ n\beta\tau + \beta\frac{n\beta\tau}{1-\beta} \end{pmatrix} = \frac{n\tau}{1-\beta}\begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} Y'_N \; ✓ \; \hat{=} \; Y'_T$$

Interpretemos el resultado de este según

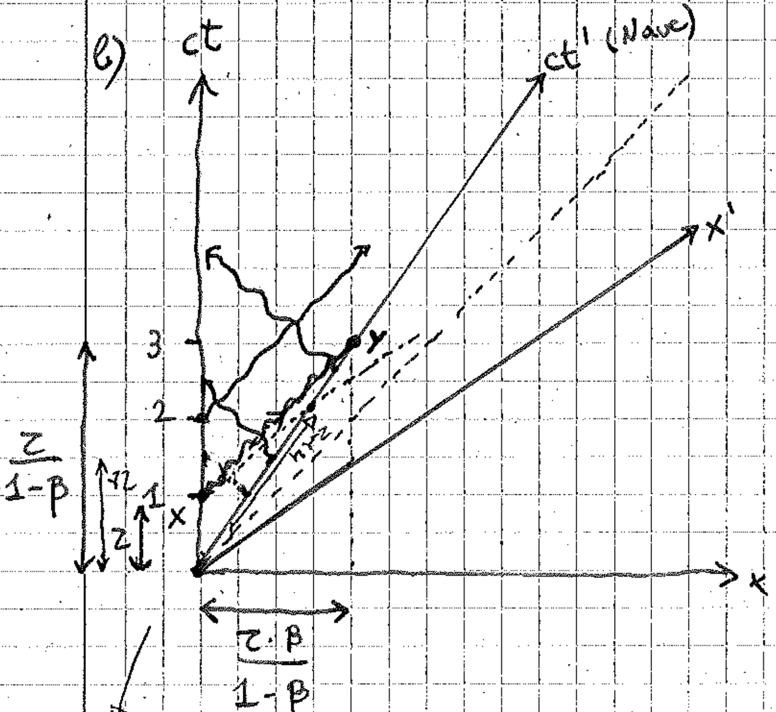
X_T → emite cada τ, S en reposo

X'_T → visto desde S', S se mueve, τ se dilata, emite en nγτ a distancia −nγτ·β

Y'_N → A partir de nγτ, fotón tarda en llegar lo que están separados entre lo que recorta el fotón (c−β)

Y_N → Desde S, el fotón tarda $n\beta\tau/1-\beta$ en alcanzar la nave. De nuevo, S' verá tiempo en S' está dilatado → multiplicar por γ.

b)



dibujo mal echo los fotones de la nave

γτ ... hipérbola de calibración

Fotones (emitidos por) de la nave: (tics)

$$X'_N(n+1) - X'_N(n) = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

$$Y_T(n+1) - Y_T(n) = \begin{pmatrix} \tau\sqrt{\frac{1+\beta}{1-\beta}} \\ 0 \end{pmatrix}$$

→ Periodicidad = Inversa de la Frecuencia con que reciben tics

Lo relacionado con efecto Doppler

$$\nu_0 = \frac{1}{\tau}$$

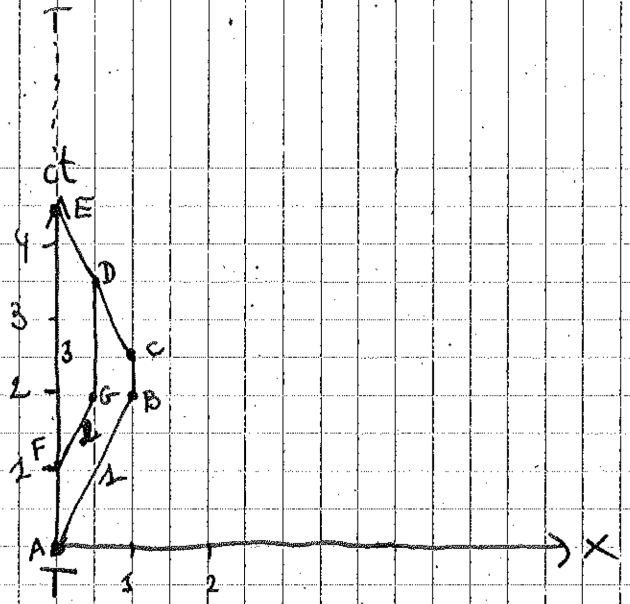
$$\nu = \nu_0\sqrt{\frac{1-\beta}{1+\beta}} < \nu_0 \text{ (Red-shift)}$$

Idem para fotones de tierra:

$$X'_T(n+1) - X_T(n) = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

$$Y'_N(n+1) - Y'_N(n) = \begin{pmatrix} \tau\sqrt{\frac{1+\beta}{1-\beta}} \\ 0 \end{pmatrix}$$

(18)



$\beta = 0,5$

$\gamma = \frac{2}{\sqrt{3}}$

Visto desde 3 (única que es inercial):

En E:

$T_1 = t'_{AB} + t'_{BC} + t'_{CE}$ → En 1 el tiempo propio $= t'$ es el menor. $t' = \frac{t}{\gamma}$

$= \frac{t_{AB}}{\gamma} + \frac{t_{BC}}{1} + \frac{t_{CE}}{\gamma}$ ... está quieta

$= \frac{2}{2/\sqrt{3}} + 0,5 + \frac{2}{2/\sqrt{3}} = 2\cdot\sqrt{3} + \frac{1}{2} = 3,86$ años

$T_2 = t'_{FG} + t_{GD} + t'_{DE} = \frac{1}{2/\sqrt{3}} + 2,5 + \frac{1}{2/\sqrt{3}} = \sqrt{3} + 2,5 = 4,23$ años
$+t'_{AF}$

$T_3 = 4,5$ años → La que más tiempo viaja, más joven es.

En D:

$\tilde{T}_1 = \frac{3}{2/\sqrt{3}} + 0,5 = \frac{3\sqrt{3}+1}{2} = 3,098$ años

$\tilde{T}_2 = \frac{1}{2/\sqrt{3}} + 2,5 = \frac{\sqrt{3}+5}{2} = 3,36$ años

$\Big\}\ \tilde{T}_2 - \tilde{T}_1 = 2 - \sqrt{3} = T_2 - T_1$

(Entre D y E van juntas, ya no desfasan sus edades entre sí pero sí respecto a 3.

(20)

$L = 1m$

A ⟶ B Σ $\qquad \vec{v} = 0{,}5\,\hat{x}$ Σ' $\qquad x' = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} x$

$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ L \end{pmatrix}$ $\rightarrow$ $A' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $B' = \begin{pmatrix} -\gamma\beta L \\ \gamma L \end{pmatrix}$

$|(B - A)_x| = L$

A' y B' no son simultáneos en S', por tanto NO es una medida de longitud.

Hay contracción de longitudes (longitud en reposo es la mayor)

Si S' se acerca a S emitiendo n destellos, en S' están espaciados τ y en S: $\gamma\tau$ (dilat. temp). Si $d_{SS'}$ en S $= v \cdot n\gamma\tau$ en S': $d' = v \cdot n\tau \rightarrow d' = d/\gamma$

↳ Por tanto, $L' = L/\gamma$

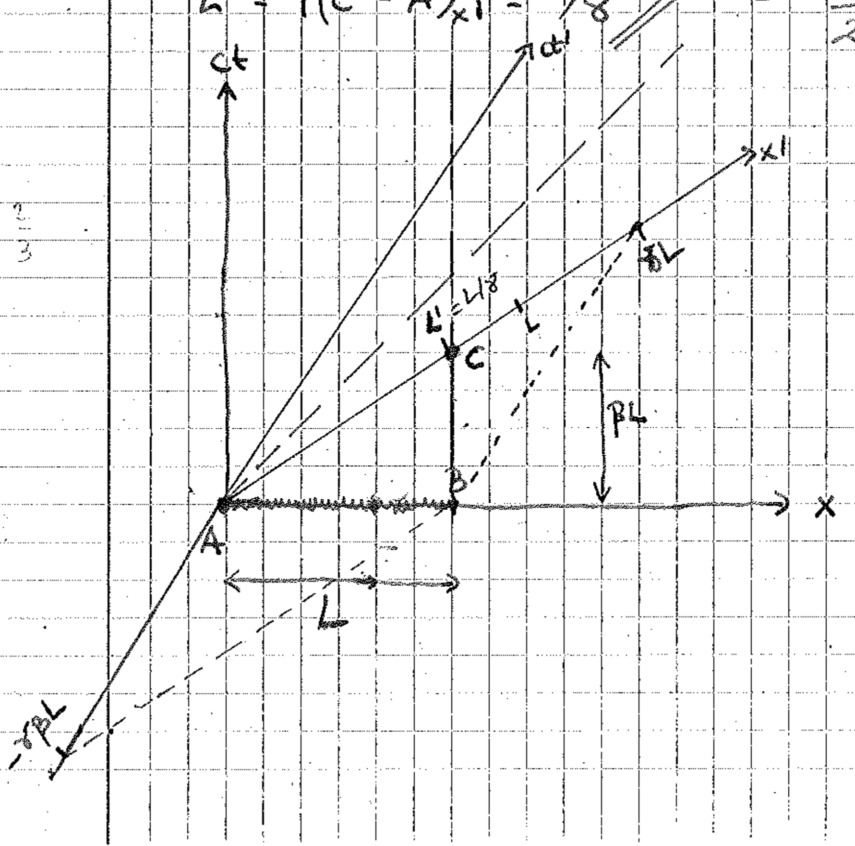
O bien: S' "espera" a que barra llegue a $t = 0$.

↳ Tras $\gamma\beta L$, B' se habrá alejado $(\gamma\beta L)\cdot\beta$

↳ $C' = \begin{pmatrix} 0 \\ \gamma L - \gamma\beta^2 L \end{pmatrix} = \gamma L \begin{pmatrix} 0 \\ 1 - \beta^2 \end{pmatrix} = \gamma L \begin{pmatrix} 0 \\ \frac{1}{\gamma^2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{L}{\gamma} \end{pmatrix}$

↳ $C = \Lambda^{-1} C' = \begin{pmatrix} L\beta \\ L \end{pmatrix}$

$L' = |(C' - A')_x| = L/\gamma$ $= \frac{\sqrt{3}}{2}\,m$



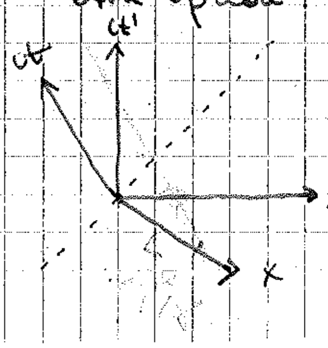
$\beta = \frac{2}{3}$

$A(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$

$B(t) = \begin{pmatrix} t \\ L \end{pmatrix}$

Hay que usar la hipérbola para calibrar el eje X' respecto a L. (analíticamente sólo γL y L/γ).

Otra opción:



(21)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$v'_x = \frac{dx'}{d(t')} = \gamma \cdot \frac{dx}{dt'} - \gamma\beta c \frac{dt}{dt'} = \gamma \frac{dt}{dt'}(v_x - \beta c) = \frac{\gamma (v_x - \beta c)}{dt'/dt}$$

$$t' = \gamma\left(t' - \frac{\beta}{c}x\right) \rightarrow \frac{dt'}{dt} = \gamma\left(1 - \frac{\beta}{c}v_x\right)$$

$$\hookrightarrow v'_x = \frac{v_x - \beta c}{1 - \frac{\beta}{c}v_x}$$

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = \frac{v_y}{1 - \frac{\beta}{c}v_x}$$

análogo

$$v'_z = \frac{v_z}{1 - \frac{\beta}{c}v_x}$$

Σ_1 : $\vec{v}_1 = (\alpha c, 0, 0)$

Σ_2 : $\beta_2 = +\frac{1}{4}$ → todo en x, omito vectores

Σ_3 : $\beta_3 = \frac{1}{2}$

Σ_4 : $\beta_4 = \frac{3}{4}$

$$v_2 = \left(\frac{\alpha c - \frac{1}{4}c}{1 - \frac{1}{4c}\alpha c}\right) = c\left(\frac{\alpha - 1/4}{1 - \alpha/4}\right) = c\left(\frac{4\alpha - 1}{4 - \alpha}\right)$$

$$v_3 = c\left(\frac{2\alpha - 1}{2 - \alpha}\right) \quad ; \quad v_4 = c\left(\frac{\alpha - 3/4}{1 - \frac{3}{4}\alpha}\right) = c\left(\frac{4\alpha - 3}{4 - 3\alpha}\right)$$

También se pueden obtener a partir de velocidades relativas entre 2 y 3, 2 y 4, 3 y 4, ...

como si $\alpha = \beta_3 = \frac{1}{2}$

$$v_{32} = v_2(\alpha = \beta_3) = c\left(\frac{2-1}{4-\frac{1}{2}}\right) = \frac{2c}{7}$$

$$v_{23} = v_3(\alpha = \beta_2) = c\left(\frac{\frac{1}{2}-1}{2-\frac{1}{4}}\right) = -\frac{2}{7}c = -v_{32}$$

$$v_{42} = v_2(\alpha = \beta_4) = c\left(\frac{3-1}{4-3/4}\right) = \frac{2c}{13/4} = \frac{8}{13}c$$

$$v_{24} = c\,\frac{1-3}{4-3/4} = -\frac{8}{13}c$$

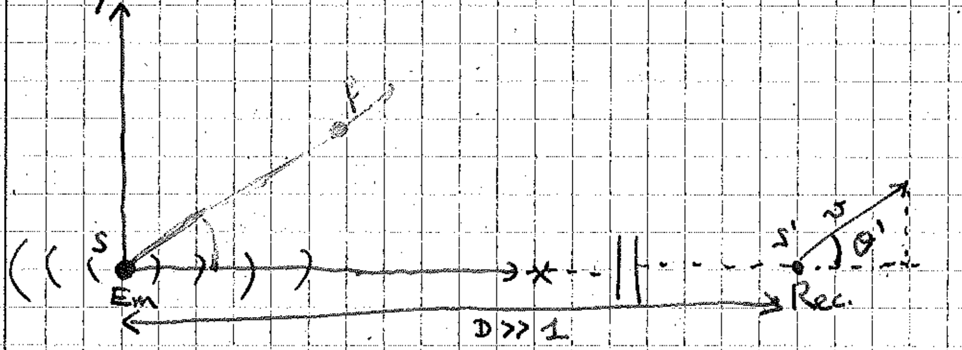
$$v_{43} = c\left(\frac{\frac{3}{2}-1}{2-3/4}\right) = c\,\frac{1/2}{5/4} = c\,\frac{2}{5} = -v_{34}$$

Por ejemplo, $v_2 = c\left(\frac{4\alpha-1}{4-\alpha}\right)$

$$v_3 = \frac{v_2 - v_{32}}{1 - \frac{v_{32}v_2}{c^2}} = c\,\frac{\frac{4\alpha-1}{4-\alpha} - \frac{2}{7}c}{1-\frac{2}{7}\left(\frac{4\alpha-1}{4-\alpha}\right)} = c\left(\frac{28\alpha - 7 - 8 + 2\alpha}{28 - 7\alpha - 8\alpha + 2}\right)$$

$$= c\left(\frac{30\alpha - 15}{30 - 15\alpha}\right) = c\left(\frac{2\alpha - 1}{2-\alpha}\right)$$

(22) → Ver 18 para el caso $\theta = 0$ y alejándose (red-shift)



especio ~~recorrido~~ entre pulsos

1^{er} pulso : $e_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ llega a receptor en $r_1 = \begin{pmatrix} D/c \\ D \\ 0 \end{pmatrix} + \tau$ lo que se aleja

2^{o} pulso $e_2 = \begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix}$ " " " $r_2 = \begin{pmatrix} D/c + \frac{v\tau\cos\theta'}{c} \\ D + v\cos\theta'\cdot\tau \\ \simeq 0 \end{pmatrix}$

Donde suponemos D muy grande, $y \simeq 0$ en comparación.

En S' : $r_1' = \begin{pmatrix} \gamma D - \gamma\beta D \\ -\gamma\beta D + \gamma D \\ \simeq 0 \end{pmatrix}$, $r_2' = \begin{pmatrix} \gamma D + \gamma\beta\tau\cos\theta - \gamma\beta D - \gamma\beta v\tau\cos\theta + \gamma\tau \\ \ldots \\ \simeq 0 \end{pmatrix}$

Lo usamos sólo boost en eje x, despreciamos lo que se mueva en y, aun cuando $\theta = \pi/2$

Por tanto,

$$\tau' = (r_2')^0 - (r_1')^0 = \gamma \underbrace{[(r_2)^0 - (r_1)^0]}_{\text{Dilatación}} = \gamma \tau (1 + \beta \cos\theta')$$

↳ caso en que se aleja : $\beta > 0$, $\theta' < \pi/2$
acerca : $\beta > 0$, $\theta' > \pi/2$

$$\nu' = \frac{1}{\tau'} = \frac{\nu_0}{\gamma} \cdot \frac{1}{1+\beta\cos\theta'} = \frac{\nu_0}{\gamma} \frac{1}{1 + \frac{\beta\cos\theta - \beta^2}{1-\beta\cos\theta}} = \frac{\nu_0}{\gamma} \frac{1-\beta\cos\theta}{1-\beta^2} = \nu_0 \gamma (1 - \beta\cos\theta)$$

↳ θ' : medido en S', $\neq \theta$

$\cos\theta' = \dfrac{\cos\theta - \beta}{1 - \beta\cos\theta}$ → ver 23

Alternativa
Mejor: (más fácil y corto:)

$$\begin{pmatrix} E \\ p_x c \end{pmatrix}' = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x c \end{pmatrix}$$

con $p_x = p\cos\theta$
$p_x' = p'\cos\theta'$

$E = \hbar\omega$, $E' = \hbar\omega'$
$p = \hbar K$, $p' = \hbar K'$
↳ $\omega = cK$

$$\omega' = \gamma\omega - \gamma\beta K_x \cdot c$$

$$= \gamma\omega - \gamma\beta\omega\cos\theta = \gamma\omega(1 - \beta\cos\theta)$$ sale lo mismo

$$= \omega \frac{(1 - \beta\cos\theta)}{\sqrt{1-\beta^2}}$$

$$\hookrightarrow \nu = \nu_0 \frac{(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}$$

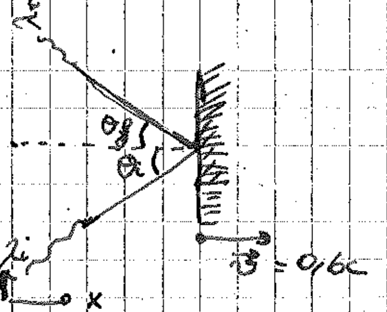
→ $\theta = 0$: $\nu = \nu_0 \sqrt{\dfrac{1+\beta}{1-\beta}}$

→ $\theta = \pi$: $\nu = \nu_0 \sqrt{\dfrac{1-\beta}{1+\beta}}$

→ $\theta = \dfrac{\pi}{2}$: $\nu = \dfrac{\nu_0}{\sqrt{1-\beta^2}}$

(23)

λ_i, θ_i con la normal



fotón descrito por:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \end{pmatrix} \rightarrow \begin{pmatrix} \omega \\ ck_x \\ ck_y \end{pmatrix} ; \quad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & \\ & & 1 \end{pmatrix} ; \quad \begin{matrix} k_x = k\cos\theta_i \\ k_y = k\sin\theta_i \end{matrix}$$

$$\omega = ck \qquad \omega' = ck'$$

$$\begin{pmatrix} \omega' \\ ck'_x \\ ck'_y \end{pmatrix} = \begin{pmatrix} \gamma\omega - \gamma\beta \overset{\omega}{\widetilde{kc}}\cos\theta_i \\ (-\gamma\beta\omega + \gamma kc\cos\theta_i)c \\ k_y c \end{pmatrix} = \begin{pmatrix} \gamma\omega(1-\beta\cos\theta_i) \\ \gamma\omega(\cos\theta_i - \beta) \\ k_y c \end{pmatrix}$$

$$k'_x = k'\cos\theta'_i \rightarrow \cos\theta'_i = \frac{k'_x}{\omega'/c} = \frac{\cos\theta_i - \beta}{1-\beta\cos\theta_i}$$

↳ En S' se cumple la ley de la reflexión:

$\theta'_f = \theta'_i$; k'_x cambia de signo, k_y no

$$\hookrightarrow \begin{pmatrix} \omega' \\ -ck_x' \\ ck_y' \end{pmatrix} \rightarrow \begin{pmatrix} \omega_f \\ ck_{xf} \\ ck_{yf} \end{pmatrix} = \Lambda^{-1}\left(\ \right)' = \begin{pmatrix} \gamma\omega' - \gamma\beta ck_x' \\ \gamma\beta\omega' - \gamma ck_x' \\ ck_y \end{pmatrix}$$

$$k_x = k_{xf} = k_f \cdot \cos\theta_f$$

$$\omega_f = \gamma(\gamma\omega)(1-\beta\cos\theta_i) - \gamma\beta(\gamma\omega)(\cos\theta_i - \beta)$$
$$= \gamma^2\omega(1 - \beta\cos\theta_i - \beta\cos\theta_i + \beta^2) \qquad ?$$
$$= \gamma^2\omega(1+\beta^2 - 2\beta\cos\theta_i) \quad // \quad \lambda_f = \frac{2\pi}{\omega_f}\cdot c \ //$$

$$\cos\theta_f = \frac{k_{xf}}{k_f} = \frac{ck_{xf}}{\omega_f} = \frac{\gamma\beta(\gamma\omega)(1-\beta\cos\theta_i) - \gamma^2\omega(\cos\theta_i - \beta)}{\omega_f}$$

$$= \frac{1}{\omega_f}\cdot\gamma^2\omega(\beta - \beta^2\cos\theta_i - \cos\theta_i + \beta) = \frac{\gamma^2\omega}{\omega_f}(2\beta - \cos\theta_i(1+\beta^2))$$

$$= \frac{(2\beta - (1+\beta^2)\cos\theta_i)}{(1+\beta^2 - 2\beta\cos\theta_i)} \ //$$

Si haz incide por la derecha, cambiar θ_i por $\theta_i + \pi$.

(24) $(\phi, A) \to$ cuadrivector

$$\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix} \to \begin{pmatrix}\phi'\\ \vec{A}'\end{pmatrix} = \Lambda\begin{pmatrix}\phi\\ \vec{A}\end{pmatrix} \overset{\text{boost en eje } x}{=} \begin{pmatrix}\gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}\phi\\ A_x\\ A_y\\ A_z\end{pmatrix}$$

$$\phi' = \gamma\phi - \gamma\beta A_x \;;\; A_x' = -\gamma\beta\phi + \gamma A_x \;;\; A_y' = A_y \;;\; A_z' = A_z$$

$$\vec{E}' = -\vec{\nabla}'\phi' - \frac{1}{c}\frac{\partial}{\partial t'}\vec{A}'$$

$$E_x' = -\frac{\partial\phi'}{\partial x'} - \frac{1}{c}\frac{\partial}{\partial t'}A_x' = -\frac{\partial\phi'}{\partial x}\cdot\frac{\partial x}{\partial x'} - \frac{1}{c}\frac{\partial}{\partial t}A_x'\cdot\frac{\partial t}{\partial t'}$$

$$\frac{\partial x'}{\partial x} = \left(\frac{\partial x}{\partial x'}\right)^{-1} = \gamma - \frac{\gamma\beta c}{v} = \gamma\left(1 - \frac{\beta c}{v}\right)$$

$$\frac{\partial t'}{\partial t} = \left(\frac{\partial t}{\partial t'}\right)^{-1} = \gamma - \frac{\gamma\beta}{c}v = \gamma\left(1 - \beta\frac{v}{c}\right)$$

$$\frac{\partial\phi}{\partial x'} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\phi}{\partial t}\cdot\frac{\partial t}{\partial x'} = \gamma\frac{\partial\phi}{\partial x} + \frac{\gamma\beta}{c}\frac{\partial\phi}{\partial t}$$

$$\frac{\partial A_x}{\partial x'} = \gamma\frac{\partial A}{\partial x} + \frac{\gamma\beta}{c}\frac{\partial A_x}{\partial t}$$

$$\frac{\partial\phi}{\partial t'} = \gamma\beta c\frac{\partial\phi}{\partial x} + \gamma\frac{\partial\phi}{\partial t} \;;\; \frac{\partial A_x}{\partial t'} = \gamma\beta c\frac{\partial A_x}{\partial x} + \gamma\frac{\partial A_x}{\partial t}$$

$$E_x' = -\gamma\frac{\partial\phi}{\partial x'} + \gamma\beta\frac{\partial A_x}{\partial x'} + \frac{1}{c}\gamma\beta\frac{\partial\phi}{\partial t'} - \frac{\gamma}{c}\frac{\partial A_x}{\partial t'}$$

$$= -\gamma^2\frac{\partial\phi}{\partial x} - \cancel{\frac{\gamma^2\beta}{c}\frac{\partial\phi}{\partial t}} + \cancel{\gamma^2\beta\frac{\partial A}{\partial x}} + \gamma^2\frac{\beta^2}{c}\frac{\partial A_x}{\partial t} + \gamma^2\beta^2\frac{\partial\phi}{\partial x} + \cancel{\frac{\gamma^2\beta}{c}\frac{\partial\phi}{\partial t}}$$

$$- \cancel{\gamma^2\beta\frac{\partial A_x}{\partial x}} - \frac{\gamma^2}{c}\frac{\partial A_x}{\partial t} = -\frac{\partial\phi}{\partial x}\underbrace{(\gamma^2 - \gamma^2\beta^2)}_{=1} + \frac{\partial A_x}{\partial t}\underbrace{(\gamma^2\beta^2 - \gamma^2)}_{=-1}\cdot\frac{1}{c}$$

$$= -\frac{\partial\phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = E_x$$

$A_y = A_y'$; $y = y'$

$$E_y' = -\frac{\partial}{\partial y'}\phi' - \frac{1}{c}\frac{\partial}{\partial t'}A_y' = -\frac{\partial}{\partial y}\phi' - \frac{1}{c}\frac{\partial}{\partial t'}A_y = -\gamma\frac{\partial\phi}{\partial y} + \gamma\beta\frac{\partial A_x}{\partial y}$$

$$- \frac{1}{c}\frac{\partial A_y}{\partial x}\beta c\gamma - \frac{1}{c}\frac{\partial A_y}{\partial t}\gamma = \gamma E_y + \gamma\beta\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)$$

$$= \gamma E_y - \gamma\beta B_z = \gamma(E_y - \beta B_z)$$

análogo a E'_y

$$E'_z = -\frac{\partial}{\partial z'}\phi - \frac{\partial}{\partial t'}A_z = -\gamma\frac{\partial\phi}{\partial z} + \gamma\beta\frac{\partial A_x}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial x}\beta c\gamma - \frac{1}{c}\frac{\partial A_z}{\partial t}\gamma$$

$$= \gamma E_z + \gamma\beta\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = \gamma(E_z + \beta B_y)$$

$$B'_x = \epsilon_{xjk}\partial'_j A'_k = \frac{\partial A'_z}{\partial y'} - \frac{\partial A'_y}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$$

$$B'_y = \frac{\partial A'_x}{\partial z'} - \frac{\partial A'_z}{\partial x'} = \frac{\partial A'_x}{\partial z} - \frac{\partial A_z}{\partial x'} = -\gamma\beta\frac{\partial\phi}{\partial z} + \gamma\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\gamma - \frac{\partial A_z}{\partial t}\gamma\beta$$

$$= \gamma B_y + \gamma\beta E_z = \gamma(B_y + \beta E_z)$$

$$B'_z = \frac{\partial A'_y}{\partial x'} - \frac{\partial A'_x}{\partial y'} = \frac{\partial A_y}{\partial x'} - \frac{\partial A'_x}{\partial y} = \frac{\partial A_y}{\partial x}\gamma + \frac{\partial A_y}{\partial t}\gamma\beta + \gamma\beta\frac{\partial\phi}{\partial y}$$

$$-\gamma\frac{\partial A_x}{\partial y} = \gamma B_z - \gamma\beta E_y = \gamma(B_z - \beta E_y)$$

Es más apropiado trabajar con el tensor campo electromagnético, que se transforma según Lorentz.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \; ; \; F'^{\mu\nu} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho F^{\lambda\rho}$$

Si $\beta = \cancel{\frac{B_z}{E_y}}$ ó $\frac{E_y}{B_z}$ ó $-\frac{B_y}{E_z}$ ó $-\frac{E_z}{B_y}$ → hay campos que se anulan

25

$A \to B + C$

S.C.M:

$p_A = (m_A, 0) = (E_b, \vec{p}) + (E_c, -\vec{p})$ (ángulo arbitrario)

↓

$$m_A^2 = m_b^2 + m_c^2 + 2E_bE_c + 2|\vec{p}|^2 = \quad (E_c = m_A - E_b)$$

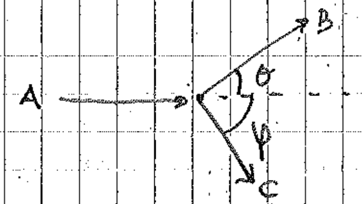
$$= m_b^2 + m_c^2 + 2E_b m_A - 2\underbrace{(E_b^2 - |\vec{p}|^2)}_{=m_b^2}$$

↳ $E_b = \dfrac{m_A^2 + m_b^2 - m_c^2}{2m_A}$ $\quad \} \; |\vec{p}| = \sqrt{E^2 - m^2} = \dfrac{1}{2m_A}\lambda^{1/2}(m_A^2, m_b^2, m_c^2)$

↳ $E_c = m_A - E_b = \dfrac{m_A^2 + m_c^2 - m_b^2}{2m_A}$

S. LAB:

$$(E_A, \vec{p}_A) = (E_b, \vec{p}_b) + (E_c, \vec{p}_c)$$



$$m_A^2 = m_b^2 + m_c^2 + 2E_bE_c - 2|\vec{p}_b|\cdot\vec{p}_c$$

$$= m_b^2 + m_c^2 + 2E_b(E_A - E_b) - 2\vec{p}_b(\vec{p}_A - \vec{p}_b)$$

$$= m_b^2 + m_c^2 + 2E_bE_A - 2(E_b^2 - \vec{p}_b^{\,2}) - 2\vec{p}_b\vec{p}_A$$

$$= m_c^2 - m_b^2 + 2\underbrace{p_b \cdot p_a}_{\text{cuadrivectores}}$$

$$t \equiv (p_a - p_b)^2 = m_A^2 + m_b^2 - 2p_bp_a = m_c^2$$

$$2|p_b||p_a|\cdot\cos\theta = m_c^2 - m_b^2 - m_A^2 + 2E_bE_A$$

$$\to \cos\theta = \frac{m_c^2 - m_b^2 - m_A^2 + 2E_bE_A}{2|\vec{p}_b|\cdot|\vec{p}_a|} \quad ; \quad m_i, E_a, p_a \text{ fijados}$$

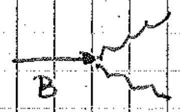
$$\cos\theta\,(E_b)$$

Idem para φ cambiando b por c. O directamente:

$$m_A^2 = m_b^2 + m_c^2 - 2E_bE_c - 2|\vec{p}_b||\vec{p}_c|\cdot\cos(\theta+\varphi)$$

$$E_c = E_A - E_b \; ; \; |\vec{p}_c| = |\vec{p}_A - \vec{p}_b|$$

Si B se desintegra, fórmulas similares:



$$(E_b, \vec{p}_b) = (E_1, \vec{p}_1) + (E_2, \vec{p}_2)$$

$$E_b = |\vec{p}_1| + |\vec{p}_2|$$

$$\vec{p}_b = |\vec{p}_1 + \vec{p}_2|$$

$$m_b^2 = 2E_1E_2 - 2\vec{p}_1\cdot\vec{p}_2 = 2|\vec{p}_1|\,|\vec{p}_2|\cdot(1 - \cos(\theta+\varphi))$$

$$= 2|\vec{p}_1|(E_b - |\vec{p}_1|)\, 2\sin^2\left(\frac{\theta+\varphi}{2}\right)$$

$$\frac{m_b^2}{4\sin^2\left(\frac{\theta+\varphi}{2}\right)} = |\vec{p}_1|E_b - |\vec{p}_1|^2 \quad \leftarrow \; |\vec{p}_1| = E_1$$

$$\to |\vec{p}_1|^2 - |\vec{p}_1|E_b$$

Mejor:

$p_c = p_a - p_b$

$\hookrightarrow m_c^2 = m_a^2 + m_b^2 + 2E_aE_b - 2|\vec{p_a}|\cdot|\vec{p_b}|\cdot\cos\theta$

$\cancel{2}|\vec{p_a}||\vec{p_b}|\cos\theta = \frac{m_a^2+m_b^2-m_c^2}{2} - \cancel{2}E_aE_b \quad ; \; (\;)^2$

$(E_a^2-m_a^2)(E_b^2-m_b^2)\cos^2\theta = \Big(\frac{\overbrace{m_a^2+m_b^2-m_c^2}^{=M^2}}{4}\Big)^2 + E_a^2E_b^2 - E_aE_b(M^2)$

$E_b^2\left((E_a^2-m_a^2)\cos^2\theta - E_a^2\right) + E_aE_bM^2 - m_b^2\cos^2\theta\,(E_a^2-m_a^2) - \frac{M^4}{4} = 0$

$E_b = \frac{-E_aM^2 \pm \sqrt{E_a^2M^4 + 4(p_A^2\cos^2\theta - E_a^2)(m_b^2\cos^2\theta\, p_A^2 + \frac{M^4}{4})}}{2(p_A^2\cos^2\theta - E_a^2)} > 0$

$= \frac{+E_aM^2 + \left(E_a^2M^4 + 4p_A^2\cos^2\theta\frac{M^4}{4} - E_a^2M^4 - E_a^2(4m_b^2\cos^2\theta p_A^2) + 4p_A^4\cos^4\theta\, m_b^2\right)^{1/2}}{2(E_a^2-p_A^2\cos^2\theta)}$

$= \frac{E_aM^2}{2(E_a^2-p_A^2\cos^2\theta)} + \left(p_A^2\cos^2\theta M^4 - 4p_A^2\cos^2\theta\, m_b^2(E_a^2 - p_A^2\cos^2\theta)\right)^{1/2}$

$= \frac{E_AM^2 + M^2p_A\cos\theta\left(1 - 4\frac{m_b^2}{M^4}(E_a^2-p_A^2\cos^2\theta)\right)^{1/2}}{2(E_A^2-p_A^2\cos^2\theta)}$

$= \frac{M^2(E_A + p_A\cos\theta(1-\ldots)^{1/2})}{2(E_A^2+p_A^2\cos\theta)} \qquad \circ\; |\vec{p_b}| = \sqrt{E_b^2-m_b^2}$

Idem con c)

b) $B \to \gamma\gamma$

En S.C.M $\to (m_B, 0) = (E_{\gamma_1}, \vec{p_1}) + (E_{\gamma_2}, -\vec{p_2})$

$\hookrightarrow E_{\gamma_1} = E_{\gamma_2} = \frac{m_B}{2}$

S.LAB: $\to$ usar fórmula anterior con $m_b = m_c = 0 \quad b\to\gamma,\; A\to B$

$E_{\gamma_1} = \frac{m_A^2(E_A + p_A\cos\theta)}{2(E_A^2 - p_A^2\cos^2\theta)} = \frac{E_A(1+\beta\cos\theta)}{2(\gamma^2)(1-\beta^2\cos^2\theta)} \neq E_{\gamma_2}$

$\hookrightarrow$ Extremos: $\alpha = 0,\; \alpha = \pi$

forward $(\alpha = 0) \to E_\gamma = \frac{E_A}{2}(1+\beta)$

backward $(\alpha = \pi) \to E_\gamma = \frac{E_A}{2}(1-\beta)$

(26) $A + B \to C + D$

S. Lab. $p_A + p_B = p_c + p_d$

$$(E_A, \vec{p_A}) + (m_b, 0) = (E_c, \vec{p_c}) + (E_D, \vec{p_D})$$

Buscamos E_A^L threshold.

$$s = (p_A + p_B)^2 = (E_A^L + m_b, \vec{p_A})^2 = (p_A^{CM} + p_B^{CM})^2$$ invariante relativista

E_A^L th implica s_{th}

↳ Fácil de visualizar en S.C.M

$$p_A + p_b = (E_a + E_b, 0) = (\gamma_c m_c + \gamma_d m_d, 0) = p_c + p_d$$

↳ Es mínimo cuando $\gamma_{c,d} = 1$ (se crean en reposo

↳ $p_A^2 + p_B^2 + 2 p_A p_B \Big|_L = s = m_A^2 + m_b^2 + 2 E_a^L m_b$

$$s_{th} = m_A^2 + m_b^2 + 2 E_a^L{}_{th}\, m_b = (m_c + m_d)^2$$

↳ $$E_a^L{}_{th} = \frac{(m_c + m_d)^2 - m_A^2 - m_b^2}{2 m_b}$$

$E_{a_{th}}^{CM} = \;?$

↓

$$m_A^2 + m_b^2 + 2 E_{a\,th}^{CM} E_{b\,th}^{CM} + 2|\vec{p}|^2 = (m_c + m_D)^2$$

$$m_A^2 + m_b^2 + 2 E_A^{CM}{}_{th} (m_c + m_D - E_A^{CM}{}_{th}) + 2|\vec{p}|^2 = (m_c + m_D)^2$$

↳ $$-m_A^2 + m_b^2 + 2 E_A^{CM}{}_{th} (m_c + m_D) = ''$$

↳ $$E_A^{CM}{}_{th} = \frac{(m_c + m_D)^2 + m_A^2 - m_b^2}{2(m_c + m_D)}$$

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Ver fórmulas de 26, con $m_A = m_B$

En S.C.M:

$$(E_1, \vec{p}) + (E_2, -\vec{p}) \overset{\text{threshold}}{=} (2m_p + (n+m)m_\pi, 0)$$

$$E_1 = E_2 = E \rightarrow E_{th}^{cm} = m_p + (n+m)\cdot\frac{m_\pi}{2}$$

$$\gamma_{th} m_p \rightarrow \gamma_{th} = 1 + \frac{(n+m)}{2}\cdot\frac{m_\pi}{m_p}$$

En S. Lab: (ver 26)

$$s_{th} = m_p^2 + m_p^2 + 2E_{1\,th}^L m_p = (2m_p + (n+m)m_\pi)^2$$

$$\hookrightarrow E_{1\,th}^L = \frac{(2m_p + (n+m)m_\pi)^2 - 2m_p^2}{2m_p} = m_p + 2(n+m)m_\pi + \frac{(n+m)^2 m_\pi^2}{2m_p}$$

$$\gamma_{th} = 1 + 2(n+m)\frac{m_\pi}{m_p} + \left(\frac{n+m}{2}\right)^2 \cdot \left(\frac{m_\pi}{m_p}\right)^2$$

Hay simetría $n-m$ por indistinguibilidad de partículas.

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$$m \xrightarrow{\gamma} M \quad \Rightarrow \quad \underset{\gamma'}{\xrightarrow{M'}}$$

$$(\gamma m, \gamma\beta m) + (M, 0) = (\gamma' M', \gamma'\beta' M')$$

$$m^2 + M^2 + 2\gamma m M = M'^2 \equiv \text{en. en reposo de part. final}$$

$$\hookrightarrow M' = (M^2 + 2\gamma M m + m^2)^{1/2} \qquad \gamma = \left[1 - \left(\frac{v_m}{c}\right)^2\right]^{-1/2}, \quad \frac{v_m}{c} = \beta$$

$$\beta' = \frac{\gamma'\beta' M'}{\gamma' M'} = \frac{\gamma\beta m}{\gamma m + M} = \frac{\beta}{1 + \frac{M}{m}\cdot\frac{1}{\gamma}}$$

El caso simétrico $\rightarrow$ intercambiar m por M. M' no cambia, es lógico porque es un invariante y un boost al simétrico no debe cambiarlo.

Caso no relativista: $M' = M + m$; $\beta' = \dfrac{\beta}{1 + \frac{M}{m}} = \dfrac{m}{M'}\beta$

$\hookrightarrow p' = p$ (choque inelástico)

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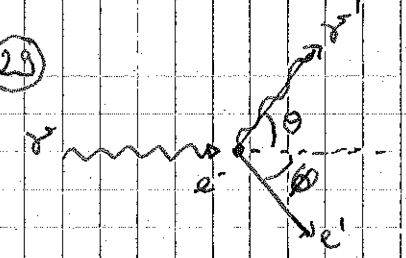

$$p_\gamma + p_e = p_{\gamma'} + p_{e'}$$

$\hookrightarrow \overset{0}{\overset{\parallel}{p_\gamma}}{}^2 + p_e^2 + 2 p_e p_\gamma = m_e^2 + 2E_\gamma m_e = m_e^2 + 2E_{e'}E_{\gamma'} - 2|\vec{p}_{\gamma'}|\cdot|\vec{p}_{e'}|$

$\hookrightarrow E_\gamma m_e = (E_\gamma + m_e - E_{\gamma'})\cdot E_{\gamma'} - \vec{p}_{\gamma'}\cdot(\vec{p}_\gamma - \vec{p}_{\gamma'})$

$E_\gamma m_e = E_\gamma E_{\gamma'} + m_e E_{\gamma'} - \underbrace{E_{\gamma'}^2 + \vec{p}_{\gamma'}^{\,2}}_{=0} - \vec{p}_{\gamma'}\,\vec{p}_\gamma$

$= E_\gamma E_{\gamma'} + m_e E_{\gamma'} - E_{\gamma'} E_\gamma \cos\theta$

$\hookrightarrow E_{\gamma'}(E_\gamma + m_e - E_\gamma\cos\theta) = E_\gamma m_e$

$$E_{\gamma'} = \frac{E_\gamma m_e}{m_e + E_\gamma(1-\cos\theta)} = \frac{1}{\frac{1}{E_\gamma} + \frac{1}{m_e}(1-\cos\theta)}$$

$p_{\gamma'} = E_{\gamma'} \quad ; (v_{\gamma'} = c = v_\gamma)$ $\quad\hookrightarrow$ o despejar el ángulo en función de $E_{\gamma'}$

$E_{e'} = E_\gamma + m_e - E_{\gamma'} = \dfrac{\cancel{E_\gamma m_e} + E_\gamma^2(1-\cos\theta) + m_e^2 + m_e E_\gamma(1-\cos\theta) - \cancel{E_\gamma m_e}}{m_e + E_\gamma(1-\cos\theta)}$

$= \dfrac{E_\gamma(E_\gamma + m_e)(1-\cos\theta) + m_e^2}{m_e + E_\gamma(1-\cos\theta)} = \dfrac{m_e(m_e + E_\gamma(1-\cos\theta)) + E_\gamma^2(1-\cos\theta)}{m_e + E_\gamma(1-\cos\theta)}$

$= m_e + \dfrac{E_\gamma^2(1-\cos\theta)}{m_e + E_\gamma(1-\cos\theta)} = m_e + \dfrac{E_\gamma}{\frac{m_e}{E_\gamma(1-\cos\theta)} + 1}$

$E_\gamma - E_{\gamma'} = E_{e'} - m_e \;// = h(\nu - \nu') = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = hc\,\frac{\lambda' - \lambda}{\lambda\lambda'} = \frac{h}{c}\nu\nu'(\lambda' - \lambda)$

o bien, en función de φ: $\quad = \dfrac{E_\gamma E_{\gamma'}}{hc}\cdot\Delta\lambda \to \Delta\lambda = \dfrac{hc}{\frac{E_{\gamma'}}{E_\gamma}\cdot\underbrace{(m_e + E_\gamma(1-\cos\theta))}_{m_e}}\cdot(1-\cos\theta) = \dfrac{hc}{m_e}(1-\cos\theta)$

$p_\gamma - p_{e'} = p_{\gamma'} - p_e$

$\hookrightarrow m_e^2 + 2E_\gamma E_{e'} - 2|\vec{p}_\gamma|\,|\vec{p}_{e'}|\cdot\cos\varphi = m_e^2 + 0 - 2E_{\gamma'} m_e$

$2E_\gamma E_{e'} + E_{\gamma'} m_e = 2E_\gamma|\vec{p}_{e'}|\cdot\cos\varphi = E_\gamma E_{e'} + (E_\gamma + m_e - E_{e'})m_e$

$= E_\gamma E_{e'} + E_\gamma m_e - E_{e'} m_e = E_{e'}(E_\gamma - m_e) + E_\gamma m_e \quad ; ()^2$

$\hookrightarrow (E_{e'})^2(E_\gamma^2 + m_e^2 - 2E_\gamma m_e) + E_\gamma^2 m_e^2 + 2E_{e'}E_\gamma m_e(E_\gamma - m_e) = E_\gamma^2|\vec{p}_{e'}|^2\cos^2\varphi$

$(E_{e'})^2(E_\gamma^2(1-\cos^2\varphi) + m_e^2 - 2E_\gamma m_e) + E_\gamma^2 m_e^2 + 2E_{e'}E_\gamma m_e(E_\gamma - m_e) + E_\gamma^2 m_e^2\cos^2\varphi = 0$

$E_{e'}^2(E_\gamma^2\sin^2\varphi + m_e^2 - 2E_\gamma m_e) + 2E_{e'}E_\gamma m_e(E_\gamma - m_e) + E_\gamma^2 m_e^2(1+\cos^2\varphi) = 0$

$\hookrightarrow$ ecua. de 2º grado, no la resuelvo. Es + sencillo con θ.

Otra opción es $\theta + \varphi$:

$$m_e^2 + 2E_\gamma m_e = m_e^2 + 2E_e' E_\gamma' - 2E_\gamma' |\vec{p}_e'| \cdot \cos(\theta + \varphi)$$

$\vdots$

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Probemos que $\gamma \to e^+ + e^-$ está prohibido.

$$(E_\gamma, \vec{p}_\gamma) = (E_1, \vec{p_1}) + (E_2, \vec{p_2})$$

$$0 = 2m_e^2 + 2E_1E_2 - 2|\vec{p_1}||\vec{p_2}|$$

$$= 2m_e^2 + 2E_1(E_\gamma - E_1) - 2(\vec{p_1})\cdot(\vec{p_\gamma} - \vec{p_1})$$

$$0 = m_e^2 + E_1E_\gamma - 2\vec{p_1}\vec{p_\gamma} - m_e^2$$

$$E_1E_\gamma = \vec{p_1}\cdot\vec{p_\gamma} = E_\gamma \cdot |\vec{p_1}| \cos\theta$$

$$E_1 = |\vec{p_1}| \cdot \cos\theta \leq |\vec{p_1}|$$

Sin embargo $E_1 = \sqrt{|\vec{p_1}|^2 + m_e^2} \geq |\vec{p_1}|$ // Está prohibido

Esto se debe a que no podemos irnos al s. CM. del fotón (∄). Si imaginamos el proceso inverso, e^+e^- en s.CM. no pueden crear un solo fotón pues no conservaría $\vec{p}$.

LAB: $(E_\gamma, \vec{p_\gamma}) + (m_N, \vec{0}) = (E_1, \vec{p_1}) + (E_2, \vec{p_2}) + (E_N, \vec{p_N})$, $\vec{p_N} = +\vec{p_\gamma}$

($\vec{0}$: espectador, quieto en S. Lab)

CM: Energía umbral: $(E_{\gamma_{th}}, \vec{p}_{\gamma_{th}}) + (E_{N_{th}}, -\vec{p}_{\gamma_{th}}) = (m_1 + m_2 + m_N, 0)$

$\hookrightarrow m_N^2 + 2p_\gamma p_N\big|_{th} = (2m_e + m_N)^2$ → Como $p_i \cdot p_j$ es escalar invariante, lo calculamos en ← S. Lab

$$m_N^2 + 2E_{\gamma_{th}}^L m_N = (2m_e + m_N)^2$$

$$E_{\gamma\,th}^L = \frac{(2m_e + m_N)^2 - m_N^2}{2m_N} = \frac{4m_e^2 + 4m_em_N}{2m_N} = \frac{2m_e^2}{m_N} + 2m_e$$

$$= 2m_e\left(1 + \frac{m_e}{m_N}\right)$$ //

→ energía de retroceso del nucleón, menor cuanto más masivo sea N.

$m_N \to \infty$, $E_\gamma^{L\,th} = 2m_e$

En el universo primitivo (muy denso) se creaban e^+e^-. Al reducirse la densidad, los fotones ya no materializan tanto, mientras que e^+e^- siguen dando 2 fotones $\Rightarrow$ se hace la luz

En S.C.M:

$|\vec{p}_\gamma| = |\vec{p}_N|$

$$0 + m_N^2 + 2E_\gamma E_N\big|_{th} + 2|\vec{p}_\gamma|^2_{th} = (2m_e + m_N)^2 = 4m_e^2 + 4m_e m_N + m_N^2$$

$$E_\gamma(2m_e + m_N - E_\gamma) + E_\gamma^2 = 2m_e(m_e + m_N)$$

$$E_{\gamma\,th}^{CM} = \frac{2m_e(m_e + m_N)}{(2m_e + m_N)} = 2m_e - \frac{2m_e^2}{2m_e + m_N}$$

$$= 2m_e\left(1 - \frac{1}{\frac{m_N}{m_e} + 2}\right)$$

$m_N \to \infty$
$E_{th} \to 2m_e$

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a) $\mathcal{L}_1 = \dot q_1^2 + \frac{\dot q_2^2}{\mu_1+\mu_2 q_1^2} + \mu_3 \dot q_1 \dot q_2 + \mu_4 q_1^2$

$$H = \sum_i \frac{\partial \mathcal{L}}{\partial \dot q_i}\cdot \dot q_i - \mathcal{L} = (2\dot q_1 + \mu_3 \dot q_2)\cdot \dot q_1 + \left(\frac{2\dot q_2}{\mu_1+\mu_2 q_1^2} + \mu_3 \dot q_1\right)\cdot \dot q_2 - \mathcal{L}$$

$$= 2\dot q_1^2 + \mu_3 \dot q_2 \dot q_1 + \frac{2\dot q_2^2}{\mu_1+\mu_2 q_1^2} + \mu_3 \cancel{\dot q_1}\dot q_2 - \dot q_1^2 - \frac{\dot q_2^2}{\mu_1+\mu_2 q_1^2} - \mu_3 \cancel{\dot q_1}\dot q_2 - \mu_4 q_1^2$$

$$= \dot q_1^2 + \frac{\dot q_2^2}{\mu_1+\mu_2 q_1^2} + \mu_3 \dot q_1 \dot q_2 - \mu_4 q_1^2$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot q_i} \Rightarrow p_1 = 2\dot q_1 + \mu_3 \dot q_2$$

$$p_2 = \frac{2\dot q_2}{\mu_1+\mu_2 q_1^2} + \mu_3 \dot q_1$$

$$\hookrightarrow \dot q_1 = \frac{p_1 - \mu_3 \dot q_2}{2}$$

$$\hookrightarrow p_2 = \frac{2\dot q_2}{\mu_1+\mu_2 q_1^2} + \frac{\mu_3}{2}p_1 - \frac{\mu_3^2}{2}\dot q_2$$

$$\dot q_2\left(\frac{2}{\mu_1+\mu_2 q_1^2} - \frac{\mu_3^2}{2}\right) = p_2 - \frac{\mu_3}{2}p_1$$

$$\dot q_2\left(\frac{4 - \mu_3^2\mu_1 - \mu_3^2\mu_2 q_1^2}{2(\mu_1+\mu_2 q_1^2)}\right) = \frac{2p_2 - \mu_3 p_1}{2}$$

$$\dot q_2 = \frac{(2p_2 - \mu_3 p_1)\cdot(\mu_1+\mu_2 q_1^2)}{4 - \mu_3^2(\mu_1+\mu_2 q_1^2)}$$

$$\dot q_1 = \frac{p_1 - \mu_3\cdot(2p_2 - \mu_3 p_1)(\mu_1+\mu_2 q_1^2)/(4-\mu_3^2(\mu_1+\mu_2 q_1^2))}{2}$$

$$= \frac{4p_1 - \cancel{p_1\mu_3^2(\mu_1+\mu_2 q_1^2)} - \mu_3 2p_2(\mu_1+\mu_2 q_1^2) + \cancel{\mu_3^2 p_1(\dots)}}{2(4-\mu_3^2(\mu_1+\mu_2 q_1^2))}$$

$$\dot q_1 = \frac{2p_1 - \mu_3 p_2 \overbrace{(\mu_1 + \mu_2 q_1^2)}^{F}}{4 - \mu_3^2(\mu_1 + \mu_2 q_1^2)}$$

$$H = \frac{4p_1^2 - 4\mu_3 p_1 p_2 F + \mu_3^2 p_2^2 F^2}{(4-\mu_3^2 F)^2} + \frac{(2p_2 - \mu_3 p_1)^2 \cdot F^2}{(4-\mu_3^2 F)^2 \cdot F}$$

$$+ \mu_3 \frac{(2p_1 - \mu_3 p_2 F)\cdot(2p_2 - \mu_3 p_1)\cdot F}{(4-\mu_3^2 F)^2} - \mu_4 q_1^2$$

$$= \frac{1}{(4-\mu_3^2 F)^2} \cdot \Big\{ (2p_1 - \mu_3 p_2 F)^2 + (2p_2 - \mu_3 p_1)^2 F + \mu_3 (2p_1 - \mu_3 p_2 F) \cdot$$
$$\cdot (2p_2 - \mu_3 p_1) F - \mu_4 q_1^2 (4 - \mu_3^2 F)^2 \Big\}$$

$$= \frac{1}{(4-\mu_3^2 p)^2} \Big\{ 4p_1^2 + \mu_3^2 p_2^2 F^2 - \cancel{4 p_1 p_2 \mu_3 F} + 4 p_2^2 F + \mu_3^2 p_1^2 F - \cancel{4\mu_3 p_1 p_2 F}$$
$$+ \cancel{4\mu_3 p_1 p_2 F} - 2\mu_3^2 p_1^2 F - 2\mu_3^2 p_2^2 F^2 + \mu_3^3 p_1 p_2 F^2 \Big\} - \mu_4 q_1^2$$

$$= \frac{1}{(4-\mu_3^2 F)^2} \Big\{ p_1^2 (4 - \mu_3^2 F) + p_2^2 F (4 - \mu_3^2 F)$$
$$- \mu_3 p_1 p_2 F (4 - \mu_3^2 F) \Big\} - \mu_4 q_1^2$$

$$= \frac{1}{(4-\mu_3^2 F)} \cdot \Big\{ p_1^2 + p_2^2 F - \mu_3 p_1 p_2 F \Big\} - \mu_4 q_1^2$$

Procedimiento más sencillo:

$$\mathcal{L} = (\dot q_1 \quad \dot q_2) \cdot \underbrace{\begin{pmatrix} 1 & \mu_3/2 \\ \mu_3/2 & \frac{1}{F} \end{pmatrix}}_{G} \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} + \mu_4 q_1^2$$

$$G^{-1} = \frac{4}{4 - \mu_3^2 F} \begin{pmatrix} 1 & -\frac{\mu_3}{2} F \\ -\frac{\mu_3}{2} F & F \end{pmatrix}$$

$$\rightarrow H = \frac{1}{4} (p_1, \ p_2) \cdot G^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \mu_4 q_1^2 = \frac{(p_1^2 + p_2^2 F - \mu_3 p_1 p_2 F)}{4 - \mu_3^2 F} - \mu_4 q_1^2$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \text{(ecuaciones de Hamilton)}$$

$F(q_1)$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{1}{4-\mu_3^2 F}\{2p_1 - \mu_3 p_2 F\}$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = \frac{1}{4-\mu_3^2 F}\{2p_2 F - \mu_3 p_1 F\}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -\left\{-2\mu_4 q_1 - \frac{1\cdot\{\}\cdot(-\mu_3^2\cdot\mu_2\cdot 2q_1)}{(4-\mu_3^2F)^2} + \frac{(p_2^2 + \mu_3 p_1 p_2)\, 2\mu_2 q_1}{4-\mu_3^2 F}\right\}$$

$$= 2\mu_4 q_1 + \frac{-2\mu_3^2\mu_2 q_1 p_1^2 - 2\mu_3^2\mu_2 q_1 p_2^2 F + 2\mu_3^3\mu_2 q_1 p_1 p_2 F}{(4-\mu_3^2F)^2}$$

$$+ \frac{8\mu_2 q_1 p_2^2 - 2\mu_3^2\mu_2 q_1 p_2^2 F - 8\mu_3 p_1 p_2 \mu_2 q_1 + 2\mu_3^3 F p_1 p_2 \mu_2 q_1}{(4-\mu_3^2F)^2}$$

$$= 2\mu_4 q_1 + \frac{-2\mu_3^2\mu_2 q_1\cdot(p_1^2 + 2p_2^2 F) + 4\mu_3^3 F \mu_2 q_1 p_1 p_2}{(4-\mu_3^2 F)^2}$$

$$+ \frac{8\mu_2 q_1(p_2^2 - p_1 p_2 \mu_3)}{(4-\mu_3^2 F)^2} \quad //$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = 0 \to p_2 = \text{cte} \to \text{cantidad conservada} \quad //$$

b)
$$L_2 = \alpha \dot{q}_1 q_2 + \beta q_1 q_2$$

$$p_1 = \frac{\partial L_2}{\partial \dot{q}_1} = \alpha q_2$$

$$p_2 = \frac{\partial L_2}{\partial \dot{q}_2} = 0$$

$$H = p_1\cdot\dot{q}_1 - \overset{p_1}{\alpha q_2}\cdot\dot{q}_1 - \beta q_1 q_2 = -\beta q_1 q_2 \quad //$$

↳ Si no $\exists\ \dot{q}_2 \to q_2$ no es variable dinámica

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = 0 \quad ; \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} = 0 \quad ; \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1} = \beta q_2 = \alpha \dot{q}_2$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -\beta q_1 = 0 \to q_1 = 0$$

→ Problema: q_2 esconde una ligadura (multiplicador de Lagrange)

↳ $H = -\beta q_1 q_2 = -\frac{\beta}{\alpha} q_1 p_1$ → en realidad sólo ∃ un grado de libertad.

$\dot{q}_1 = \frac{\partial H}{\partial p_1} = -\frac{\beta}{\alpha} q_1 \to q_1 = q_1(0) \cdot e^{-\beta/\alpha\, t}$

$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = +\frac{\beta}{\alpha} \cdot p_1 \to p_1 = p_1(0) \cdot e^{\beta/\alpha\, t}$

↳ $q_2 = \frac{p_1}{\alpha} = q_2(0) \cdot e^{\beta/\alpha\, t}$, $q_2(0) = \frac{p_1(0)}{\alpha}$

A partir de Euler - Lagrange:

$\mathcal{L} = q_2(\alpha \dot{q}_1 + \beta q_1)$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = q_2 \alpha$; $\frac{\partial \mathcal{L}}{\partial q_1} = q_2 \beta$

↳ $\dot{q}_2 \alpha - q_2 \beta = 0 \to q_2 = q_2(0) \cdot e^{\beta/\alpha\, t}$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = 0$; $\frac{\partial \mathcal{L}}{\partial q_2} = (\alpha \dot{q}_1 + \beta q_1)$

↳ $\alpha \dot{q}_1 + \beta q_1 = 0 \to \dot{q}_1 = -\frac{\beta}{\alpha} q_1 \to q_1 = q_1(0) \cdot e^{-\frac{\beta}{\alpha} t}$

p_ϕ se conserva.

Coincide

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1) $\mathcal{L} = \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta^2 + m_2 l_1 l_2 \cos(\theta-\phi)\dot\theta\dot\phi + \frac{1}{2} m_2 l_2^2 \dot\phi^2$
$+ (m_1-m_2) g\, l_1 \cos\theta + m_2 g\, l_2 \cos\phi$

$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = (m_1+m_2) l_1^2 \dot\theta + m_2 l_1 l_2 \dot\phi \cos(\theta-\phi)$

$p_\phi = m_2 l_1 l_2 \dot\theta \cos(\theta-\phi) + m_2 l_2^2 \dot\phi$

$\mathcal{H} = p_\theta \cdot \dot\theta + p_\phi \cdot \dot\phi - \mathcal{L} = (m_1+m_2) l_1^2 \dot\theta^2 + m_2 l_1 l_2 \cos(\theta-\phi)\dot\theta\dot\phi$
$+ m_2 l_1 l_2 \cos(\theta-\phi)\dot\theta\dot\phi + m_2 l_2^2 \dot\phi^2 - \mathcal{L}$
$= \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta^2 + m_2 l_1 l_2 \cos(\theta-\phi)\dot\theta\dot\phi + \frac{1}{2} m_2 l_2^2 \dot\phi^2$
$- (m_1-m_2) g\, l_1 \cos\theta - m_2 g\, l_2 \cos\phi = T + V$

⇒ Faltaría sustituir $\dot\theta, \dot\phi$ ↳ (p_θ, p_ϕ)

→ También se puede hacer con la matriz G

$$\mathcal{L} = (\dot\theta \quad \dot\phi) \overbrace{\begin{pmatrix} \frac{1}{2}(m_1+m_2) l_1^2 & \frac{m_2}{2} l_1 l_2 \cos(\theta-\phi) \\ \frac{m_2}{2} l_1 l_2 \cos(\theta-\phi) & m_2 l_2^2 \end{pmatrix}}^{G} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} - (m_1-m_2) g l_1 \cos\theta - m_2 g l_2 \cos\phi$$

$$\mathcal{H} = (p_\theta \quad p_\phi) \cdot G^{-1} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} + \dots$$

$$\mathcal{H} = \frac{1}{2} \dot\theta p_\theta + \cancel{\frac{1}{2} \dot\phi p_\phi} - (m_1-m_2) g l_1 \cos\theta - m_2 g l_2 \cos\phi$$

2)

$$\mathcal{L} = \frac{m}{2} (\dot l^2 + l^2 \sin^2\alpha \underbrace{\dot\phi^2}_{\omega^2}) - mgl\cos\alpha = \mathcal{L}(l, \dot l)$$

$$p_l = \frac{\partial \mathcal{L}}{\partial \dot l} = m \dot l$$

$$\mathcal{H} = \frac{m \dot l^2}{2} - \frac{m}{2} l^2 \omega^2 \sin^2\alpha + mgl\cos\alpha = \frac{p_l^2}{2m} - \frac{m}{2}(l\omega\sin\alpha)^2 + mgl\cos\alpha$$
$$= \mathcal{H}(l, p_l)$$

$$G = \frac{m}{2}$$

$$\frac{\partial \mathcal{H}}{\partial p_l} = \dot l = \frac{p_l}{m_l}$$

$$\dot p_l = -\frac{\partial H}{\partial l} = + m l \omega^2 \sin^2\alpha - mg\cos\alpha = m\ddot l$$ → coincide con E-L.

3)

$$\mathcal{L} = \frac{m}{2} \left(\dot x \quad R\dot\theta \right) \cdot \underbrace{I}_{I^{-1}} \begin{pmatrix} \dot x \\ R\dot\theta \end{pmatrix} + mgx\sin\alpha$$

$$\mathcal{H} = \frac{m}{2} (\dot x^2 + R^2 \dot\theta^2) - mgx\sin\alpha$$

$$\frac{\partial \mathcal{L}}{\partial \dot x} = m\dot x \quad ; \quad \frac{\partial \mathcal{L}}{\partial \dot\theta} = mR^2\dot\theta$$

$$\to \mathcal{H} = \frac{p_x^2 + p_\theta^2 / R^2}{2m} - mgx\sin\alpha$$

O directamente $G = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}$

$$\mathcal{H} = \frac{1}{2m} (p_x \quad p_\theta) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix} \begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \dot{\theta} = cte = \omega$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \rightarrow p_\theta = cte$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = + mg\sin\alpha = m\ddot{x}$$

→ Faltaría añadir ligadura.

4)
$$\mathcal{L} = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) - mg\rho\cos\theta$$

$$= \frac{m}{2}\left(\dot{\rho} \quad \dot{\theta}\right)\begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}\begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix} - mg\rho\cos\theta$$

$$\mathcal{H} = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2}\right) + mg\rho\cos\theta$$

$$\frac{\partial \mathcal{H}}{\partial p_\rho} = \dot{\rho} = \frac{p_\rho}{m} \qquad \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{\rho^2} = \dot{\theta}$$

$$\frac{\partial H}{\partial \rho} = -\frac{p_\theta^2}{m\rho^3} + mg\cos\theta = -\dot{p}_\rho \ ; \ -\frac{\partial \mathcal{H}}{\partial \theta} = \dot{p}_\theta = -mg\rho\sin\theta$$

→ Falta ligadura

5)
$$\mathcal{L} = 2mR^2\dot{\theta}^2\sin^2\frac{\theta}{2} + 2mgR\sin^2\frac{\theta}{2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 4mR^2\dot{\theta}\sin^2\frac{\theta}{2} = p_\theta$$

$$\mathcal{H} = p_\theta\dot{\theta} - \mathcal{L} = 2mR^2\dot{\theta}^2\sin^2\frac{\theta}{2} - 2mgR\sin^2\frac{\theta}{2} = \frac{p_\theta^2}{8mR^2\sin^2\frac{\theta}{2}} - 2mgR\sin^2\frac{\theta}{2}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{4mR^2\sin^2\frac{\theta}{2}}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{p_\theta^2 \, \frac{1}{2}\cos\frac{\theta}{2}}{4mR^2\sin^3\frac{\theta}{2}} \quad //$$

(33)

$H(q,p,t) = \mathcal{L}_{\dot q}(L(q,\dot q,t))$ (Transf. de Legendre)

con respecto a ... $x \equiv q$, $u \equiv \dot p$

$K(\dot q,\dot p,t) = \mathcal{L}_q(H(q,\dot q,t)) = q\cdot\dot p - \mathcal{L}$

$dK = \frac{\partial K}{\partial \dot q}d\dot q + \frac{\partial K}{\partial \dot p}d\dot p + \frac{\partial K}{\partial t}dt = \dot p\, dq + q\, d\dot p - \left(\frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot q}d\dot q + \frac{\partial L}{\partial t}dt\right)$

$dK = 0 \rightarrow \frac{\partial K}{\partial \dot q} = -\frac{\partial L}{\partial \dot q} = -p \qquad \rightarrow \frac{\partial K}{\partial t} = -\frac{\partial L}{\partial t}$

Ecuac. del movi∅

$\rightarrow \frac{\partial K}{\partial \dot p} = q \qquad \rightarrow \dot p - \frac{\partial L}{\partial q} = 0$

(34)

3 maneras de saber si transformación es canónica:
- Función generatriz
- Corchetes de Poisson
- Condición simpléctica (jacobiano)

$Q = \log(1+\sqrt{q}\cos p)$

$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p = 2(\sqrt{q}+q\cos p)\sin p = 2\sqrt{q}\,(sp + \frac{\sqrt{q}}{2}s2p)$

$\frac{\partial Q}{\partial q} = \frac{\frac{1}{2}q^{-1/2}\cos p}{1+\sqrt{q}\cos p} = \frac{\frac{1}{2}\cos p}{\sqrt{q}+q\cos p}$

$\frac{\partial Q}{\partial p} = \frac{-\sqrt{q}\cdot\sin p}{1+\sqrt{q}\cos p}$

$\frac{\partial P}{\partial q} = 2\left(\frac{1}{2}q^{-1/2}+\cos p\right)\sin p$

$\frac{\partial P}{\partial p} = 2\sqrt{q}\,(\cos p + \sqrt{q}\cos 2p)$

$\{Q,P\} = \frac{\cos p\,(\cos p + \sqrt{q}\cos 2p)}{1+\sqrt{q}\cos p} + \frac{\sqrt{q}\sin^2 p\,(q^{-1/2}+2\cos p)}{1+\sqrt{q}\cos p}$

$= \frac{1}{1+\sqrt{q}\cos p}\cdot\left(\cos^2 p + \sqrt{q}\cos p(\cos^2 p - \sin^2 p) + \sin^2 p + 2\sqrt{q}\cos p\sin^2 p\right)$

$= \frac{1}{1+\sqrt{q}\cos p}\left(1+\sqrt{q}\cos p\,(\cos^2 p - \sin^2 p + 2\sin^2 p)\right) = 1$

↳ Es canónica

$F_3 = F_3(p, Q)$

$$q = -\frac{\partial F_3}{\partial p} = ? \rightarrow e^Q = 1 + \sqrt{q}\cos p \rightarrow q = \left(\frac{e^Q - 1}{\cos p}\right)^2$$

$$\hookrightarrow \frac{\partial F_3}{\partial p} = -q = -\frac{(e^Q - 1)^2}{\cos^2 p} \rightarrow F_3 = -(e^Q - 1)^2 \tan(p) + f(Q)$$

$$\mathcal{P} = -\frac{\partial F_3}{\partial Q} = +2(e^Q - 1)\tan p + f'(Q)$$

$$= [illegible] = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$$

$$= 2\, e^Q \sin p + \frac{e^Q - 1}{\cos p}\sin p = 2(e^Q - 1)\tan p \rightarrow$$

$$\hookrightarrow f'(Q) = 0$$

$$\hookrightarrow F_3(p, Q) = -(e^Q - 1)^2 \tan p$$

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a) $\begin{cases} Q = \log\left(\frac{\sin p}{q}\right) \\ \mathcal{P} = q / \tan p \end{cases}$

$$\frac{\partial Q}{\partial q} = \frac{q}{\sin p}\cdot\left(-\frac{\sin p}{q^2}\right) = -\frac{1}{q} \quad ; \quad \frac{\partial Q}{\partial p} = \frac{q}{\sin p}\cdot\frac{\cos p}{q} = \frac{1}{\tan p}$$

$$\frac{\partial \mathcal{P}}{\partial q} = \frac{1}{\tan p} \quad ; \quad \frac{\partial \mathcal{P}}{\partial p} = -\frac{q}{\tan^2 p}\cdot\frac{1}{\cos^2 p} = -\frac{q}{\sin^2 p}$$

$$\{Q, P\} = \frac{1}{\sin^2 p} - \frac{1}{\tan^2 p} = \frac{1 - \cos^2 p}{\sin^2 p} = \frac{\sin^2 p}{\sin^2 p} = 1$$

Es canónica ✓

b) $\begin{cases} Q = \arctan \dfrac{\alpha q}{p} \\ P = \dfrac{\alpha q^2}{2}\left(1 + \dfrac{p^2}{\alpha^2 q^2}\right) \end{cases}$

$$\frac{\partial Q}{\partial q} = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} \cdot \frac{\alpha}{p} \; ; \quad \frac{\partial Q}{\partial p} = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} (-) \frac{\alpha q}{p^2}$$

$$\frac{\partial P}{\partial q} = \alpha q \left(1 + \frac{p^2}{\alpha^2 q^2}\right) + \frac{\alpha q^2}{2} \cdot (-2) \frac{p^2}{\alpha^2 q^3} = \alpha q + \frac{p^2}{\alpha q} - \frac{p^2}{\alpha q} = \alpha q$$

$$\frac{\partial P}{\partial p} = \frac{\alpha q^2}{2} \cdot \frac{2p}{\alpha^2 q^2} = \frac{p}{\alpha}$$

$$\{Q, P\} = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} + \frac{\alpha^2 q^2 / p^2}{1+\left(\frac{\alpha q}{p}\right)^2} = 1$$

c) $\begin{cases} Q = q^{\alpha} \cos \beta p \; \rightarrow \; \dfrac{\partial Q}{\partial q} = \alpha q^{\alpha-1} \cos \beta p \; ; \; \dfrac{\partial Q}{\partial p} = -\beta q^{\alpha} \sin \beta p \\ P = q^{\alpha} \sin \beta p \; \rightarrow \; \dfrac{\partial P}{\partial q} = \alpha q^{\alpha-1} \sin \beta p \; ; \; \dfrac{\partial P}{\partial p} = \beta q^{\alpha} \cos \beta p \end{cases}$

$$\{Q, P\} = q^{2\alpha-1} \alpha\beta \cos^2 \beta p + q^{2\alpha-1} \alpha\beta \sin^2 \beta p = \alpha\beta\, q^{2\alpha-1} = 1 \quad \forall q$$

Sii $\alpha = \frac{1}{2}$; $\beta = 2$

$\begin{cases} Q = \sqrt{q} \cos 2p \\ P = \sqrt{q} \sin 2p \end{cases}$ $\rightarrow$ Cambio útil para resolver oscilador armónico.

(36)

a) $Q = \alpha\, p/q$

$P = \beta q^2$

$\frac{\partial Q}{\partial q} = -\alpha p/q^2$; $\frac{\partial Q}{\partial p} = \alpha/q$

$\frac{\partial P}{\partial q} = 2\beta q$; $\frac{\partial P}{\partial p} = 0$

$\{Q, P\} = -2\alpha\beta = 1$
↳ Condición para α y β / trans. canónica

$F_1(Q, q)$

↳ $p = \frac{\partial F_1}{\partial q} = Qq/\alpha \rightarrow F_1 = \frac{Qq^2}{2\alpha} + f(Q)$

$P = -\frac{\partial F_1}{\partial Q} = -\frac{q^2}{2\alpha} + f'(Q) = \beta q^2 \rightarrow -\frac{1}{2\alpha} = \beta$

$f'(Q) = 0$

$f = 0$ (elijo)

↓ $F_1(Q, q) = \frac{Qq^2}{2\alpha} = -\beta Q q^2$

$F_3(Q, p)$ → ~~$Q^2 P = \alpha^2 \beta\, p^2 = \alpha^2 P$~~

$q = \frac{\partial F_3}{\partial p} = -\frac{\alpha p}{Q} \rightarrow F_3 = -\frac{\alpha p^2}{2Q} + f(Q)$

$\frac{\partial F_3}{\partial Q} = \frac{\alpha p^2}{2Q^2} + f'(Q) = -P = -\beta q^2 = -\beta \frac{\alpha^2 p^2}{Q^2}$

$\rightarrow f = f' = 0 \rightarrow -\beta\alpha = \frac{1}{2}$ ✓ $\rightarrow F_3 = -\frac{\alpha p}{Q} = \mathcal{L}(F_1)$

$F_4(P, p)$

$\frac{\partial F_4}{\partial p} = -q = -\sqrt{\frac{P}{\beta}} \rightarrow F_4 = -\sqrt{\frac{P}{\beta}}\cdot p + f(P)$

$\frac{\partial F_4}{\partial P} = -Q = -\frac{\alpha p}{q} = -\frac{\alpha p}{\sqrt{\frac{P}{\beta}}} = f'(P) - \frac{1}{2}\frac{p}{\sqrt{\frac{P}{\beta}}}\left(\frac{1}{\beta}\right)$ → ∄ F_2 porque $P(q)$ únicamente.

↳ $f = 0$; $\beta = -\frac{1}{2\alpha}$ → $F_4 = -p\sqrt{\frac{P}{\beta}}$

b) $\begin{cases} Q = p + i\lambda q \\ P = \dfrac{p - i\lambda q}{2i\lambda} = \dfrac{p}{2i\lambda} - \dfrac{q}{2} \end{cases}$

$$\left.\begin{array}{ll} \dfrac{\partial Q}{\partial q} = i\lambda & ; \quad \dfrac{\partial Q}{\partial p} = 1 \\ \dfrac{\partial P}{\partial q} = -\dfrac{1}{2} & ; \quad \dfrac{\partial P}{\partial p} = \dfrac{1}{2i\lambda} \end{array}\right\} \Rightarrow \{Q,P\} = \frac{1}{2} + \frac{1}{2} = 1$$

Es canónica $\forall \lambda$

$F_1(Q,q)$

$\hookrightarrow p = \cancel{Q\lambda(i\lambda q\ \tfrac{q}{2})}\ \dfrac{\partial F_1}{\partial q} \rightarrow F_1 = Qq - \dfrac{i\lambda}{2}q^2 + f(Q)$

$$P = \frac{p}{2i\lambda} - \frac{q}{2} = \frac{Q - i\lambda q}{2i\lambda} - \frac{q}{2} = \frac{Q}{2i\lambda} - q = -\frac{\partial F_1}{\partial Q} = -q + f'(Q)$$

$\hookrightarrow f'(Q) = \dfrac{Q}{2i\lambda} \rightarrow f(Q) = \dfrac{Q^2}{4i\lambda}$

$$\Rightarrow F_1(Q,q) = \frac{Q^2}{4i\lambda} + Qq - \frac{i\lambda}{2}q^2$$

$F_2(P,q)$

$p = 2i\lambda\left(P + \dfrac{q}{2}\right) = \dfrac{\partial F_2}{\partial q} \rightarrow F_2 = 2i\lambda\left(Pq + \dfrac{q^2}{4}\right) + f(P)$

$Q = p + i\lambda q = 2i\lambda\left(P + \dfrac{q}{2}\right) + i\lambda q = 2i\lambda(P + q) = \dfrac{\partial F_2}{\partial P} = 2i\lambda q + f'(P)$

$\hookrightarrow f'(P) = 2i\lambda P \rightarrow f = i\lambda P^2$

$$\rightarrow F_2 = i\lambda\left(P^2 + 2Pq + q^2/2\right)$$

$F_3(Q,p) \rightarrow \dfrac{\partial F_3}{\partial p} = -q = -\dfrac{(Q-p)}{i\lambda} \rightarrow F_3 = -\dfrac{(Qp - p^2/2)}{i\lambda} + f(Q)$

$\dfrac{\partial F_3}{\partial Q} = -\dfrac{p}{i\lambda} + f'(Q) = -P = \dfrac{q}{2} - \dfrac{p}{2i\lambda} = \dfrac{Q-p}{2i\lambda} - \dfrac{p}{2i\lambda} = \dfrac{Q}{2i\lambda} - \dfrac{p}{i\lambda}$

$\hookrightarrow f'(Q) = \dfrac{Q}{2i\lambda} \rightarrow f(Q) = \dfrac{Q^2}{4i\lambda} \Rightarrow F_3 = (p^2 - 2Qp + Q^2/2)/2i\lambda$

$F_4(P,p)$

$\hookrightarrow \dfrac{\partial F_4}{\partial p} = -q = \left(P - \dfrac{p}{2i\lambda}\right)\cdot 2 = 2P - p/i\lambda \rightarrow F_4 = 2Pp - p^2/2i\lambda + f(P)$

$\dfrac{\partial F_4}{\partial P} = Q = p + i\lambda \cdot 2\left(\dfrac{p}{2i\lambda} - P\right) = 2p - 2i\lambda P = 2p + f'(P) \rightarrow f(P) = -i\lambda P^2$

$$F_4 = -i\lambda P^2 + 2Pp - p^2/2i\lambda$$

Resolución del oscilador armónico unidimensional:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = H(q,p)$$

a) $q^2 = P/\beta$; $p^2 = \frac{Q^2 q^2}{\alpha^2} = \frac{Q^2}{\alpha^2} \cdot P/_\beta = 4Q^2 \beta P$

$$H' = \frac{2Q^2\beta P}{m} + \frac{1}{2\beta} m\omega^2 P = H'(Q,P) = P\left(\frac{2\beta Q^2}{m} + \frac{m\omega^2}{2\beta}\right)$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -P\left(\frac{4\beta Q}{m}\right) = -\frac{4\beta}{m} QP$$

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{2\beta Q^2}{m} + \frac{m\omega^2}{2\beta} = \frac{dQ}{dt} \rightarrow \int_{Q(t_0)}^{Q(t)} \frac{dQ}{\frac{2\beta}{m}Q^2 + \frac{m\omega^2}{2\beta}} = \int_{t_0}^{t} dt$$

↳ Ver tablas ⇒ (tipo arcotangente) $t - t_0 = \frac{1}{\omega^2} \operatorname{arctg}\left(\frac{2\beta}{m\omega} Q\right)\Big|_{Q(t_0)=0 \text{ (origen)}}^{Q(t)}$

↳ $Q(t) = \frac{m\omega}{2\beta} \tan(\omega(t-t_0))$

$$\frac{dP}{dt} = -\frac{4\beta}{m} \cdot \frac{m\omega}{2\beta} \cdot \tan(\omega(t-t_0)) P = 2\omega \tan(\omega(t-t_0)) P$$

$\cos\theta = z$ $\quad -\sin\theta\, d\theta = dz$

$-\int \frac{dP}{P} = \int 2\omega \tan(\omega(t-t_0))\,dt = $ (Cambio de variable) $\quad \theta = \omega(t-t_0)$, $d\theta = \omega\,dt$

$$-\log P = \int 2\omega \frac{(-dz/\omega)}{z} \rightarrow \log P = 2\log z + \underset{\log(P_0)}{cte}$$

$$\Downarrow$$

$$P(t) = P_0 \cdot \cos^2(\omega(t-t_0))$$

Si deshacemos el cambio

$$q = \sqrt{\frac{P}{\beta}} = q_0 \cdot \cos(\omega(t-t_0))$$

$$p = -2\beta Q q = -2\beta \frac{m\omega}{2\beta} \cdot \tan(\omega(t-t_0)) \cdot q_0 \cos(\omega(t-t_0))$$

$$= -m\omega q_0 \sin(\omega(t-t_0)) = m\dot{q}$$ ✓

b) $p = \frac{Q}{2} + 2i\lambda P \quad \rightarrow p^2 = \frac{Q^2}{4} - \lambda^2 P^2 + i\lambda Q P$

$q = \left(\frac{Q}{2} - i\lambda P\right)/_{2i\lambda} = \frac{Q}{2i\lambda} - \frac{P}{} \rightarrow q^2 = -\frac{Q^2}{4\lambda^2} + P^2 - \frac{QP}{i\lambda}$

$$H' = \frac{1}{2m}\left(\frac{Q^2}{4} - \lambda^2P^2 + i\lambda QP\right) + \frac{1}{2}m\omega^2\left(P^2 - \frac{Q^2}{4\lambda^2} - \frac{QP}{i\lambda}\right)$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -\left(\frac{Q}{4m} + \frac{i\lambda P}{2m} - \frac{Q}{4\lambda^2}m\omega^2 - \frac{Pm\omega^2}{2i\lambda}\right)$$

$$= -\frac{Q}{4}\left(\frac{1}{4m} - \frac{m\omega^2}{4\lambda^2}\right) + \frac{iP}{2}\left(\frac{\lambda}{m} + \frac{m\omega^2}{\lambda}\right) = -Q\cdot\frac{a}{2} + P\cdot b$$

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{1}{2m}(-2\lambda^2P + i\lambda Q) + \frac{1}{2}m\omega^2\left(2P - \frac{Q}{i\lambda}\right) =$$

$$= \frac{iQ}{2}\left(\frac{\lambda}{m} + \frac{m\omega^2}{\lambda}\right) + P\left(m\omega^2 - \frac{\lambda^2}{m}\right) = Q\cdot b\lambda + P\frac{2a\lambda^2}{\lambda^2}$$

$$\ddot{Q} = \dot{Q}b - \dot{P}\frac{2a\lambda^2}{\lambda^2} = \dot{Q}b + \frac{2a\lambda^2}{\lambda^2}\left(Q\frac{a}{2} + Pb\right)$$

$$= \dot{Q}b + \frac{a^2\lambda^2}{\lambda^2}Q + \frac{2ab\lambda^2}{\lambda^2}(Qb - \dot{Q})\cdot\frac{1}{2a\lambda^2} = \dot{Q}b + \frac{a^2\lambda^2}{\lambda^2}Q + b(Qb - \dot{Q})$$

$$= \dot{Q}b - \dot{Q}b + \frac{\lambda^2 a^2}{\lambda^4}Q + b^2Q = \left(\frac{a^2\lambda^2}{\lambda^2} + b^2\right)Q$$

$$a = \frac{1}{2}\left(\frac{1}{m} - \frac{m\omega^2}{\lambda^2}\right) \rightarrow \lambda^2a^2 = \frac{1}{4}\left(\frac{\lambda^2}{m^2} - 2\omega^2 + \frac{m^2\omega^4}{\lambda^2}\right)$$

$$b = \frac{i}{2}\left(\frac{\lambda}{m} + \frac{m\omega^2}{\lambda}\right) \rightarrow b^2 = -\frac{1}{4}\left(\frac{\lambda^2}{m^2} + \frac{m^2\omega^4}{\lambda^2} + 2\omega^2\right)$$

$a^2\lambda^2 + b^2 = -\omega^2$

$\Rightarrow \quad \ddot{Q} + \omega^2 Q = 0 \quad \rightarrow$ Oscilador armónico

$$Q = A\cos(\omega t + \delta)$$

$$P = \frac{Qb - \dot{Q}}{2a\lambda^2}$$

(37)

$$V = mgz$$

$$T = \frac{1}{2}m\dot{z}^2$$

$$\mathcal{L} = T - V = \frac{m}{2}\dot{z}^2 - mgz \rightarrow p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

$$\mathcal{H} = T + V = \frac{m\dot{z}^2}{2} + mgz = p_z \cdot \dot{z} - \mathcal{L} \checkmark$$

$$= \frac{p^2}{2m} + mgq = P \qquad \begin{pmatrix} q \equiv z \\ p_z \equiv p \end{pmatrix}$$

$$\rightarrow F_4(P, p) \quad / \quad H' = P$$

$$\frac{\partial F_4}{\partial p} = -q \qquad \frac{\partial H'}{\partial Q} = -\dot{P} = 0 \rightarrow P(t) = cte$$

$$\frac{\partial F_4}{\partial P} = Q \qquad \frac{\partial H'}{\partial P} = \dot{Q} = 1 \rightarrow Q(t) = t + C_1$$

$$q = \frac{P - p^2/2m}{mg} \rightarrow F_4 = -\frac{1}{mg}\left(Pp - p^3/6m\right) + f(P)$$

$$\frac{\partial F_4}{\partial P} = -\frac{p}{mg} + f'(P) = Q$$

→ tengo libertad para elegir $f = 0$, y la transformación sigue siendo canónica ↳ (única solución fácil para que lo sea).

$$\begin{cases} Q = -\dfrac{p}{mg} \\ P = mgq + p^2/2m \end{cases} \Rightarrow \{Q, P\} = \begin{vmatrix} 0 & -\frac{1}{mg} \\ mg & p/m \end{vmatrix} = 1 \checkmark$$

$$F_4 = \frac{1}{mg}\left(\frac{p^3}{6m} - Pp\right)$$

$$P(t) = cte = E = mgq + p^2/2m = mg\,q(0) + p(0)^2/2m$$

$$Q(t) = t + C_1 = -p/mg \rightarrow p(t) = -mgt + p_0$$

$$q(t) = \frac{P - p^2/2m}{mg} = q(0) + \frac{p(0)^2}{2m^2g} - \frac{(mgt)^2}{2m^2g} - \frac{p_0^2}{2m^2g} + \frac{2p_0mgt}{2m^2g}$$

$$= q(0) - \frac{1}{2}gt^2 + \underbrace{\frac{p_0}{m}}_{v_0} \cdot t$$

(38) $q = Q + \frac{P}{m} t = Q + \frac{P}{m} t$

$p = P \qquad \{Q, P\} = \begin{vmatrix} 1 & -\frac{t}{m} \\ 0 & 1 \end{vmatrix} = 1$ ✓

$H = \frac{p^2}{2m} + V(q)$

$F_1(q, Q) \to p = \frac{\partial F_1}{\partial q} = P = (q - Q) \cdot \frac{m}{t} \to F_1 = \left(\frac{q^2}{2} - Qq\right)\frac{m}{t} + f(Q)$

$P = -\frac{\partial F_1}{\partial Q} = q \frac{m}{t} - f'(Q) = q\frac{m}{t} - Q\frac{m}{t}$

$\hookrightarrow f'(Q) = Q\frac{m}{t} \to f(Q) = \frac{Q^2 m}{2t} \to F_1 = \left(\frac{q^2}{2} - Qq + \frac{Q^2}{2}\right)\frac{m}{t}$

$F_1 = \frac{1}{2}(q - Q)^2 \frac{m}{t}$ $\qquad q - Q = \frac{P}{m} t$

$\hookrightarrow F_1(p) = p^2 \frac{t}{m}$

$F_2(P, q)$

$p = \frac{\partial F_2}{\partial q} = P \to F_2 = Pq + f(P)$

$Q = \frac{\partial F_2}{\partial P} = q - \frac{P}{m} t = q + f'(P) \to f(P) = -\frac{P^2}{2m} t$

$\to F_2 = Pq - \frac{P^2}{2m} t$

$F_3(Q, p) \to \frac{\partial F_3}{\partial p} = -q = -Q - \frac{P}{m} t = -Q - \frac{p}{m} t$

$F_3 = -Qp - \frac{p^2}{2m} t + f(Q)$

$\frac{\partial F_3}{\partial Q} = -P = -p = -p + f'(Q) \to f = 0 \to F_3 = -p\left(Q + \frac{P}{2m} t\right)$

$F_4(P, p)$

$\frac{\partial F_4}{\partial p} = -q = -Q - \frac{P}{m} t \to \nexists F_4$ porque $P = p$ (sólo depende de p, no de q)

Analizamos sólo F_1:

$H' = H + \frac{\partial F_1}{\partial t} = \frac{p^2}{2m} + V(q) + \frac{m}{2}(q - Q)^2 \cdot \left(-\frac{1}{t^2}\right)$

$= \frac{(q-Q)^2 m^2}{2m\, t^2} + V(q) - \frac{m}{2t^2}(q - Q)^2 = V(q) = V\left(Q + \frac{P}{m} t\right)$

Si $V(q) = -q$

$H'(Q,P) = -Q - \frac{P}{m} t$

$\frac{\partial H'}{\partial Q} = -\dot{P} = -1 \rightarrow \dot{P} = 1 \rightarrow P = t + P_0$

$\frac{\partial H'}{\partial P} = \dot{Q} = -\frac{t}{m} \rightarrow Q = -\frac{t^2}{2m} + Q_0$

$P = p$ ✓

Para $H = \frac{p^2}{2m} - q = H(q,p)$

$\frac{\partial H'}{\partial q} = -\dot{p} = -1 \rightarrow \dot{p} = 1 \rightarrow p = t + p_0$

$\frac{\partial H'}{\partial p} = \dot{q} = \frac{p}{m} = \frac{t}{m} + \frac{p_0}{m} t \rightarrow q = \frac{t^2}{2m} + \frac{p_0}{m} t + q_0$

$Q = q - \frac{P}{m} t = \frac{t^2}{2m} + \frac{p_0 t}{m} + q_0 - \frac{t^2}{m} - \frac{p_0 t}{m}$
$= q_0 - \frac{t^2}{2m} \longrightarrow Q_0 = q_0$

(39)

$G(q,P) = q^2 P \quad ; \quad F_2(q,P) = Pq + \epsilon G(q,P)$

$H(q,p)$?

$\delta_G H = \epsilon \{H, G\} = 0$ → invariante bajo dicha transformación (simetría

↳ Como analizamos a orden ϵ, podemos tomar $P = p$

$$\hookrightarrow \begin{vmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \\ 2qP & q^2 \end{vmatrix} = q^2 \frac{\partial H}{\partial q} - 2qP \frac{\partial H}{\partial p} = 0 \rightarrow q \frac{\partial H}{\partial q} = 2\underset{p}{P} \frac{\partial H}{\partial p}$$

H desarrollable en serie de potencias : $H = \sum_{n,m} c_{nm} q^n p^m$

$\frac{\partial H}{\partial q} = c_{nm}\, n q^{n-1} p^m \; ; \; \frac{\partial H}{\partial p} = c_{nm}\, m q^n p^{m-1} \Rightarrow q \frac{\partial H}{\partial q} = c_{nm} q^n p^m \cdot n$

$2m = n \quad 2p\frac{\partial H}{\partial p} = 2c_{nm} q^n p^m \cdot m$

$H = c_m (q^2 p)^m = H(q^2 p) = H(G)$

↳ Debe tener dependencia de la forma de G: $H(q^2 p)$

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$Q = Q(q,t)$

$P = P(q,p)$

Transformación canónica: $\{Q,P\} = 1$

$$\begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\partial Q}{\partial q} & 0 \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q} \cdot \frac{\partial P}{\partial p} = 1$$

$\hookrightarrow \frac{\partial P}{\partial p} = \frac{1}{\frac{\partial Q}{\partial q}} = f(q,t)$ en general

$\hookrightarrow P = f(q,t) \cdot p + cte \rightarrow$ lineal en p

---

$$\begin{cases} Q = q + \epsilon q^3 = Q(q) \\ P = p - 3\epsilon q^2 p = P(q,p) \end{cases}$$

o bien: $\{Q,P\} = 1$

$$\begin{vmatrix} 1+3\epsilon q^2 & 0 \\ -6\epsilon q p & 1-3\epsilon q^2 \end{vmatrix} = 1 - 9\epsilon^2 q^4 = 1 + O(\epsilon^2) \approx 1$$

Debe cumplir $\frac{\partial P}{\partial p} = \frac{1}{\frac{\partial Q}{\partial q}}$

$$\left.\begin{aligned} \frac{\partial P}{\partial p} &= 1 - 3\epsilon q^2 \\ \frac{\partial Q}{\partial q} &= 1 + 3\epsilon q^2 \end{aligned}\right\} \frac{1}{\frac{\partial Q}{\partial q}} = \frac{1}{1+3\epsilon q^2} = \frac{1}{1+x} \overset{\text{Taylor}}{\simeq} 1 - x \underbrace{+ O(x^2)}_{0} = 1 - 3\epsilon q^2 = \frac{\partial P}{\partial p}$$

infinitesimal: $\epsilon \to 0$, $x = 3\epsilon q^2 \to 0$

$\rightarrow$ Por tanto la transformación infinitesimal es canónica.

Hallamos una función generatriz, también infinitesimal

$F_2(q,P) = P \cdot q + \epsilon G(P,q) + O(\epsilon^2)$

$\hookrightarrow$ Debe ser de esta forma para que cuando $\epsilon \to 0$, F_2 genere la transformación identidad.

$\hookrightarrow Q = \frac{\partial F_2}{\partial P} \overset{\epsilon \to 0}{=} q \quad ; \quad p = \frac{\partial F_2}{\partial q} \overset{\epsilon \to 0}{=} P$ ✓

Para $\epsilon \neq 0$, pero pequeño: desprecio $\Theta(\epsilon^2)$

$$Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{\partial G}{\partial P} = q + \epsilon q^3 \quad Q - q = \delta q = \frac{\partial G}{\partial P}\cdot\epsilon + \Theta(\epsilon^2)$$

$$\frac{\partial G}{\partial P} = q^3 \rightarrow G = q^3 P + h(q) \rightarrow F_2 = qP + \epsilon(q^3 P + h(q))$$

$$p = \frac{\partial F_2}{\partial q} = \left[3q^2 P + \frac{\partial h(q)}{\partial q}\right]\cdot\epsilon + P = \frac{P}{1-3\epsilon q^2} \overset{\epsilon\to 0}{\simeq} P(1+3\epsilon q^2) + \Theta(\epsilon^2)$$

$$\frac{\partial h}{\partial q} = 0 \rightarrow h \overset{\text{elijo}}{=} 0 \qquad P - p = \delta p = -\frac{\partial G}{\partial q}\cdot\epsilon + \Theta(\epsilon^2)$$

$$\Rightarrow F_2 = qP + \epsilon q^3 P = qP(1+\epsilon q^2)$$

Transformación es simetría del stma $\Leftrightarrow$ H invariante $\Leftrightarrow$ Cantidad conservada

$G = q^3 p$ es el generador infinitesimal de la transforma

La función hamiltoniana queda invariante si $G(q,p)$ es una constante a lo largo de las trayectorias físicas.

$H(q,p)$

$$H(Q,P) = H(q,p) + \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p \qquad \epsilon\to 0;\ P = p + \epsilon f \qquad \epsilon\frac{\partial G}{\partial P} = \epsilon\frac{\partial G}{\partial p} + \Theta(\epsilon^2)$$

$$\delta_G H = H(Q,P) - H(q,p) = \frac{\partial H}{\partial q}\frac{\partial G}{\partial P}\epsilon - \frac{\partial H}{\partial p}\frac{\partial G}{\partial q}\epsilon = \epsilon\{H,G\}$$

$\updownarrow$ (vimos en teoría)

$$\frac{dG}{dt} = \{G,H\} + \frac{\partial G}{\partial t} \rightarrow \text{si } G = \text{cte}(t) \rightarrow \delta_G H = 0$$

(ver prob. 4.1)

Podremos usar $\frac{\partial G}{\partial P} = \frac{\partial G}{\partial p}$ al haber un ϵ previo.

$$\left.\frac{dG}{dt}\right|_{\text{trayec. física}} = 0 \rightarrow \epsilon\{G,H\} = 0$$

nos quedamos a $\Theta(\epsilon)$

$$\{G,H\} = \begin{vmatrix} 3q^2 P & q^3 \\ \frac{\partial H}{\partial q} & \frac{\partial H}{\partial P} \end{vmatrix} = 3q^2 P\frac{\partial H}{\partial P} - q^3\frac{\partial H}{\partial q} = 0$$

$$3P\frac{\partial H}{\partial P} = q\frac{\partial H}{\partial q} \rightarrow H \propto q^n P^m \rightarrow 3Pm\,q^n P^{m-1} = q\,n\,q^{n-1}P^m$$

$$3m = n \rightarrow H \propto q^{3m}P^m = (q^3 P)^m = G^m \Rightarrow f(G) = \sum c_m G^m \text{ (Taylor)}$$

$H = f(G) \rightarrow$ lógico $\{G, f(G)\} = 0$ por propiedad de corchetes.

(45)

1 grado de libertad
Stma conservativo (E = cte)

$H(q,p) = p^2/2m + V(q) = E$

$p = \pm\sqrt{(E - V(q))2m} \longrightarrow p' = \frac{p}{\sqrt{2m}} = \pm\sqrt{E - V(q)}$

Espacio de fases $\rightarrow$ $p'(q)$ para distintos E

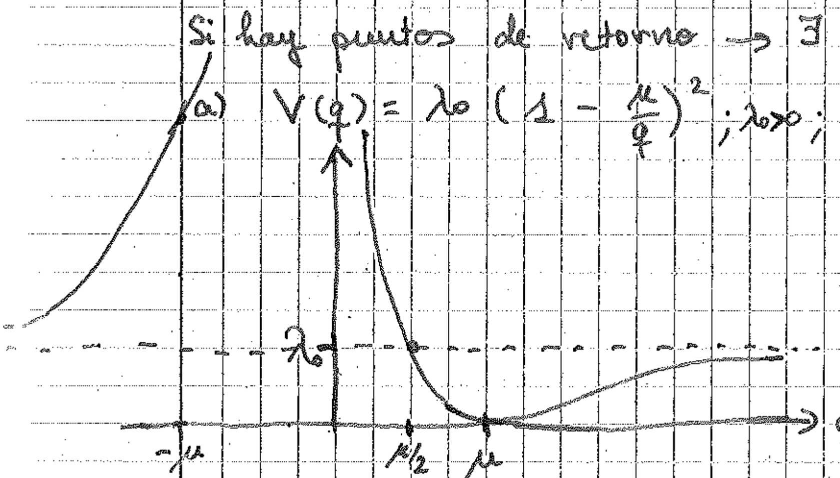
Punto elíptico $\rightarrow$ Alrededor de él hay movi✗ acotado (elipses), estado ligado

Punto hiperbólico

Separatrices: unen puntos de equilibrio inestable; además $E_{curva} = V_{max}$ potencial en el máximo local

Si hay puntos de retorno $\rightarrow$ $\exists$ movimiento vibracional.

a) $V(q) = \lambda_0 \left(1 - \frac{\mu}{q}\right)^2$, $\lambda_0 > 0$; $V(q) \geq 0 \rightarrow V(\mu)$ es un mínimo



$V(\mu) = 0$ $\quad V\left(\frac{\mu}{2}\right) = \lambda_0$

$V(\infty) = +\lambda_0$ (por debajo)

$V(0^{\pm}) = +\infty$

$V(-\mu) = 4\lambda_0$

$V(-\infty) = +\lambda_0$ (por arriba)

$\frac{\partial V}{\partial q} = 0 \Rightarrow \mu = q_{min}$

$E = K\lambda_0$, $K \in \mathbb{N}$

- Si $E < 0$ $\rightarrow$ prohibido (imaginario)
- Si $q > 0$ y $\underset{K<1}{E < \lambda_0}$ $\rightarrow$ movimiento acotado $\rightarrow$ movi✗ vibracional

  $\hookrightarrow$ Puntos de retorno: $E = V(q_r)$; $K < 1$

  $\lambda_0 K = \lambda_0\left(1 - \frac{\mu}{q_r}\right)^2 \rightarrow \pm\sqrt{K} = 1 - \frac{\mu}{q_r} \longleftrightarrow q_r = \frac{\mu}{1 \mp \sqrt{K}}$

- Si $K > 1$

  $\hookrightarrow$ semiacotado, sólo hay un punto de retorno (hay dos soluciones pero en $q = 0$ hay un ∞ que no pueden atravesar clásicamente).

- Si $K = 1$ $\rightarrow$ 2º pto de retorno en el ∞, tarda $t = \infty$ en llegar.

(He supuesto $\lambda_0 > 0$) $\rightarrow$ si no, no estaría acotado inferiormente.

se puede representar en un programa con una $z = f(x,y)$, set contour

$z \equiv K = \tilde{p}^2 + V(q)$; $y \equiv \tilde{p}$, $x \equiv q$

$\tilde{p} = \frac{p'}{\sqrt{\lambda_0}} = \pm\sqrt{K - \left(1-\frac{\mu}{q}\right)^2}$; $\tilde{p}_{max}(q=\mu) = \pm\sqrt{K}$

si $K=0$, $q=\mu \to \tilde{p} = 0$

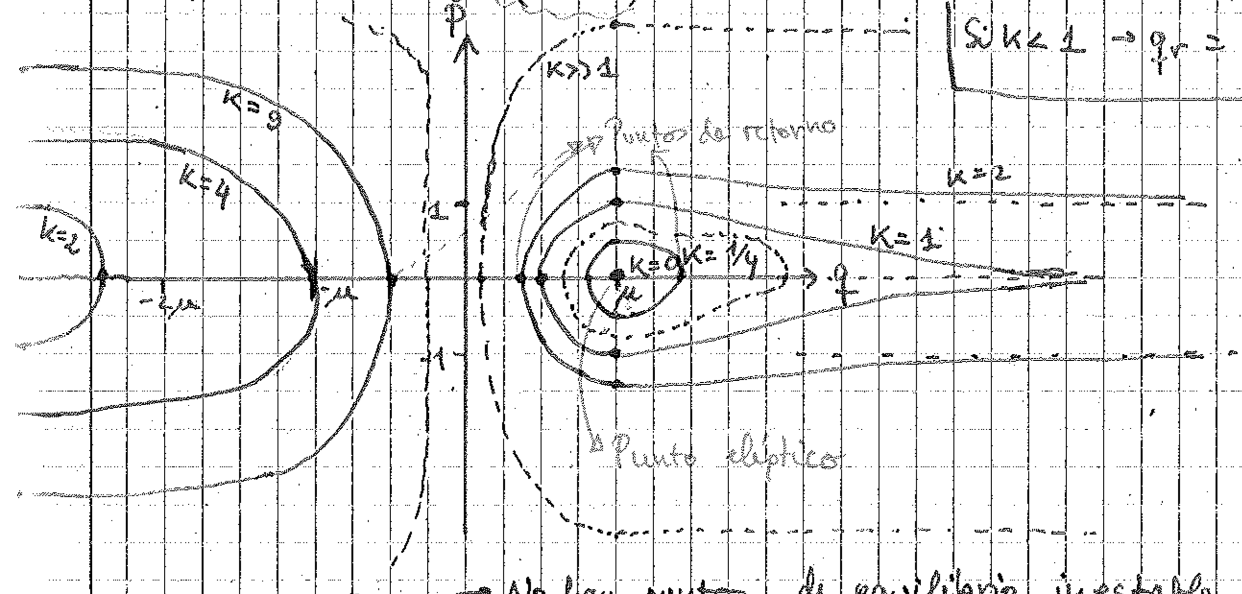
$\tilde{p}(q \to \pm\infty) = \pm\sqrt{K-1}$, para $K>1$, si $<1 \to \exists\, q_r$ (max)

si $K \gg 1 \to \tilde{p}_{max} \simeq \tilde{p}(q \to \pm\infty)$

Si $q>0 \to q_{min} = \mu/(1+\sqrt{K})$

Si $q<0 \to q_{max} = \mu/(1-\sqrt{K})$ para $K>1$

Si $K<1 \to q_r = \frac{\mu}{1 \mp \sqrt{K}}$



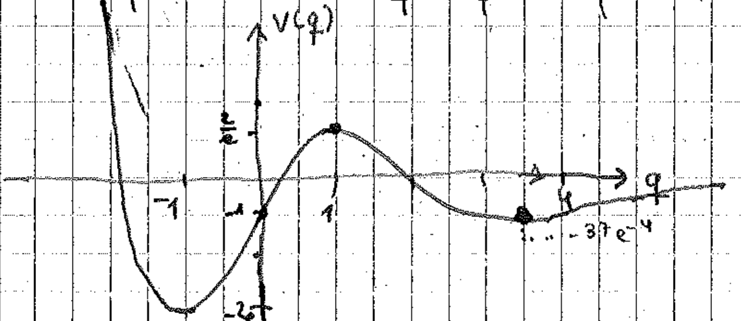
→ No hay puntos de equilibrio inestable (estaría en ±∞)

↳ ∄ separatriz / puntos hiperbólicos

$K < 1 \to$ Oscilaciones

b)

$V(q) = e^{-q}(-q^3 + q^2 + 3q - 1)$



$V(0) = -1$

$V(1) = \frac{1}{e}(-1+1+3-1) = \frac{2}{e}$

$V(+\infty) = 0$

$V(-\infty) = +\infty$

$V(-1) = e(1+1-3-1) = -2e$

$\frac{\partial V}{\partial q} = e^{-q}(+q^3 - q^2 - 3q + 1 - 3q^2 + 2q + 3) = 0$

Mínimo ↳ $e^{-q}(q^3 - 4q^2 - q + 4) = 0 = e^{-q}(q^2(q-4) - (q-4))$

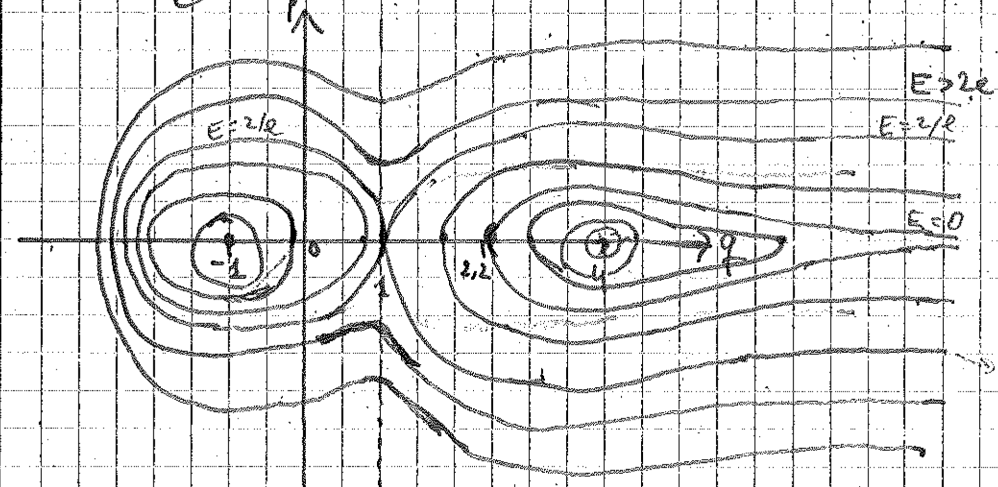
$= e^{-q}(q-4)(q+1)(q-1) = 0 \to$ Extremos en $\pm 1, 4$

$V(1) = 2/e$; $V(-1) = -2e$; $V(4) = -37 \cdot e^{-4}$ → Por tanto -1 y 4 son mínimos, 1 es máximo

$p' = \pm\sqrt{E - V(q)}$

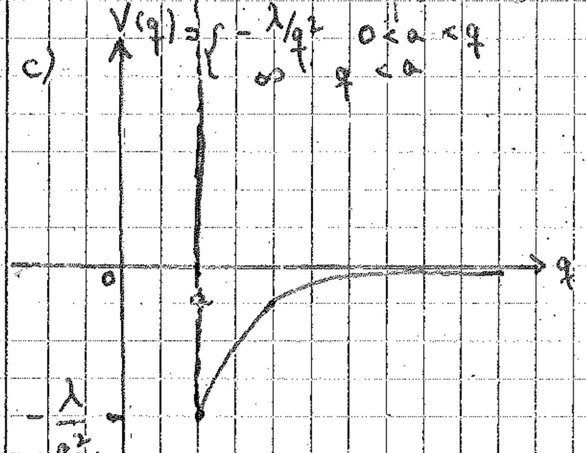
si $E < -2e \to$ Prohibido

Si $E < -37e^{-4}$ → soluciones sólo en pozo de la izqda (acotado)
Si $-37e^{-4} < E < \frac{2}{e}$ → " en ambos pozos (") } oscilaciones
Si $E > \frac{2}{e}$ → Movimiento semiacotado (punto de retorno en $q<0$)

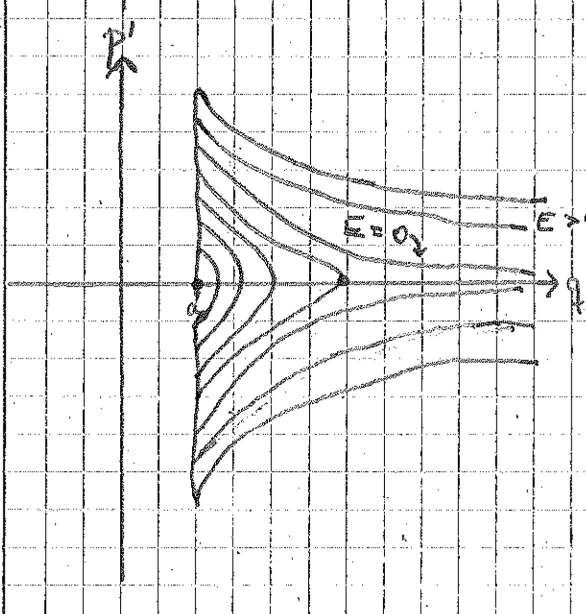


$q = 4$ y -1 → Puntos elípticos } $p = 0$
$q = 1$ → Punto hiperbólico

c) $V(q) = \begin{cases} -\lambda/q^2 & 0<a<q \\ \infty & q<a \end{cases}$



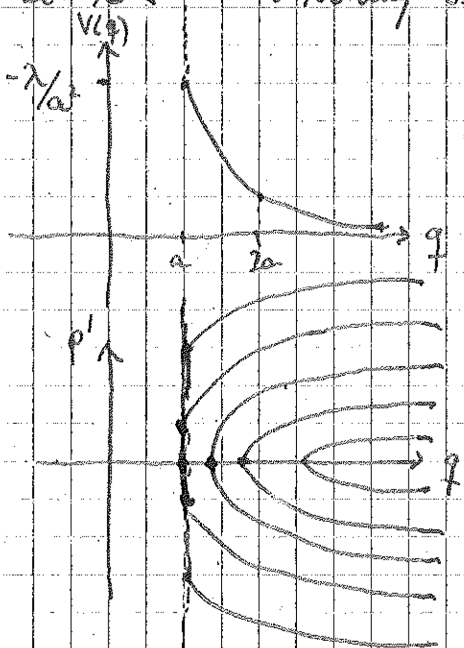
Si $\lambda > 0$
Soluciones sólo para $q > a$ y $E > -\frac{\lambda}{a^2}$
Si $-\frac{\lambda}{a^2} < E < 0$ → movimiento acotado entre a y $q_{ret} > a$.
Si $E > 0$ → movimiento semiacotado entre $q = a$ y $q \to +\infty$
→ Discontinuidad en $q = a$



Punto elíptico en $q = a$, $p = 0$
Si $-\frac{\lambda}{a^2} < E < 0$
↳ $q_{ret_1} = a$
$q_{ret_2} = \sqrt{-\frac{\lambda}{E}}$ } Oscilaciones

$E > 0$ siempre

Si $\lambda < 0$ ↳ No hay oscilaciones (semiacotado)



$q_{ret} = \sqrt{-\frac{\lambda}{E}}$ si $E < -\frac{\lambda}{a^2}$

$q_{ret} = a$ si $E > \frac{\lambda}{a^2}$

↳ Discontinuidad en p'

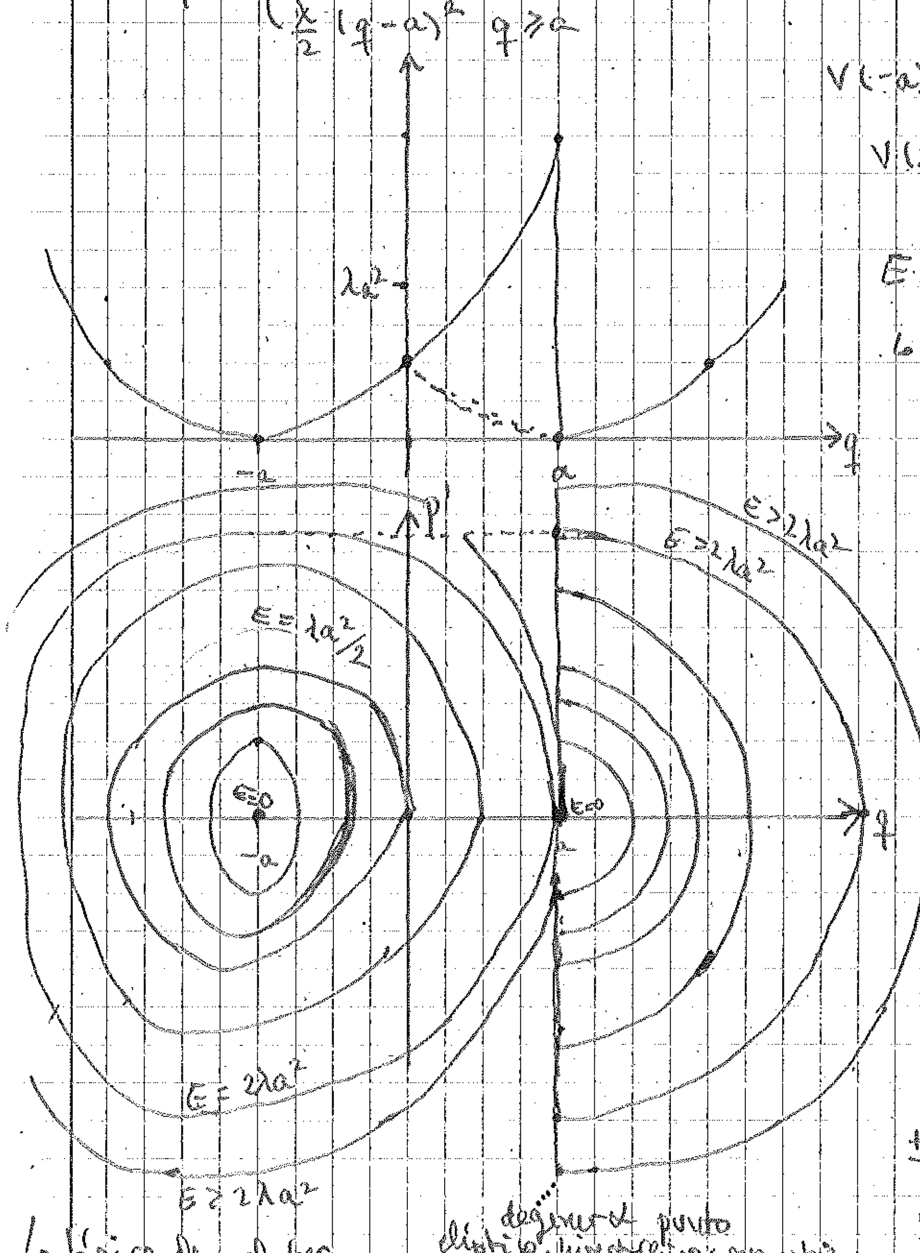
∘ No hay puntos elíp/hiperb./separatrices

d) $V(q) = \begin{cases} \frac{\lambda}{2}(q+a)^2 & q<a \\ \frac{\lambda}{2}(q-a)^2 & q \geq a \end{cases}$

$\lambda > 0$ / V acotado inferiormente

$V(-a) = 0$; $V(a) = \begin{cases} 2\lambda a^2 \\ 0 \end{cases}$ Discontinuidad

$V(2a) = \frac{\lambda a^2}{2} = V(0) = V(-2a)$



$E > 0$ necesariamente

↳ Movimiento de oscilador armónico en pozo izqda.

↳ Círculos/Elipses espacio fase si $\lambda = 2$

↳ En pozo dcho → discontinuid ↳ semicírculos

Oscilaciones ∀ q

$\underbrace{q = -a,\ q = a}_{E=0}$ → Puntos elípticos

$E > 2\lambda a^2$ → cambia de pozo

Puntos retorno:

$E = \frac{\lambda}{2}(q_{ret} \pm a)^2$

$\pm\sqrt{\frac{2E}{\lambda}} = q_{ret} \pm a$

$q_{ret} = \pm\left(\sqrt{\frac{2E}{\lambda}} - a\right)$

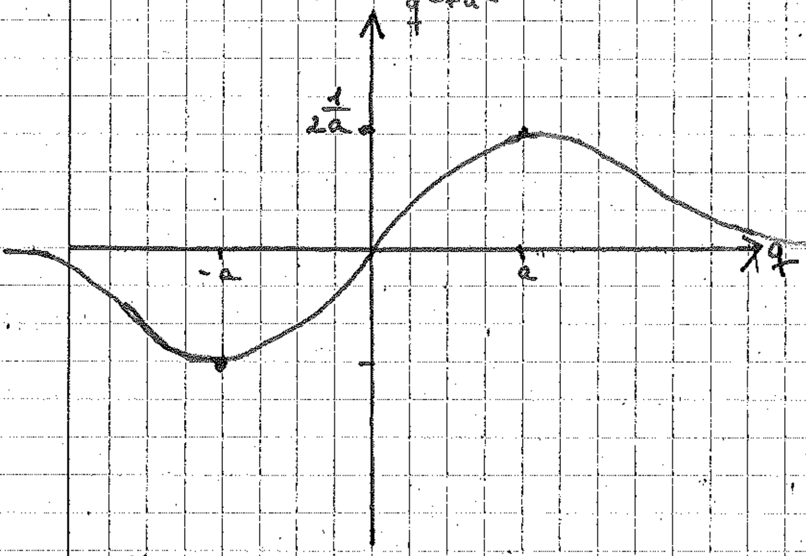
además de $q = a$.

↳ típico de rebotes

degenerado punto elíptico, hiperbólico, separatriz
↳ discontinuidad

Si $a=0$ → potencial armónico 

e) $V(q) = \dfrac{q}{q^2+a^2}$



$V(0)=0$

$V(q) = -V(-q)$ (impar)

$V(a) = \dfrac{1}{2a}$

$V'(q) = \dfrac{1}{q^2+a^2} - \dfrac{2q^2}{(q^2+a^2)^2}$

$= \dfrac{q^2+a^2-2q^2}{(q^2+a^2)^2} = 0$

↳ $q_{\substack{max \\ min}} = \pm a$

$V(+\infty) = 0$

$V''(q)=0$ → en $q \simeq 1{,}7a$

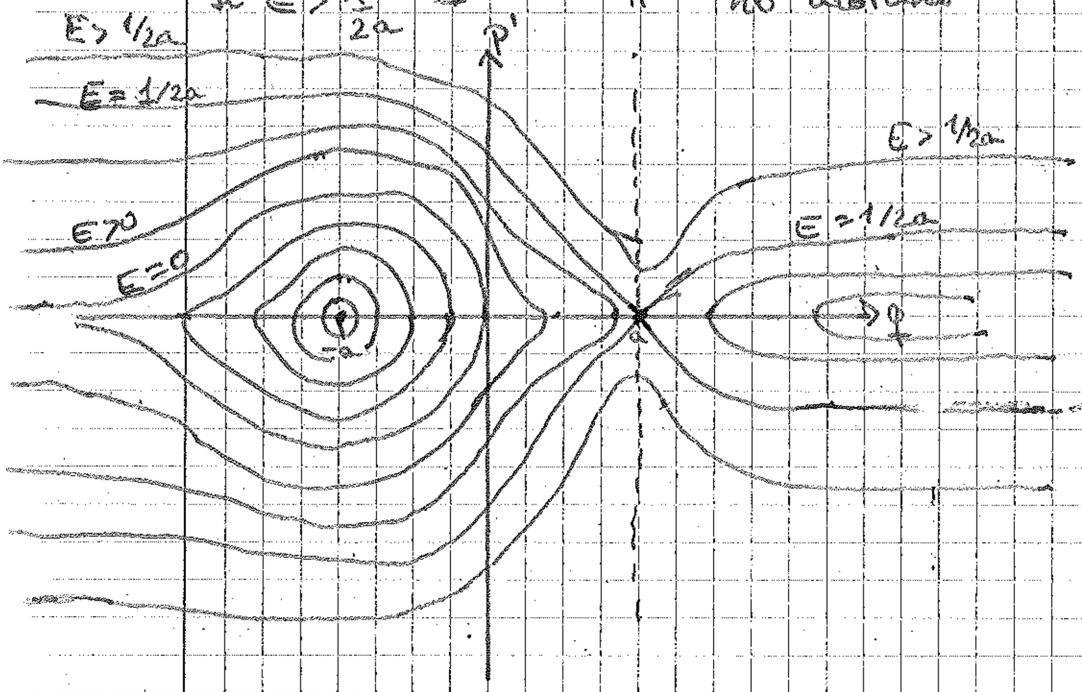
$V(a)$ → punto eq. inestable

Si $-1/2a < E < 0$ → soluciones sólo en $q<0$ → oscilaciones

Si $E < -\dfrac{1}{2a}$ → Prohibido

Si $0 < E < \dfrac{1}{2a}$ → Movimiento semiacotado (1 punto retorno)

Si $E > \dfrac{1}{2a}$ → " no acotado



oscilas si $q<0$ / $E<0$

Punto elíptico en $q=-a$, $p'=0$

Punto hiperbólico en $q=a$, $p'=0$

$E\leftarrow 0$ en ∞

Separatriz

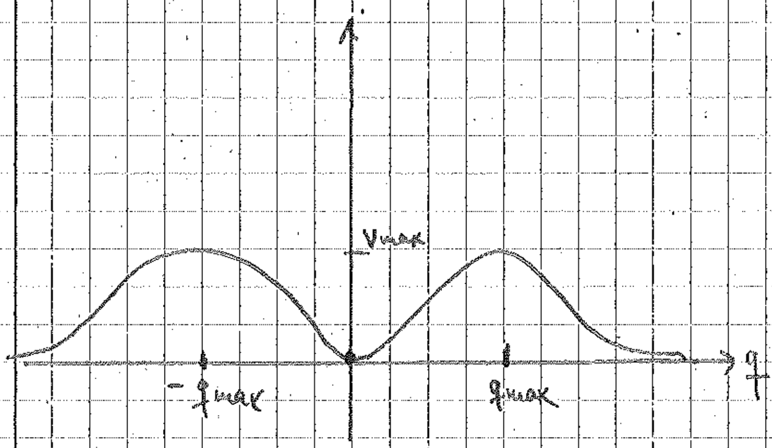
$I = \frac{1}{2\pi}\oint_{\text{curva}} p\,dq$

Separatriz → $E = V_{max}$

f) $V(q) = \dfrac{q^2 e^{-\lambda \left(\frac{q}{a}\right)^2}}{q^2 + a^2}$

$V(0) = 0$
$V(a) = \frac{1}{2} e^{-\lambda}$
$V(-q) = V(q)$
$V(\pm\infty) = 0$ si $\lambda > 0$

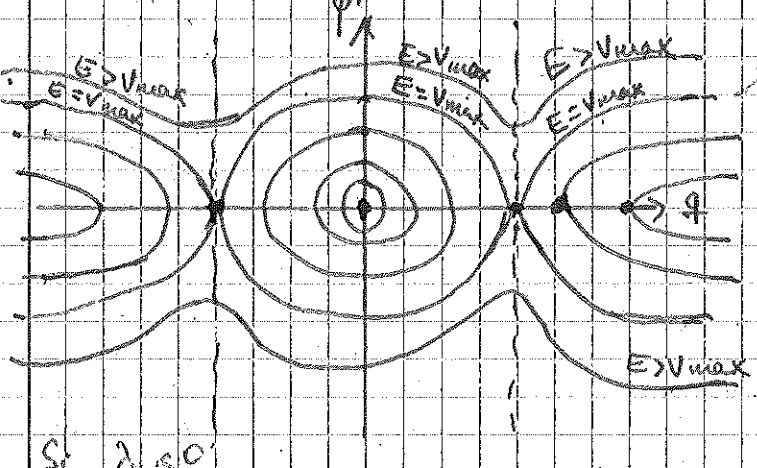
→ Represento por ordenador
→ 3 extremos



Si $E < 0$ → Prohibido

Si $V_{max} > E > 0$ $\begin{cases} \text{si } q \in [-q_{max}, q_{max}] \to \text{Oscilaciones} \\ \text{si } q \notin \to \text{semiacotado} \end{cases}$

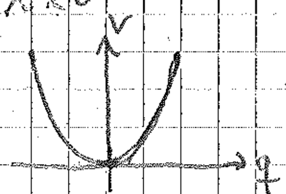
Si $E > V_{max}$ → no acotado



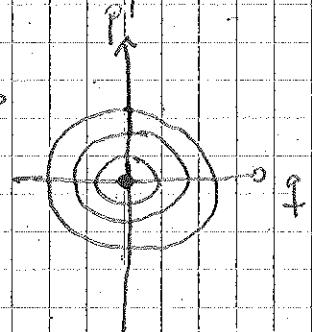
2 puntos hiperbólicos ($\pm q_{max}$) → 1 separatriz

1 punto elíptico en $q = 0$

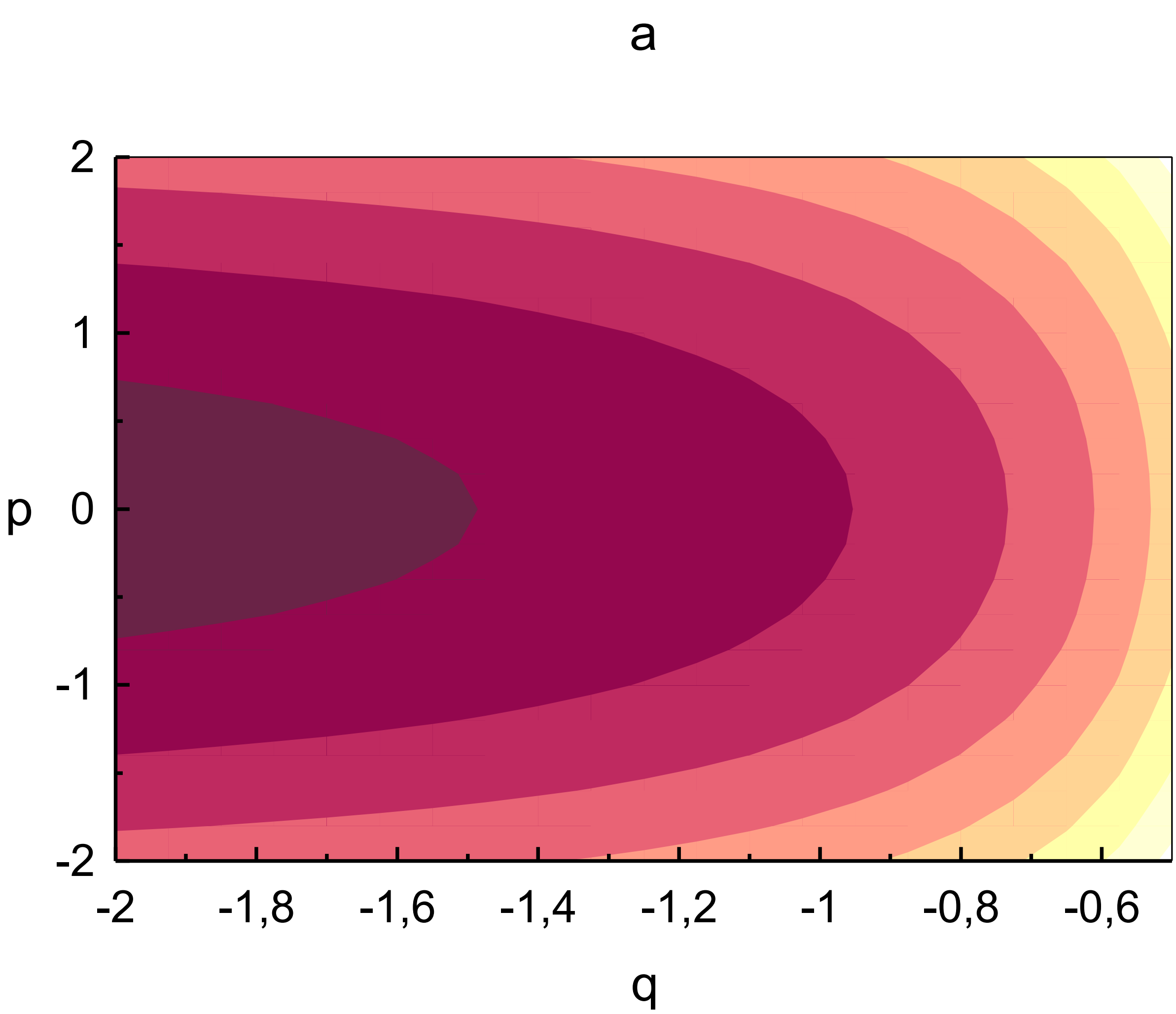
Si $\lambda < 0$



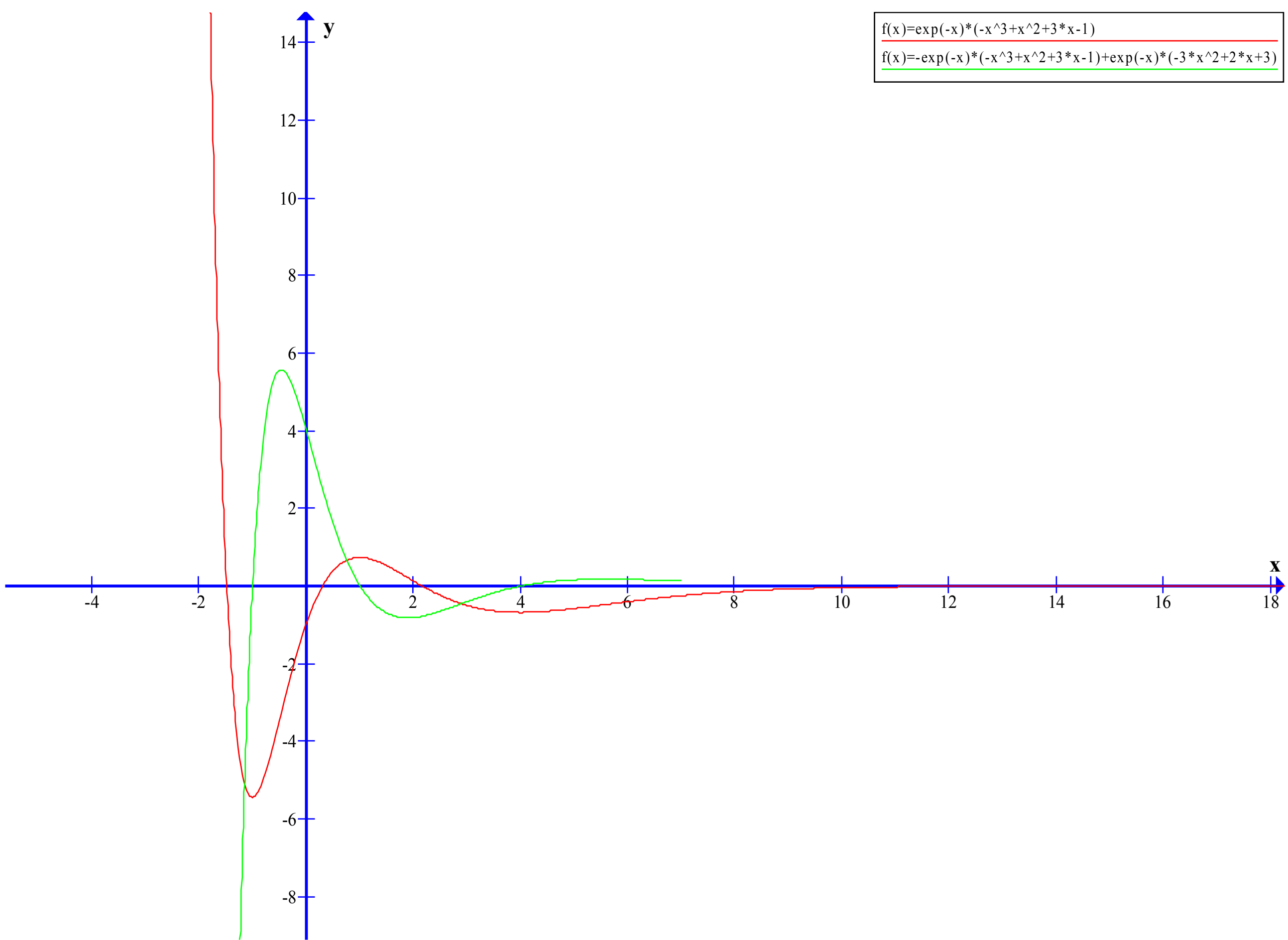
⇒



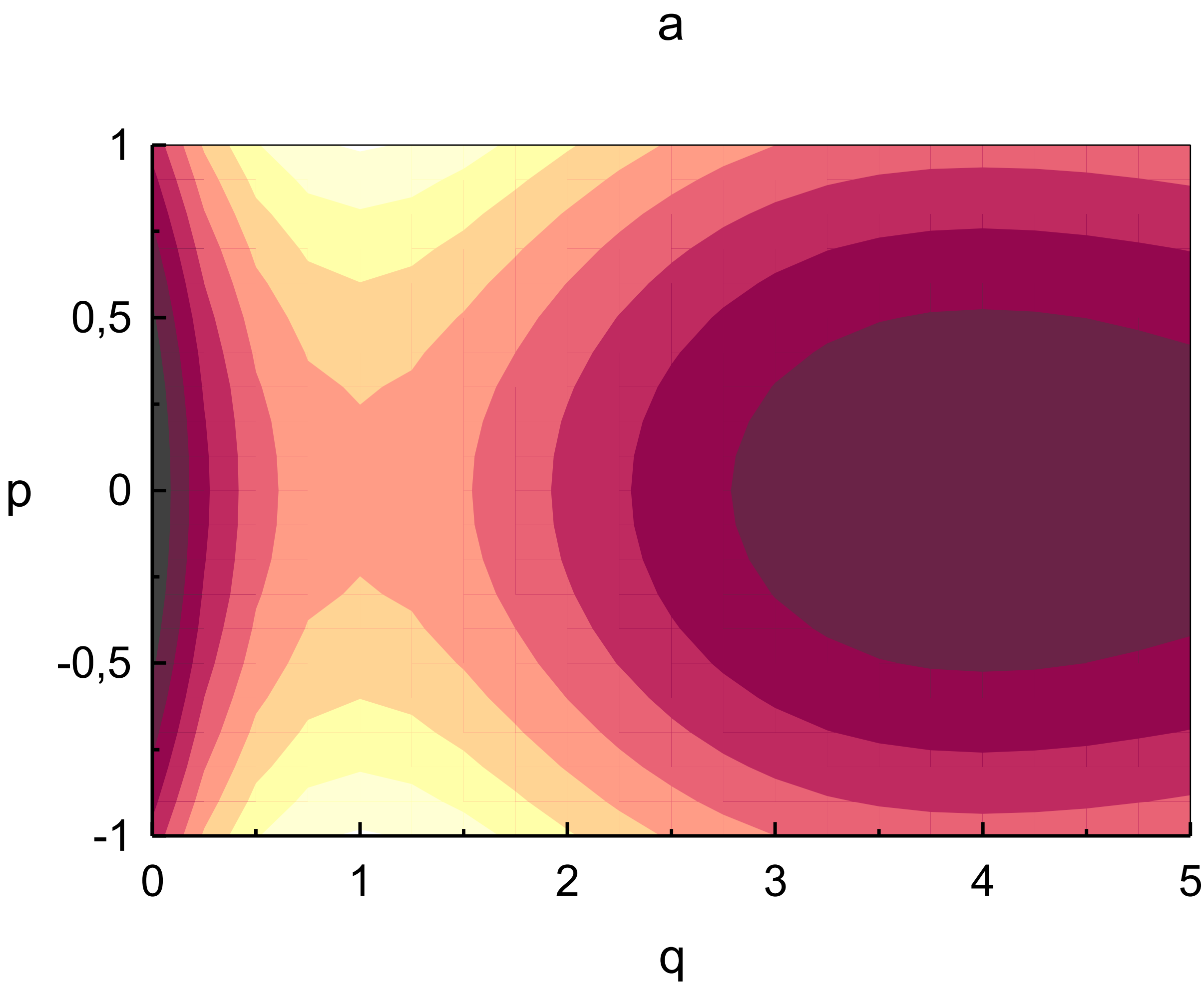
→ 1 punto elíptico



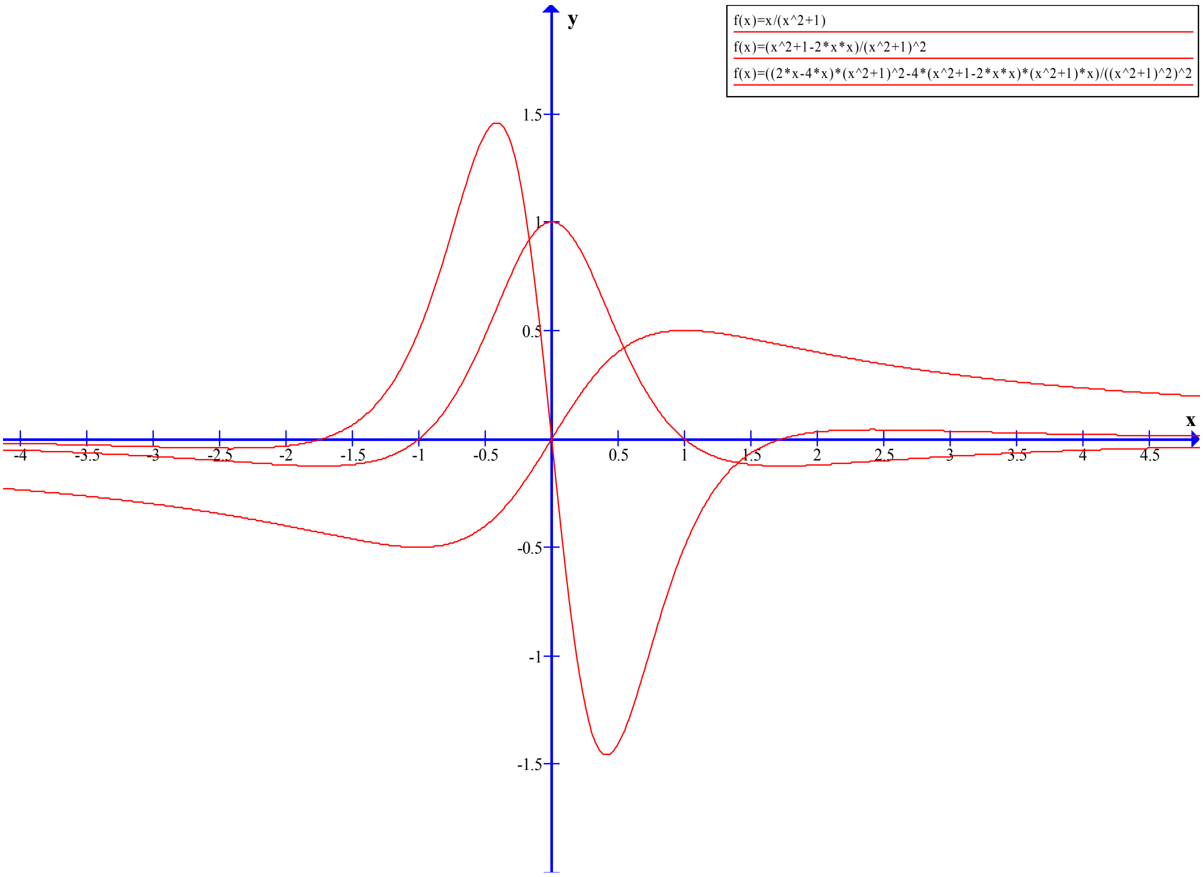
45a), b)
(Gráficas con Kyplot)



45b)

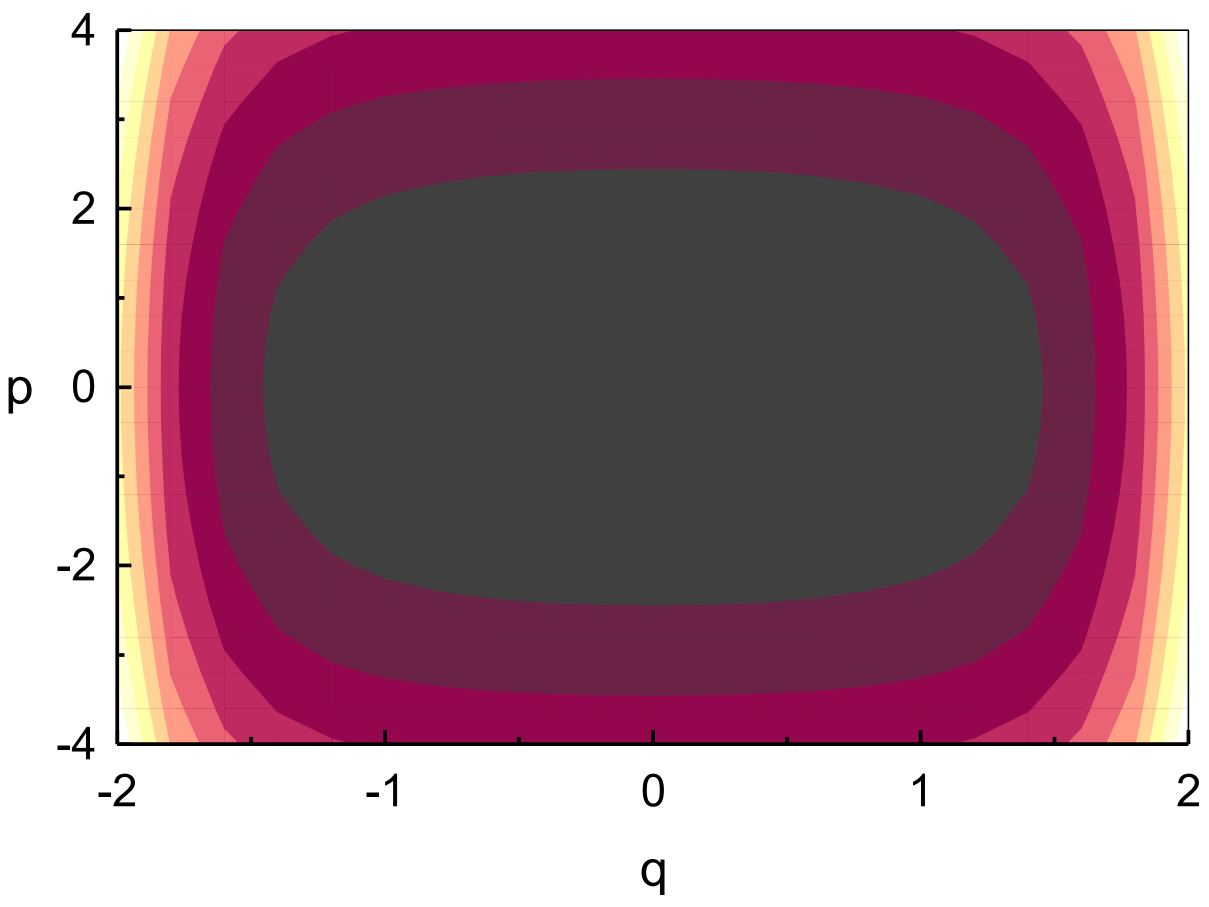


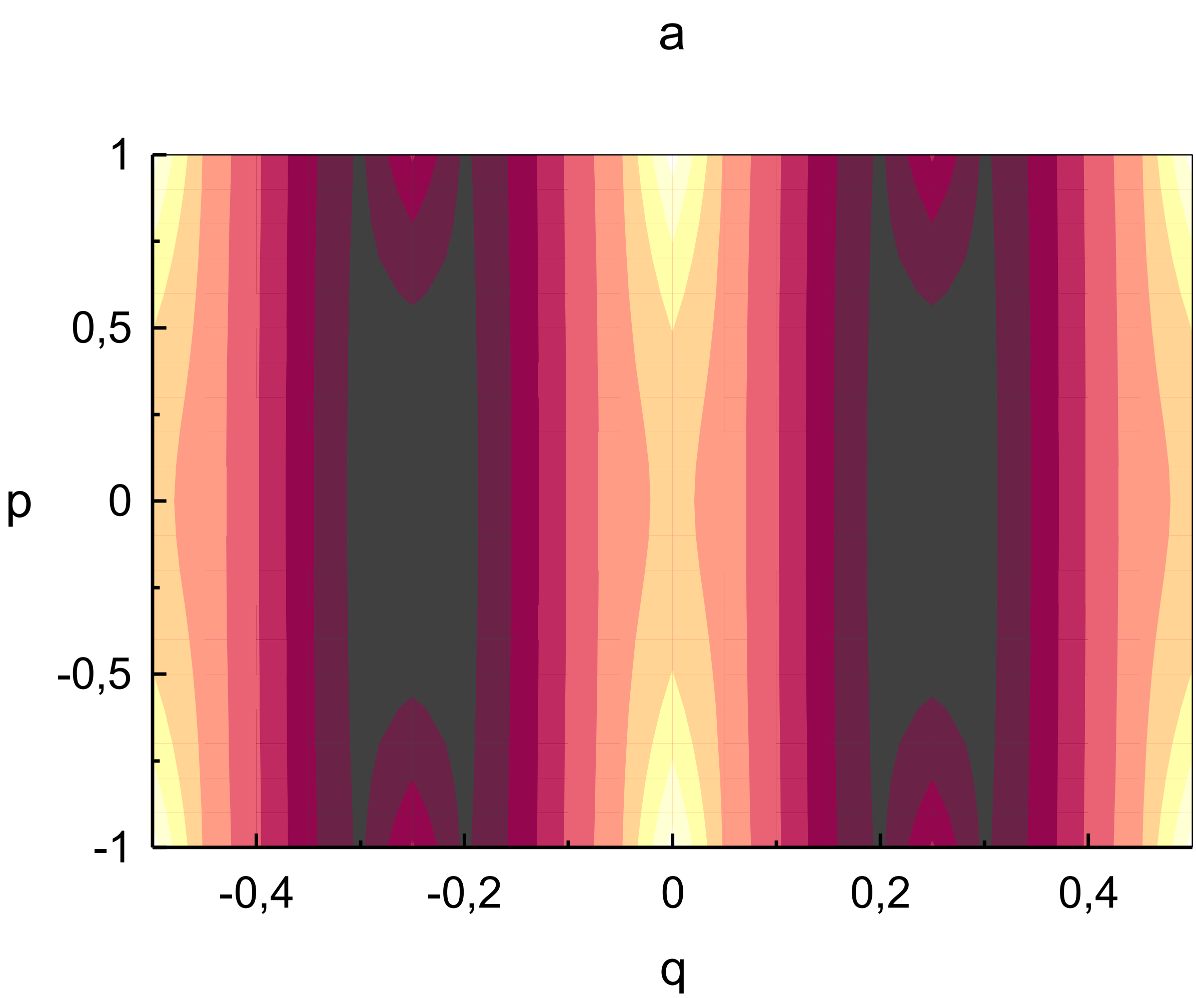
45c)



45e),f)

a





46c)

46c) y 47 --> archivos de Mathematica en mural.uv.es/ferhue

(41)

$$H = \frac{p^2 - q^{-2}}{2}$$

$$D = \frac{pq}{2} - Ht = \frac{pq}{2} - \frac{p^2 - q^{-2}}{2}t$$

ver teoría

$$\frac{dD}{dt} = \{D, H\} + \frac{\partial D}{\partial t} = \{D, H\} - H$$

$$\frac{\partial D}{\partial q} = \frac{p}{2} - q^{-3} \cdot t \quad ; \quad \frac{\partial D}{\partial p} = \frac{q}{2} - p \cdot t$$

$$\frac{\partial H}{\partial q} = q^{-3} \quad ; \quad \frac{\partial H}{\partial p} = p$$

$$\{D, H\} = \frac{p^2}{2} - pq^{-3}t - \frac{q^{-2}}{2} + pq^{-3}t = \frac{p^2 - q^{-2}}{2} = H$$

$$\frac{\partial D}{\partial t} = -H \Rightarrow \{D, H\} - H = H - H = 0 = \frac{dD}{dt} \rightarrow \text{cte. de movim}$$

O bien:

$$\frac{dD}{dt} = \{D, H\} + \frac{\partial D}{\partial t} = \left\{\frac{pq}{2}, H\right\} - \{H, H\} \cdot t + \frac{\partial}{\partial t}\left(\frac{pq}{2}\right) - \frac{\partial}{\partial t}(Ht)$$

$$= \left\{\frac{pq}{2}, \frac{p^2 - q^{-2}}{2}\right\} - H = \begin{vmatrix} p/2 & q/2 \\ q^{-3} & p \end{vmatrix} - H = H - H = 0$$

(42)

$$H = q_1 p_1 - q_2 p_2 - \lambda q_1^2 + \mu q_2^2$$

$$K_1 = \frac{p_1 - \lambda q_1}{q_2} \quad ; \quad K_2 = q_1 q_2$$

$$\frac{\partial K_i}{\partial t} = 0$$

$$\{K_1, H\} = \left\{\frac{p_1}{q_2} - \frac{\lambda q_1}{q_2}, q_1 p_1 - q_2 p_2 - \lambda q_1^2 + \mu q_2^2\right\} \overset{\{f,g\}}{\downarrow}= \frac{\partial f_i}{\partial q_i} \cdot \frac{\partial g_i}{\partial p_i} - \frac{\partial f_i}{\partial p_i}\frac{\partial g_i}{\partial q_i}$$

$$= -\frac{\lambda}{q_2} \cdot q_1 - \frac{1}{q_2} \cdot (p_1 - 2\lambda q_1) + \frac{p_1 - \lambda q_1}{q_2^2} \cdot (-q_2) - 0 \cdot (-p_2 + 2\mu q_2)$$

$$= \frac{1}{q_2}(-\lambda q_1 - p_1 + 2\lambda q_1 + p_1 - \lambda q_1) = 0 \rightarrow K_1 \text{ es cte de movimiento}$$

$\{K_2, H\} = \{q_1 q_2,\ q_1 p_1 - q_2 p_2 - \lambda q_1^2 + \mu q_2^2\}$

$= q_2 \cdot q_1 - 0\cdot(p_1 - 2\lambda q_1) + q_1\cdot(-q_2) - 0\cdot(-q_2)$

$= q_2 q_1 - q_1 q_2 = 0$ $\rightarrow$ K_2 es cte de movimiento.

Por teorema de Poisson:

$\{H, \{K_1, K_2\}\} + \{K_1, \overset{0}{\{K_2, H\}}\} + \{K_2, \overset{0}{\{H, K_1\}}\} = 0$

$\hookrightarrow \{K_1, K_2\} =$ cte ó $f(H)$

$\{\frac{p_1 - \lambda q_1}{q_2}, q_1 q_2\} = -\frac{\lambda}{q_2}\cdot 0 - \frac{1}{q_2}\cdot q_2 + \frac{\lambda q_1}{q_2^2}\cdot 0 - 0\cdot 0 = -1$

$\hookrightarrow$ No es una cte. relevante pues no involucra a las variables p_i, q_i.

Analicemos la simetría de H. Salvo signos y constantes, hay simetría bajo el intercambio $1 \Leftrightarrow 2$

Por tanto, cabe esperar que si en K_1, K_2 intercambiamos $1 \Leftrightarrow 2$ haya también ctes de movimiento.
$\hookrightarrow$ y cambiamos λ por μ

$K_3 = \frac{p_2 - \mu q_2}{q_1}$; $K_4 = q_2 q_1 = K_2$

$\{K_3, H\} = \frac{p_2 - \mu q_2}{q_1^2}\cdot q_1 - 0\cdot(p_1 - 2\lambda q_1) + \left(-\frac{\mu}{q_1}\right)\cdot(-q_2) - \frac{1}{q_1}\cdot(-p_2 + 2\mu q_2)$

$= \cancel{-\frac{p_2}{q_1}} + \mu\frac{q_2}{q_1} + \mu\frac{q_2}{q_1} - 2\mu q_2 + \frac{p_2}{q_1} = 0$ ✓ cte de movi∅

$\{K_1, K_2\} = -1$ $\rightarrow \{\{K_1, K_2\}, H\} = 0 \rightarrow$ cte. mov. (sin utilidad)

$\{K_1, K_3\} = \left\{\frac{p_1 - \lambda q_1}{q_2}, \frac{p_2 - \mu q_2}{q_1}\right\} = -\frac{\lambda}{q_2}\cdot 0 + \frac{1}{q_2}\cdot\frac{(p_2 - \mu q_2)}{q_1^2}$

$+ \frac{p_1 - \lambda q_1}{(-)q_2^2}\cdot\frac{1}{q_1} - 0\cdot\left(-\frac{\mu}{q_1}\right) = \frac{1}{(q_1 q_2)^2}\left(q_2 p_2 - \mu q_2^2 + p_1 q_1 + \lambda q_1^2\right).$

$= -\frac{H}{K_2^2} = f(H)$ ∴ $K_2 \rightarrow$ cte de movi∅ sí útil, se puede reescribir $H = K_2(K_1 - K_3)$

$\Rightarrow \{\{K_1, K_3\}, H\} = \{f(H), H\} = 0$ $\rightarrow$ cte. de movi∅

$\{K_2, K_3\} = \{q_1 q_2, \frac{p_2 - \mu q_2}{q_1}\} = q_2\cdot 0 - 0\cdot\frac{p_2 - \mu q_2}{q_1^2} + q_1\cdot\frac{1}{q_1} - 0\cdot\left(-\frac{\mu}{q_1}\right) = 1$

$\{\{K_2, K_3\}, H\} = 0 \rightarrow \{K_2, K_3\}$ cte de movimiento (sin utilidad).

(143) $f(r^2, p^2, r\cdot p)$ $\quad \vec{r} = (x_1, x_2, x_3)$, $r_i = x_i$

$\vec{F} = f_1\vec{r} + f_2\vec{p} + f_3\, \vec{r}\times\vec{p}$

(convenio de índices de Einstein)

$(\vec{L})_i = (\vec{r}\times\vec{p})_i = \epsilon_{ijk}\, r_j p_k$

a) $\{f, (\vec{L})_i\} = \{f, \epsilon_{ijk} r_j p_k\} = \epsilon_{ijk}\left(\frac{\partial f}{\partial x_\ell}\cdot\frac{\partial (r_j p_k)}{\partial (p_\ell)} - \frac{\partial f}{\partial p_\ell}\cdot\frac{\partial (r_j p_k)}{\partial x_\ell}\right)$ $\quad r_j = x_j$

$= \epsilon_{ijk}\left(\frac{\partial f}{\partial x_\ell}\, r_j\, \delta_{\ell k} - \frac{\partial f}{\partial p_\ell}\cdot p_k\, \delta_{\ell j}\right) = \epsilon_{ijk}\left(\frac{\partial f}{\partial r_k}\cdot r_j - \frac{\partial f}{\partial p_j}\, p_k\right)$

$\frac{\partial f}{\partial r_k} = \frac{\partial f}{\partial r^2}\cdot\frac{\partial r^2}{\partial r_k} + \frac{\partial f}{\partial (rp)}\cdot\frac{\partial (rp)}{\partial r_k} + 0 = \frac{\partial f}{\partial r^2}\cdot 2r_k + \frac{\partial f}{\partial (rp)}\cdot p_k$ $\quad r^2 = (r_i)^2$, $rp = r_i\cdot p_i$

$\frac{\partial f}{\partial p_j} = \frac{\partial f}{\partial p^2}\cdot\frac{\partial p^2}{\partial p_j} + \frac{\partial f}{\partial (rp)}\cdot\frac{\partial (rp)}{\partial p_j} + 0 = \frac{\partial f}{\partial p^2}\cdot 2p_j + \frac{\partial f}{\partial (rp)}\cdot r_j$

$\{f, L_i\} = \epsilon_{ijk}\left(\left(\frac{\partial f}{\partial r^2}\cdot 2r_k r_j + \frac{\partial f}{\partial (rp)}\cdot p_k r_j\right) - \left(\frac{\partial f}{\partial p^2}\cdot 2p_j p_k + \frac{\partial f}{\partial (rp)}\cdot r_j p_k\right)\right)$

↳ ~~Intercambio j con k en 2º término y cambio el signo~~:
$(\epsilon_{ijk} A_{jk} = \epsilon_{ikj} A_{kj} = -\epsilon_{ijk} A_{kj})$

$\{f, L_i\} = \epsilon_{ijk}\left(\frac{\partial f}{\partial r^2}\cdot 2 r_k r_j + \frac{\partial f}{\partial p^2}\cdot 2p_j p_k\right)$

Simétr. x Antis. $= 0$ $\quad$ conmutan

$\epsilon_{ijk}\, r_k r_j = \epsilon_{ijk}\frac{r_k r_j}{2} + \epsilon_{ijk}\frac{r_k r_j}{2} = \frac{\epsilon_{ijk}}{2}(r_k r_j - r_j r_k) = 0$

Idem para p.

Por tanto $\rightarrow \{f, L_i\} = 0 \rightarrow \{f, \vec{L}\} = 0$

b) $\{F_i, L_j\} = \epsilon_{ijk} F_k$

$F_i = f_1 r_i + f_2 p_i + f_3\, \epsilon_{ijk} r_j p_k$

$= \{f_1 r_i, L_j\} + \{f_2 p_i, L_j\} + \{f_3 \epsilon_{i\ell k} r_\ell p_k, L_j\}$

$\{AB, C\} = \frac{\partial (AB)}{\partial r_i}\cdot\frac{\partial C}{\partial p_i} - \frac{\partial (AB)}{\partial p_i}\cdot\frac{\partial C}{\partial r_i} = A\frac{\partial B}{\partial r_i}\frac{\partial C}{\partial p_i} - A\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial r_i}$ (regla cadena)

$+ B\frac{\partial A}{\partial r_i}\frac{\partial C}{\partial p_i} - B\frac{\partial A}{\partial p_i}\frac{\partial C}{\partial r_i} = A\cdot\{B, C\} + B\{A, C\}$

En este caso, $A = f_n$, $C = L_j$ $\rightarrow \{A, C\} = \{f_n, L_j\} = 0$ (por A)

$n = 1, 2, 3$ (número, no es coordenada)
↳ todas de tipo f.

Por tanto se simplifica a:

$$\{F_i, L_j\} = f_1\{r_i, L_j\} + f_2\{p_i, L_j\} + f_3\, \epsilon_{i\ell k}\overbrace{\{r_\ell p_k, L_j\}}^{\text{"}\{L_i,L_j\}}$$

Corchetes de Poisson fundamentales

$$\{r_i, L_j\} = \epsilon_{jmn}\{r_i, r_m p_n\} = \epsilon_{jmn}\cdot\left[r_m\{r_i, p_n\} + p_n\underbrace{\{r_i, r_m\}}_{0}\right]$$

$$= \epsilon_{jmn} r_m\left[\frac{\partial r_i}{\partial r_s}\frac{\partial p_n}{\partial p_s} - \frac{\partial r_i}{\partial p_s}\frac{\partial p_n}{\partial r_s}\right] = \epsilon_{jmn} r_m(\delta_{is}\cdot\delta_{ns} - 0) \quad \delta_{is}\delta_{ns} = \delta_{in}$$

$$= \epsilon_{jmi} r_m = \epsilon_{ijm} r_m$$

$$\{p_i, L_j\} = \epsilon_{jmn}\{p_i, r_m p_n\} = \epsilon_{jmn}\left[r_m\underbrace{\{p_i, p_n\}}_{0} + p_n\{p_i, r_m\}\right]$$

$$= \epsilon_{jmn} p_n\left[0 - \delta_{im}\right] = -\epsilon_{jin} p_n = +\epsilon_{ijn} p_n$$

$$\{r_\ell p_k, L_j\} = r_\ell\{p_k, L_j\} + p_k\{r_\ell, L_j\} = r_\ell\cdot\epsilon_{kjn}p_n + p_k\,\epsilon_{\ell jn} r_n$$

$$= \epsilon_{kjn}\cdot r_\ell p_n + \epsilon_{\ell jn}\, p_k r_n$$

$$\{F_i, L_j\} = f_1\cdot\epsilon_{ijn} r_n + f_2\,\epsilon_{ijn} p_n + f_3\,\epsilon_{i\ell k}(\epsilon_{kjn} r_\ell p_n + \epsilon_{\ell jn} p_k r_n)$$

Cambio $k \to n$, $n \to \ell$, $\ell \to k$

$$= \epsilon_{ijn}(f_1 r_n + f_2 p_n) + f_3(\epsilon_{i\ell k}\epsilon_{kjn} + \epsilon_{kj\ell}\epsilon_{ikn})\, r_\ell p_n$$

Utilizamos la propiedad $\epsilon_{imj}\,\epsilon_{km\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} \Rightarrow$

$$\hookrightarrow \epsilon_{i\ell k}\epsilon_{kjn} + \epsilon_{kj\ell}\epsilon_{ikn} = +\epsilon_{ik\ell}\,\epsilon_{jkn} - \epsilon_{jk\ell}\,\epsilon_{ikn}$$

$$= \delta_{ij}\delta_{\ell n} - \delta_{in}\delta_{\ell j} - \delta_{ji}\delta_{\ell n} + \delta_{jn}\delta_{\ell i} = \delta_{jn}\delta_{\ell i} - \delta_{in}\delta_{\ell j} = \epsilon_{jki}\,\epsilon_{\ell kn}$$

$$\{L_i, L_j\} = \epsilon_{ikj}\,\epsilon_{\ell kn}\, r_\ell p_n = -\epsilon_{ikj} L_k = +\epsilon_{ijk} L_k$$

k por n (mudos)

$$\Rightarrow \{F_i, L_j\} = \epsilon_{ijk}(f_1 r_k + f_2 p_k + f_3 L_k) = \epsilon_{ijk}\cdot F_k$$

(44)

$$H = \frac{p^2}{2m} - \lambda m t q = H(q,p,t)$$

$\hookrightarrow S(q, P, t) \rightarrow H' = H + \frac{\partial S}{\partial t} = H\left(q, p = \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t}$

$\rightarrow H' = \frac{\left(\frac{\partial S}{\partial q}\right)^2}{2m} - \lambda m t q + \frac{\partial S}{\partial t} = 0(t)$ origen E

ecuación de Schrödinger

$S(q,t) = f(t)\, q + g(t)$

$\frac{\partial S}{\partial q} = f'(t) \cdot t$; $\frac{\partial S}{\partial t} = q \cdot f'(t) + g'(t)$

$\hookrightarrow \frac{f'(t)^2 \cdot t^2}{2m} - \lambda m t q + q f'(t) + g'(t) = 0(t)$

$\hookrightarrow f'(t) = \lambda m t \rightarrow f(t) = \frac{\lambda m t^2}{2} + f_0$

$\hookrightarrow -\frac{f^2(t)}{2m} = g'(t) = -\frac{\lambda^2 m^2}{4 \cdot 2m} t^4 - \frac{\lambda m t^2 f_0}{2m} - \frac{f_0^2}{2m}$

$\hookrightarrow g(t) = -\frac{\lambda^2 m t^5}{40} - \frac{\lambda t^3 f_0}{6} - \frac{f_0^2}{2m} t + g_0$

$S(q,t) = \left(\frac{\lambda m t^2}{2} + f_0\right) q - \frac{\lambda^2 m t^5}{40} - \frac{\lambda f_0 t^3}{6} - \frac{f_0^2 t}{2m} + g_0$

$S = S(q, t, f_0, g_0)$, con $f_0, g_0 \rightarrow$ ctes de integración $= \alpha, \alpha'$
$= S(q, t, f_0) + g_0 \rightarrow$ Identifico $E = f_0$ (α' aditiva)

$Q = \frac{\partial S}{\partial f_0} = B = q - \frac{\lambda t^3}{6} - \frac{f_0 t}{m} \leftrightarrow q = B + \frac{\lambda t^3}{6} + \frac{f_0 t}{m}$

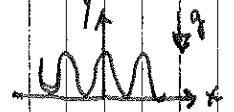
$p = \frac{\partial S}{\partial q} = A = f(t) = \frac{\lambda m t^2}{2} + f_0 = \dot{q} \cdot m \rightarrow$ $\left.\begin{array}{l} q(0) = B = 0 \\ p(0) = f_0 = m v_0 = A \end{array}\right\} q = \frac{\lambda t^3}{6} + v_0 t$

porque $H' = 0 \rightarrow \dot{Q} = \dot{P} = 0$

$Q = 0$, $P = A = m v_0$

$\frac{\partial S}{\partial ...}$

(45) → en página -31-



(46)

$$y(x) = \Lambda_0 \cos^2 \frac{2\pi x}{\lambda_0} \;\to\; \dot y = 2\Lambda_0 \cos \frac{2\pi x}{\lambda_0}(-)\sin\frac{2\pi x}{\lambda_0}\cdot\frac{2\pi}{\lambda_0}\dot x = -\frac{\Lambda_0 2\pi}{\lambda_0}\dot x \sin\frac{4\pi x}{\lambda_0}$$

a) $$\mathcal{L} = \frac{1}{2} m(\dot x^2+\dot y^2) + mgy$$

$$= \frac{1}{2} m\dot x^2\left(1+\frac{\Lambda_0^2 4\pi^2}{\lambda_0^2}\sin^2\frac{4\pi x}{\lambda_0}\right) - mg\Lambda_0\cos^2\left(\frac{2\pi x}{\lambda_0}\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot x} = m\dot x\left(1+4\pi^2\frac{\Lambda_0^2}{\lambda_0^2}\sin^2\frac{4\pi x}{\lambda_0}\right)$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot x}\right) = m\ddot x\left(1+4\pi^2\frac{\Lambda_0^2}{\lambda_0^2}\sin^2\frac{4\pi x}{\lambda_0}\right) + m\dot x\, 4\pi^2\frac{\Lambda_0^2}{\lambda_0^2}\cdot\sin\frac{8\pi x}{\lambda_0}\cdot\frac{4\pi\dot x}{\lambda_0}$$

$$= m\ddot x\left(\quad '' \quad\right) + m\dot x^2\, 16\pi^3\frac{\Lambda_0^2}{\lambda_0^2}\sin\left(\frac{8\pi x}{\lambda_0}\right)$$

$$\frac{\partial\mathcal{L}}{\partial x} = \left(\frac{2\pi^2\Lambda_0^2}{\lambda_0^2}\cdot\frac{4\pi}{\lambda_0}\cdot\sin\frac{8\pi x}{\lambda_0}\right)\cdot m\dot x^2 = m\dot x^2\, 8\pi^3\frac{\Lambda_0^2}{\lambda_0^2}\sin\frac{8\pi x}{\lambda_0} + 2\pi mg\frac{\Lambda_0}{\lambda_0}\sin\frac{4\pi x}{\lambda_0}$$

$$+ mg\Lambda_0\cos\left(\frac{4\pi x}{\lambda_0}\right)\cdot\frac{2\pi}{\lambda_0}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot x}\right) - \frac{\partial\mathcal{L}}{\partial x} = m\ddot x\left(1+4\pi^2\frac{\Lambda_0^2}{\lambda_0^2}\sin^2\frac{4\pi x}{\lambda_0}\right) - 2\pi mg\frac{\Lambda_0}{\lambda_0}\sin\frac{4\pi x}{\lambda_0} = 0$$

$$+ m\dot x^2\, 8\pi^3\frac{\Lambda_0^2}{\lambda_0^2}\sin\frac{8\pi x}{\lambda_0}$$

$\downarrow$ $2\pi\,\Lambda_0/\lambda_0 \equiv b$

$$+ \dot x^2\, 2\pi b^2\sin\frac{8\pi x}{\lambda_0}$$

$$\ddot x\left(1+\left(b\sin\frac{4\pi x}{\lambda_0}\right)^2\right) - gb\sin\frac{4\pi x}{\lambda_0} = 0$$

O bien, si $f_n \equiv 2\pi\frac{\Lambda_0}{\lambda_0}\sin\left(\frac{4\pi x\cdot n}{\lambda_0}\right) \to \mathcal{L} = \frac{1}{2}m\dot x^2(1+f_1^2) - mg\Lambda_0\cos^2\left(\frac{2\pi x}{\lambda_0}\right)$

↳ $\ddot x(1+f_1^2) + \dot x^2\cdot 4\pi^2\frac{\Lambda_0}{\lambda_0}\cdot f_2 - g f_1 = 0$ ec. E-L.

b) $$H = \frac{\partial\mathcal{L}}{\partial\dot x}\cdot\dot x - \mathcal{L} = m\dot x^2(1+f_1^2) - \frac{m\dot x^2}{2}(1+f_1^2) + mg\Lambda_0\cos^2\left(\frac{2\pi x}{\lambda_0}\right)$$

$$= \frac{m\dot x^2}{2}(1+f_1^2) + mg\Lambda_0\cos^2\left(\frac{2\pi x}{\lambda_0}\right) = T+V = E \checkmark = H(x,\dot x)$$

$H(x,p) \to p = m\dot x(1+f_1^2)$

↳ $$H(x,p) = \frac{p^2}{2m(1+f_1^2)} + mg\Lambda_0\cos^2\left(\frac{2\pi x}{\lambda_0}\right)$$

$$\left.\begin{aligned} \dot x &= \frac{\partial H}{\partial p} = \frac{p}{m(1+f_1^2)} \\ \dot p &= -\frac{\partial H}{\partial x} = \frac{p^2\cdot 2f_1\cdot\frac{2\pi^2\Lambda_0}{\lambda_0}\cdot\cos\left(\frac{4\pi x}{\lambda_0}\right)\cdot\frac{4\pi}{\lambda_0}}{2m(1+f_1^2)^2} + mg\Lambda_0\frac{2\pi}{\lambda_0}\sin\left(\frac{4\pi x}{\lambda_0}\right) \end{aligned}\right\}\text{ Ecs. de Hamilton}$$

$$= \frac{p^2 f_2\, 4\pi^2\Lambda_0}{m(1+f_1^2)^2\lambda_0^2} + mg f_1$$

c) $p(x; E)$

$H(x,p) = E \Rightarrow p^2 = \left(E - mg\lambda_0 \cos^2\left(\frac{2\pi x}{\lambda_0}\right)\right)\cdot 2m\,(1+f_1'^2)$

↳ Representar $E(x,p)$ como $z = f(x,y)$ con líneas de contorno con ordenador.

Los puntos de retorno ($p=0$) son:

$$E = mg\lambda_0 \cos^2\left(\frac{2\pi x_{ret}}{\lambda_0}\right) \rightarrow \text{Pozos periódicos (equiespaciados)}$$

↳ Espacio de fases periódico

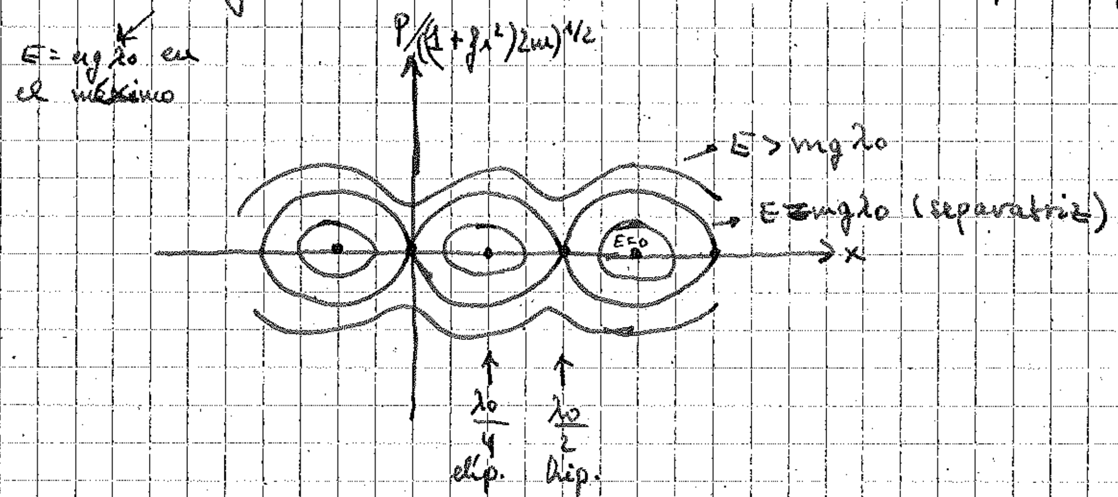
Potencial $V(x) = +mg\lambda_0 \cos^2 \frac{2\pi x}{\lambda_0}$

$V'(x_{eq}) = -mg f_1' = 0 \rightarrow f_1 = 0 \rightarrow x_{ret} = n\frac{\lambda_0}{4}$

Si n es par → máximo V → punto hiperbólico

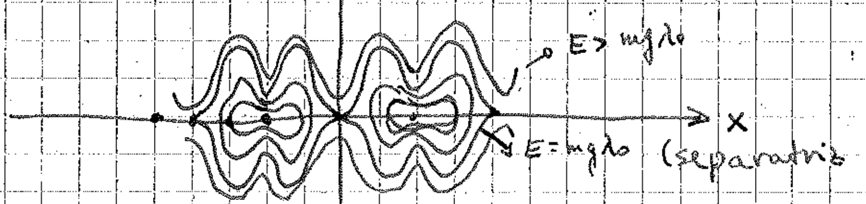
Si n es impar → mínimo V → " elíptico

El factor $(1+f_1'^2)$ no altera el carácter de estos puntos, pues justamente $f_1 = 0$ en ellos. Si no hubiese ese factor, el diagrama sería:

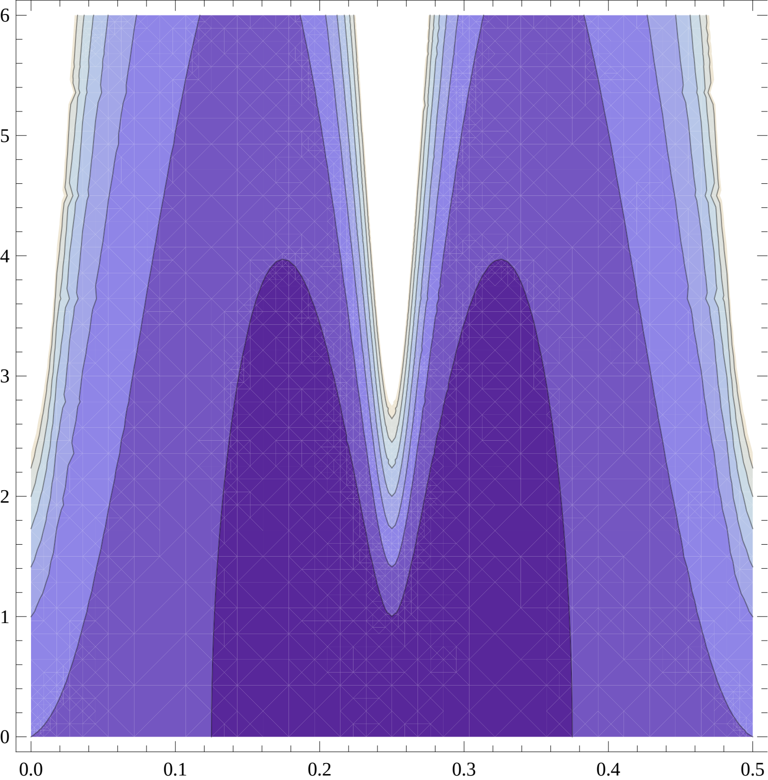


↳ Representa una red cristalina. Si los e^- tienen $E >$ pozos de las bandas, entonces hay conductividad.

Si representamos con ordenador $p(x)$, se obtiene:

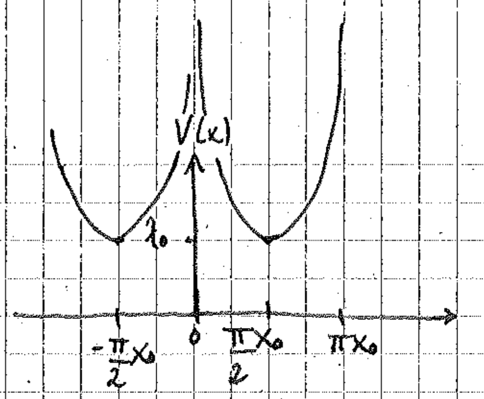


donde se aprecia la modulación de frecuencia doble ($2\pi \rightarrow 4\pi$) introducida por el factor $1 + f_1'^2$.



0
1
2
3
4
5
6
0.0
0.1
0.2
0.3
0.4
0.5

(47) $V(x) = \lambda_0 \sin^{-2}\left(\frac{x}{x_0}\right)$



$$H = \frac{p^2}{2m} + V = E$$

$$p = \frac{\partial W}{\partial x} = \sqrt{2m(E-V)} \rightarrow W = \int \sqrt{2m(E-V)}\, dx$$

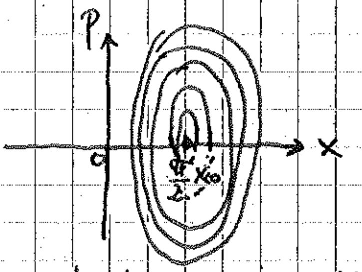
W: función característica de Hamilton

O bien:

$$H' = 0 = \left(\frac{\partial S}{\partial x}\right)^2 \Big/ 2m + V + \frac{\partial S}{\partial t} = H - E = \left(\frac{\partial S}{\partial x}\right)^2 \Big/ 2m + V - E$$

$\hookrightarrow S = W(q) - E t$

$$W = \int \sqrt{2m\left(E - \frac{\lambda_0}{\sin^2\left(\frac{x}{x_0}\right)}\right)} \Rightarrow$$



dos puntos de retorno a una energía dada anulan W ya que E=V

$\hookrightarrow \sin^2\left(\frac{x_{ret}}{x_0}\right) = \frac{\lambda_0}{E} \rightarrow x_{ret\pm} = x_0 \operatorname{asin}\left(\pm\sqrt{\frac{\lambda_0}{E}}\right)$

Al ser movimientos oscilatorios y no poder saltar de pozo (barrera infinita), es lícito utilizar las variables ángulo-acción.

$\hookrightarrow$ Analizamos oscilación en $\pi/2\, x_0$

$$I = \frac{1}{2\pi}\oint p \cdot dx \;,\; J = 2\pi I = \oint p \cdot dx = \oint \frac{\partial W}{\partial x} \cdot dx =$$

simétrico alrededor $x_0 \pi/2$

$$= \oint \sqrt{2m(E - V(x))} \cdot dx = 2 \int_{x_{ret+}}^{x_{ret-}} dx \sqrt{2m\left(E - \frac{\lambda_0}{\sin^2\left(\frac{x}{x_0}\right)}\right)} = 4 \int_{x_0 \operatorname{asin}\left(\sqrt{\frac{\lambda_0}{E}}\right)}^{\pi/2\, x_0} \sqrt{2mE - \frac{2m\lambda_0}{\sin^2\frac{x}{x_0}}}\, dx$$

una oscilación ida y vuelta

$-\sin j\, dj = \sin\alpha \cos\varphi\, d\varphi \;,\; du = \frac{1}{\cos^2\varphi} d\varphi$

$\Rightarrow$ Cambios de variable: $j = \frac{x}{x_0}$, $\cos\alpha = \sqrt{\frac{\lambda_0}{E}}$, $\cos j = \sin\alpha \sin\varphi$, $u = \tan\varphi$

$$J = 4\sqrt{2m} \int \left(E - \frac{\lambda_0}{\sin^2 j}\right)^{1/2} x_0\, dj = 4\sqrt{2m}\, x_0 \int \left(E - \frac{E\cos^2\alpha}{1 - \sin^2\alpha \sin^2\varphi}\right)^{1/2} dj$$

$$= 4x_0\sqrt{2mE} \int \left(\frac{1 - \sin^2\alpha \sin^2\varphi - \cos^2\alpha}{1 - \sin^2\alpha \sin^2\varphi}\right)^{1/2} \cdot \frac{\sin\alpha \cdot \cos\varphi \cdot d\varphi}{-\sin j}$$

$$= 4x_0\sqrt{2mE}\,(-) \int \frac{\sin^2\alpha \cos^2\varphi\, d\varphi}{\left(1 - \sin^2\alpha \sin^2\varphi\right)^{1/2}} \cdots (-) \int \frac{\sin^2\alpha \cos^4\varphi\, du}{1 - \sin^2\alpha \sin^2\varphi}$$

$$\frac{1}{1+u^2} = \cos^2\varphi$$

$$\hookrightarrow \int \frac{\sin^2\alpha \cos^4\varphi \, du}{1-\sin^2\alpha\sin^2\varphi} = \int \frac{\sin^2\alpha \, du}{(1+u^2)\left(\frac{1}{\cos^2\varphi} - \sin^2\alpha\sin^2\varphi\right)}$$

$$= \int \frac{\sin^2\alpha \; du}{(1+u^2)(1+u^2-\sin^2\alpha\, u^2)} = \int \frac{\sin^2\alpha \quad du}{(1+u^2)(1+u^2\cos^2\alpha)} \rightarrow$$ descomponer en fracciones simples

$$= \int \left(\frac{A}{1+u^2} + \frac{B}{1+u^2\cos^2\alpha} \right) du \rightarrow A(1+u^2\cos^2\alpha) + B(1+u^2) = \sin^2\alpha$$

$$\hookrightarrow \left.\begin{array}{l} A\cos^2\alpha + B = 0 \\ A + B = \sin^2\alpha \end{array}\right\} \quad A\cos^2\alpha + \sin^2\alpha - A = 0$$

$$A = 1$$

$$\hookrightarrow B = -\cos^2\alpha$$

$$= \int \left(\frac{1}{1+u^2} - \frac{\cos^2\alpha}{1+u^2\cos^2\alpha} \right) du$$

Límites: $x \Big|_{x_0 \operatorname{asin}\sqrt{\frac{\lambda_0}{E}}}^{\frac{\pi}{2}x_0} \rightarrow j \Big|_{\operatorname{asin}\sqrt{\frac{\lambda_0}{E}}}^{\pi/2} \rightarrow \varphi \Big|_{\pi/2}^{0} \rightarrow u \Big|_{\infty}^{0}$

porque $\cos(\operatorname{asin} x) = \sin(\operatorname{acos} x)$

$$J = -4x_0\sqrt{2mE} \int_{\infty}^{0} \left(\frac{1}{1+u^2} - \frac{\cos^2\alpha}{1+u^2\cos^2\alpha} \right) du$$

$$= + 4x_0\sqrt{2mE} \Big[\operatorname{arctg} u - \cos\alpha \operatorname{arctg}(\cos\alpha \cdot u) \Big]_0^{\infty}$$

$$= x_0 4\sqrt{2mE} \left(\frac{\pi}{2} - 0 - \sqrt{\frac{\lambda_0}{E}} \overbrace{\operatorname{arctg}\left(\sqrt{\frac{\lambda_0}{E}}\cdot\infty\right)}^{\pi/2} + 0 \right)$$

$$= 2\pi\sqrt{2m}\, x_0 \left(\sqrt{E} - \sqrt{\lambda_0} \right) \quad x_0 \left(\sqrt{E} - \sqrt{\lambda_0} \right)$$

$$\hookrightarrow I = \frac{J}{2\pi} = \sqrt{2m}\, x_0 \left(\sqrt{E} - \sqrt{\lambda_0}\right) \leftrightarrow E = \left(\frac{I}{x_0\sqrt{2m}} + \sqrt{\lambda_0} \right)^2$$

$$\dot{\varphi} = \frac{\partial H}{\partial I} = \frac{\partial E}{\partial I} = \omega(I) = \left(\frac{I}{x_0\sqrt{2m}} + \sqrt{\lambda_0} \right) \cdot \frac{2}{x_0\sqrt{2m}}$$

$$= \frac{I}{2m x_0^2} + \sqrt{\frac{2\lambda_0}{m}} \cdot \frac{1}{x_0} \quad // \quad \rightarrow \varphi = \omega(t-t_0) \text{ ya que } \dot{I} = 0$$

$$= \frac{(\sqrt{E} - \sqrt{\lambda_0})}{\sqrt{2m}\; x_0} + \frac{2\sqrt{\lambda_0}}{\sqrt{2m}\, x_0} = \frac{\sqrt{E} + \sqrt{\lambda_0}}{\sqrt{2m}\; x_0} \quad //$$

(48)

$$\begin{cases} x(\theta) = R(\theta + \sin\theta) \\ y(\theta) = R(1-\cos\theta) \end{cases} \quad \rightarrow \quad \begin{array}{l} \dot{x} = R(1+\cos\theta)\cdot\dot{\theta} \\ \dot{y} = R(\cancel{0} + \sin\theta)\dot{\theta} = R\sin\theta\cdot\dot{\theta} \end{array}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}R^2\dot{\theta}^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)$$

$$= \frac{m}{2}R^2\dot{\theta}^2\cdot 2(1+\cos\theta) = 2mR^2\dot{\theta}^2\cos^2\frac{\theta}{2}$$

$$V = mgy = mgR(1-\cos\theta) = 2mgR\sin^2\frac{\theta}{2} = 2mgR - 2mgR\cos^2\frac{\theta}{2}$$

$$\mathcal{L} = T - V = 2mR^2\dot{\theta}^2\cos^2\frac{\theta}{2} + 2mgR\cos^2\frac{\theta}{2} - 2mgR$$

$$= 2mR(R\dot{\theta}^2+g)\cos^2\frac{\theta}{2} - 2mgR$$

$$H = \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\cdot\dot{\theta} - \mathcal{L} = 2mR^2\cdot 2\dot{\theta}^2\cos^2\frac{\theta}{2} - \mathcal{L} = 2mR(R\dot{\theta}^2-g)\cos^2\frac{\theta}{2} + 2mgR = T+V = E$$

$$p = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 4mR^2\dot{\theta}\cos^2\frac{\theta}{2}$$

$$\hookrightarrow H = 2mR\left(\frac{R\,p^2}{\left(16m^2\cos^4\frac{\theta}{2}\right)R^4} - g\right)\cos^2\frac{\theta}{2} + 2mgR =$$

$$= \frac{p^2}{8mR^2\cos^2\frac{\theta}{2}} + 2mgR\sin^2\frac{\theta}{2} = E$$

$$\hookrightarrow p = \left[\left(E - 2mgR\sin^2\frac{\theta}{2}\right)\cdot 8mR^2\cos^2\frac{\theta}{2}\right]^{1/2}$$

$$I = \frac{1}{2\pi}\oint p\cdot d\theta$$

Puntos de retorno: $E = 2mgR\sin^2\frac{\theta_{ret}}{2} \rightarrow \theta_{ret} = 2\,\mathrm{asin}\pm\sqrt{\frac{E}{2mgR}}$

$$\hookrightarrow I = \frac{1}{2\pi}\cdot 2\int_{\theta_-}^{\theta_+} p\,d\theta \overset{\text{simétrico}}{=} \frac{2}{\pi}\int_0^{\theta_+}\left[\left(E - 2mgR\sin^2\frac{\theta}{2}\right)8mR^2\cos^2\frac{\theta}{2}\right]^{1/2}d\theta$$

$$= \frac{4}{\pi}\int_0^{\theta_+}\left[E - 2mgR\sin^2\frac{\theta}{2}\right]^{1/2}\sqrt{2m}\,R\cos\frac{\theta}{2}\,d\theta \;=\; \sin\frac{\theta}{2} = z \rightarrow d\theta\,\frac{1}{2}\cos\frac{\theta}{2} = dz$$

$$= \frac{4\sqrt{2m}\cdot R}{\pi}\int_0^{+r}\left[E - 2mgR\,z^2\right]^{1/2}\cdot 2dz = \frac{8R\sqrt{2m}}{\pi}\int\sqrt{E-2mgRz^2}\,dz$$

$$= \frac{8R}{\pi}\sqrt{2m}\sqrt{2mgR}\int_0^{+r}\sqrt{\frac{E}{2mgR}-z^2}\,dz \overset{\text{ver tablas}}{=} \frac{16Rm\sqrt{gR}}{\pi}\left(\frac{z}{2}\sqrt{\frac{E}{2mgR}-z^2} + \frac{E}{4mgR}\,\mathrm{asin}\left(\frac{z}{\sqrt{\frac{E}{2mgR}}}\right)\right)_0^{+r}$$

$$= \frac{16Rm\sqrt{gR}}{\pi}\left[0+0+\frac{\sqrt{\frac{E}{2mgR}}}{2}\cdot 0+\frac{E}{4mgR}\cdot \operatorname{asin}(1)\right]$$

$$= \frac{16Rm\sqrt{gR}}{\pi}\cdot\frac{E}{4mgR}\cdot\frac{\pi}{2} = 2E\sqrt{\frac{R}{g}} = I$$

$$\hookrightarrow E = \frac{I}{2}\sqrt{\frac{g}{R}} \quad \nearrow E = \omega I$$

$$\dot\psi = \omega(I) = \frac{\partial E}{\partial I} = \frac{1}{2}\sqrt{\frac{g}{R}}$$ → Frecuencia de oscilación

↓ coincide con la obtenida en el ejercicio 5.

$$\psi = \frac{1}{2}\sqrt{\frac{g}{R}}\cdot(t-t_0)$$

(49)

a) • $H_1 = \frac{p^2}{9q^2} + q^2 = E$

$\hookrightarrow p^2 = (E-q^2)\cdot 9q^2 \rightarrow p = \pm 3q\sqrt{E-q^2}$

$p(q_{ret}) = 0 \rightarrow q_{ret} = \pm\sqrt{E}$ (en $q=0$, $p, E=0$)

• $H_2 = q^2p^2 + \log^2 q = E$

$\hookrightarrow p = \pm\frac{1}{q}\sqrt{E-\log^2 q} \rightarrow q_{ret} = e^{\pm\sqrt{E}}$

b)

• $$I_1 = \frac{1}{2\pi}\oint p\cdot dq = \frac{2}{2\pi}\int_{-\sqrt{E}}^{+\sqrt{E}}|p|dq = \frac{4}{2\pi}\int_0^{+\sqrt{E}}|p|dq = \frac{2}{\pi}\int_0^{+\sqrt{E}}3q\sqrt{E-q^2}\,dq$$

$$= \frac{6}{\pi}\int_0^{+\sqrt{E}} q\sqrt{E-q^2}\,dq = \frac{6}{\pi}(-)(E-q^2)^{3/2}\cdot\frac{1}{3}\Big|_0^{+\sqrt{E}} = (-)\frac{2}{\pi}\left(0-E\right)^{3/2}$$

$$= \frac{2E^{3/2}}{\pi} \rightarrow E_1 = \left(\frac{\pi}{2}I_1\right)^{2/3} \rightarrow \omega_1 = \frac{\partial E_1}{\partial I_1} = \left(\frac{\pi}{2}\right)^{2/3}\frac{2}{3}I_1^{-1/3} = \left(\frac{\pi}{2}\right)^{2/3}\cdot\frac{2}{3}\cdot\left(\frac{2E^{3/2}}{\pi}\right)^{-1/3}$$

$$= \left(\frac{\pi}{2}\right)^{2/3}\cdot\frac{2}{3}\cdot 2^{-1/3}\cdot\frac{E^{-1/2}}{\pi^{-1/3}} = \frac{\pi}{3\sqrt{E}}$$

$$\varphi = \frac{\partial W}{\partial I} = \frac{\partial W}{\partial E}\cdot\frac{\partial E}{\partial I} = \frac{\partial W}{\partial E}\cdot\omega$$

W: función característica de Hamilton; $p = \frac{\partial W}{\partial q}$

• H_2:

$$I_2 = \frac{1}{2\pi}\oint p\cdot dq = \frac{2}{\pi}\int_{e^0}^{e^{+\sqrt{E}}} \frac{1}{q}\cdot\sqrt{E-\log^2 q}\; dq = \frac{2}{\pi}\int_0^{+\sqrt{E}}\sqrt{E-z^2}\cdot dz \qquad z = \log q \to dz = \frac{1}{q}\cdot dq$$

ver tablas

$$= \frac{2}{\pi}\cdot\left[\frac{z}{2}\sqrt{E-z^2} + \frac{E}{2}\,\mathrm{asin}\left(\frac{z}{\sqrt{E}}\right)\right]_0^{+\sqrt{E}} = \frac{2}{\pi}\cdot\left(\frac{E}{2}\cdot\frac{\pi}{2} - 0\right)$$

$$= \frac{E}{2} \to E = 2I$$

$$\to \omega = \frac{\partial E}{\partial I} = 2$$

c)

$$p = \frac{\partial W}{\partial q} \to W = \int p\cdot dq$$

• H_1:

$$W = \pm\int 3q\sqrt{E-q^2}\,dq = \pm 3(-)\frac{1}{3}(E-q^2)^{3/2} = \mp (E-q^2)^{3/2}$$

$$\varphi = \frac{\partial W}{\partial I} = \frac{\partial W}{\partial E}\cdot\frac{\partial E}{\partial I} = \omega\cdot(\mp)\frac{3}{2}(E-q^2)^{1/2} = \mp\frac{\pi}{3\sqrt{E}}\cdot\frac{3}{2}\sqrt{E-q^2}$$

$$= \mp\frac{\pi}{2}\sqrt{1-\frac{q^2}{E}} = \omega(t-t_0)$$

$$\hookrightarrow \frac{4\varphi^2}{\pi^2} = 1 - \frac{q^2}{E} \to q = \pm\sqrt{E}\left(1-\frac{4\varphi^2}{\pi^2}\right)^{1/2}$$

$$\hookrightarrow q(t) = \pm\sqrt{E}\left(1-\left(\frac{2\cdot\pi}{\pi\, 3\sqrt{E}}(t-t_0)\right)^2\right)^{1/2} = \pm\sqrt{E}\left(1-\frac{4(t-t_0)^2}{9E}\right)^{1/2}$$

$$p(t) = \frac{\partial W}{\partial q} = \pm 3q\sqrt{E-q^2} = 3\sqrt{E}\left(1-\dots\right)^{1/2}(\pm)\frac{2\cdot(t-t_0)}{3}$$

$$= \pm 2\sqrt{E}(t-t_0)\cdot\Big(1-\underbrace{\frac{4(t-t_0)^2}{9E}}_{\alpha}\Big)^{1/2}$$

Comprobación:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{2p}{9q^2} = \pm\frac{2}{9}\frac{\sqrt{E}\,2\,(t-t_0)\,(1-\alpha)^{1/2}}{E\cdot(1-\alpha)} = \pm\frac{4}{9\sqrt{E}}\frac{(t-t_0)}{(1-\alpha)^{1/2}}$$

$$\dot{q} = \pm\frac{1}{2}\sqrt{E}\,(1-\alpha)^{-1/2}\cdot\frac{8}{9E}(t-t_0) = \pm\frac{4}{9\sqrt{E}}(t-t_0)/(1-\alpha)^{1/2}$$

$$\dot p = -\frac{\partial H}{\partial q} = \frac{2p^2}{9q^3} - 2q$$

$$= \pm\frac{2}{9}\cdot\frac{4(t-t_0)^2(1-\alpha)\cdot E}{E(1-\alpha)\cdot\sqrt{E}\,(1-\alpha)^{1/2}} \mp 2\sqrt{E}\,(1-\alpha)^{1/2}$$

$$= \pm\frac{8}{9}\,\frac{(t-t_0)^2}{E^{1/2}(1-\alpha)^{1/2}} \mp 2\sqrt{E}\,(1-\alpha)^{1/2}$$

$$\dot p = \pm 2\sqrt{E}(1-\alpha)^{1/2} \mp 2\sqrt{E}(t-t_0)^2\cdot\frac{8}{9E}\cdot\frac{1}{2}\cdot(1-\alpha)^{-1/2}$$

$$= \pm 2\sqrt{E}(1-\alpha)^{1/2} \mp \frac{8}{9\sqrt{E}}\cdot\frac{(t-t_0)^2}{(1-\alpha)^{1/2}}$$

El orden $\pm$ de los signos en p está al revés.

• H_2

$$W = \pm\int\frac{1}{q}\sqrt{E-\log^2 q}\,dq \overset{\text{ver b)}}{=} \pm\left[\frac{\log(q)}{2}\sqrt{E-\log^2 q} + \frac{E}{2}\,\operatorname{asin}\left(\frac{\log q}{\sqrt{E}}\right)\right]$$

$$\varphi = \frac{\partial W}{\partial I} = \frac{\partial W}{\partial E}\cdot\frac{\partial E}{\partial I} = \pm\left[\frac{\log(q)}{4\sqrt{E-\log^2 q}} + \frac{1}{2}\,\operatorname{asin}\left(\frac{\log q}{\sqrt{E}}\right)\right]\cdot 2\omega$$

$$= \overset{\text{no demuestro}}{\ldots} = \operatorname{arctg}\left(\frac{\log q}{\sqrt{E-\log^2 q}}\right) = 2\cdot(t-t_0)$$

$$\tan^2\varphi = \frac{\log^2 q - E + E}{E-\log^2 q} = -1 + \frac{E}{E-\log^2 q}$$

$$\frac{E}{E-\log^2 q} = 1+\tan^2\varphi = 1+\frac{\sin^2\varphi}{\cos^2\varphi} = \frac{1}{\cos^2\varphi}$$

$$\hookrightarrow E-\log^2 q = E\cos^2\varphi \rightarrow E(1-\cos^2\varphi) = E\sin^2\varphi = \log^2 q$$

$$\log q = \pm\sqrt{E}\sin\varphi \rightarrow q(t) = e^{\pm\sqrt{E}\sin\varphi} = e^{\pm\sqrt{E}\sin(2(t-t_0))}$$

$$\varphi(t) = \pm\frac{1}{q}\sqrt{E-\log^2 q} = \pm e^{\mp\sqrt{E}\sin\varphi}\sqrt{E-E\sin^2\varphi}$$

$$= \pm\sqrt{E}\cdot\cos(2(t-t_0))\cdot e^{\mp\sqrt{E}\sin(2(t-t_0))}$$

Check:

$$\dot q = \frac{\partial H_2}{\partial p} = 2q^2p = 2\cdot e^{\pm 2\sqrt{E}\sin\varphi}(\pm)\sqrt{E}\cdot\cos\varphi\; e^{\mp\sqrt{E}\sin\varphi}$$

$$= \pm 2\, e^{\pm\sqrt{E}\sin\varphi}\cdot\cos\varphi$$

$$\dot q = e^{\pm\sqrt{E}\sin\varphi}(\pm\sqrt{E}\cos\varphi)\cdot 2 \quad ✓$$

$$\dot{p} = -\frac{\partial H_2}{\partial q} = 2qp^2 + \frac{2}{q}\log q$$

$$= 2\, e^{\pm\alpha} \cdot E\cos^2\varphi \; e^{\mp 2\alpha} + 2e^{\mp\alpha}\,(\pm\alpha)$$

$$\dot{p} = \pm\sqrt{E}\cdot\cos\varphi\cdot e^{\mp\alpha}\cdot(\mp)\sqrt{E}\cos\varphi\cdot 2 \mp \sqrt{E}\sin\varphi\cdot 2e^{\mp\alpha}$$ ✓

signos al revés en p?

d)

- H_1

$$I = \frac{2}{\pi}E^{3/2} \longrightarrow \omega = \frac{\partial E}{\partial I} = \left(\frac{\partial I}{\partial E}\right)^{-1} = \left(\frac{3\sqrt{E}}{\pi}\right)^{-1} = \frac{\pi}{3\sqrt{E}}$$

$$E = \left(\frac{\pi}{2}I\right)^{2/3} = \left(\frac{3\sqrt{E}}{2}\omega I\right)^{2/3}$$

$$\hookrightarrow E = \frac{3}{2}\omega I \quad // \qquad (\omega(E,I))$$

- H_2

$$E = 2I = \omega I$$

Aquí ω sí es fijo.