Ejercicios de Óptica Cuántica

- Sea el problema de la interacción semiclásica de un átomo de dos niveles con un campo monocromático. Se trata de

(a) Obtener la ecuaciones de evolución de la amplitud de probabilidad en la imagen de interacción. Para ello, utilizad la siguiente transformación unitaria: $U = \exp(iA)$ con .

(b) Resolved las ecuaciones obtenidas. Discutid las oscilaciones de Rabi.

- El Hamiltoniano que describe, en la imagen de Schröedinger, la dinámica de un ión atrapado en interacción con un campo electromagnético clásico es

$$\hat{H} = \hbar\left[\omega_T \hat{b}^\dagger\hat{b} + \frac{1}{2}\omega_A\hat{\sigma}_z + \frac{1}{2}\Omega_L(\hat{\sigma}_+ + \hat{\sigma}_-)\left(\hat{F}e^{-i\omega_L t} + \hat{F}^\dagger e^{i\omega_L t}\right)\right]$$

donde

$$\hat{F} = \exp\left[i\eta\left(\hat{b} + \hat{b}^\dagger\right)\right], \quad \eta = k\sqrt{\frac{\hbar}{2\omega_T m}}.$$

Escribidlo en la imagen de interacción. Para ello usad la transformación unitaria $U' = exp[-iH_0t/\hbar]$.

Óptica Cuántica - Problemas a entregar

a) Modelo semiclásico de átomo de 2 niveles y campo monocromático.

Partimos de:

$$|\psi(t)\rangle = \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} \rightarrow 2 \text{ niveles } \begin{cases} |0\rangle \\ |1\rangle \end{cases}$$

y

$$H = \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\vec{\mu}_{01}\cdot\vec{E} \\ -\vec{\mu}_{10}\cdot\vec{E} & \frac{\hbar\omega_0}{2} \end{pmatrix} \qquad \begin{aligned} &\vec{\mu}_{10} = \vec{\mu}_{01}^{\,*} \\ &\vec{E} = \frac{1}{2}\vec{e}\cdot\varepsilon\cdot e^{i\omega t} + c.c \\ &\quad \rightarrow \vec{e} \in \mathbb{R}^3 \end{aligned}$$

en imagen de Schrödinger.

$$i\hbar \begin{pmatrix} \dot{c}_0 \\ \dot{c}_1 \end{pmatrix} = H \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

↳ Realizamos un cambio de imagen:

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = U \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \quad \text{con} \quad U = \begin{pmatrix} e^{-i\frac{\omega}{2}t} & \\ & e^{+i\omega/2\, t} \end{pmatrix} \begin{matrix} \rightarrow \text{diagonal} \\ \rightarrow \text{unitaria} \end{matrix}$$

$$|\psi\rangle_I = U|\psi\rangle_S \qquad UU^+ = I \qquad \rightarrow U^+ \equiv A \text{ en enunciado}$$

La ec. de Schrödinger queda:

$$i\hbar \begin{pmatrix} \frac{d}{dt}c_0 \\ \frac{d}{dt}c_1 \end{pmatrix} = H\cdot\overbrace{U^+U}^{I}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = HU^+\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

$$i\hbar\frac{d}{dt}|\psi\rangle_S = i\hbar\frac{d}{dt}\left(U^+|\psi_I\rangle\right) = i\hbar\frac{d}{dt}\begin{pmatrix} b_0\cdot e^{+i\frac{\omega}{2}t} \\ b_1\cdot e^{-i\frac{\omega}{2}t} \end{pmatrix}$$

$$\Rightarrow i\hbar \begin{pmatrix} e^{+i\frac{\omega}{2}t}\left(+\frac{i\omega}{2}b_0 + \dot{b}_0\right) \\ e^{-i\frac{\omega}{2}t}\left(-i\frac{\omega}{2}b_1 + \dot{b}_1\right) \end{pmatrix} = HU^+\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \qquad U\cdot|$$

$$i\hbar \begin{pmatrix} e^{-i\frac{\omega}{2}t} & \\ & e^{+i\omega/2\, t} \end{pmatrix}\cdot \quad \text{''} \quad = UHU^+\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

$$\hookrightarrow i\hbar\begin{pmatrix} \dot{b}_0 \\ \dot{b}_1 \end{pmatrix} + \frac{\hbar\omega}{2}\begin{pmatrix} -b_0 \\ +b_1 \end{pmatrix} = UHU^+\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

$$UHU^+ = \begin{pmatrix} e^{-i\frac{\omega}{2}t} & 0 \\ 0 & e^{+i\frac{\omega}{2}t} \end{pmatrix}\cdot\begin{pmatrix} -\frac{\hbar\omega_0}{2}\cdot e^{+i\frac{\omega}{2}t} & -\vec{\mu}_{01}\cdot\vec{E}\cdot e^{-i\frac{\omega}{2}t} \\ -\vec{\mu}_{10}\cdot\vec{E}\cdot e^{+i\frac{\omega}{2}t} & \frac{\hbar\omega_0}{2}\cdot e^{-i\frac{\omega}{2}t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & -\vec{\mu}_{01}\cdot\vec{E}\cdot e^{-i\omega t} \\ -\vec{\mu}_{10}\cdot\vec{E}\cdot e^{+i\omega t} & \frac{\hbar\omega_0}{2} \end{pmatrix}$$

$$\to i\hbar\begin{pmatrix}\dot b_0\\ \dot b_1\end{pmatrix}=\begin{pmatrix}\frac{\hbar}{2}(\omega-\omega_0) & -\vec{\mu_{01}}\cdot\vec{E}\cdot e^{-i\omega t}\\ -\vec{\mu_{10}}\cdot\vec{E}\cdot e^{i\omega t} & -\frac{\hbar}{2}(\omega-\omega_0)\end{pmatrix}\begin{pmatrix}b_0\\ b_1\end{pmatrix};\ \delta\equiv\omega-\omega_0$$

$$\vec{\mu_{01}}\cdot\vec{E}\cdot e^{-i\omega t}=\frac{1}{2}\vec{\mu_{01}}\cdot\vec{e}\,(\mathcal{E}e^{i\omega t}+\mathcal{E}^*e^{-i\omega t})\cdot e^{-i\omega t}$$

$$=\frac{1}{2}\vec{\mu_{01}}\cdot\vec{e}\,(\mathcal{E}+\mathcal{E}^*e^{-2i\omega t})$$

Por la RWA (rotating wave approximation, eliminamos $e^{-2i\omega t}$)

$$\vec{\mu_{01}}\cdot\vec{E}\cdot e^{-i\omega t}\simeq\frac{1}{2}\vec{\mu_{01}}\cdot\vec{e}\cdot\mathcal{E}\equiv\frac{\hbar}{2}\Omega$$ ∴ frecuencia de Rabi

$$\vec{\mu_{10}}\cdot\vec{E}\cdot e^{i\omega t}=\frac{\hbar}{2}\Omega^*$$ (demo similar)

$$\to i\hbar\begin{pmatrix}\dot b_0\\ \dot b_1\end{pmatrix}=\underbrace{\frac{\hbar}{2}\begin{pmatrix}\delta & -\Omega\\ -\Omega^* & -\delta\end{pmatrix}}_{\overline{H}}\begin{pmatrix}b_0\\ b_1\end{pmatrix}$$

Resolvemos sistema con operador de evolución temporal:

$$\begin{pmatrix}b_0(t)\\ b_1(t)\end{pmatrix}=\exp\left(-\frac{i}{\hbar}\overline{H}t\right)\cdot\begin{pmatrix}b_0(0)\\ b_1(0)\end{pmatrix}$$

Para ello, hay que diagonalizar $\overline{H}$:

$$\begin{vmatrix}\frac{\hbar\delta}{2}-\lambda & -\frac{\hbar\Omega}{2}\\ -\frac{\hbar\Omega^*}{2} & -\frac{\hbar\delta}{2}-\lambda\end{vmatrix}=\underbrace{-\left(\frac{\hbar\delta}{2}-\lambda\right)\left(\frac{\hbar\delta}{2}+\lambda\right)}_{-\left(\frac{\hbar^2}{4}\delta^2-\lambda^2\right)}-\frac{\hbar^2}{4}|\Omega|^2=0$$

$$\to\lambda_\pm=\pm\frac{\hbar}{2}\sqrt{\delta^2+|\Omega|^2}=\pm\frac{\hbar}{2}\tilde{\Omega}$$ ∴ frecuencia generalizada de Rabi

Vectores propios:

$$\lambda_+:\quad \frac{\hbar}{2}(\delta-\tilde{\Omega})x-\frac{\hbar}{2}\Omega y=0\ \to|+\rangle=\frac{1}{\sqrt{|\Omega|^2+(\delta-\tilde{\Omega})^2}}\begin{pmatrix}\Omega\\ \delta-\tilde{\Omega}\end{pmatrix}$$

$$\lambda_-:\ -\frac{\hbar}{2}\Omega^*x-\frac{\hbar}{2}(\delta-\tilde{\Omega})y=0\ \to|-\rangle=\frac{1}{\sqrt{|\Omega|^2+(\delta-\tilde{\Omega})^2}}\begin{pmatrix}\delta-\tilde{\Omega}\\ -\Omega^*\end{pmatrix}$$

$\langle+|+\rangle=1$; $\langle+|-\rangle=0$
$\langle-|-\rangle=1$

$$\exp\left(-\frac{i}{\hbar}\overline{H}t\right)=\exp\left(-\frac{i}{\hbar}\lambda_+t\right)|+\rangle\langle+|+\exp\left(-\frac{i}{\hbar}\lambda_-t\right)|-\rangle\langle-|$$

$$=e^{-i\frac{\tilde{\Omega}}{2}t}|+\rangle\langle+|+e^{i\frac{\tilde{\Omega}}{2}t}|-\rangle\langle-|$$

$$|+\rangle\langle+| = \frac{1}{|\Omega|^2+(\delta-\tilde{\Omega})^2}\begin{pmatrix} |\Omega|^2 & \Omega(\delta-\tilde{\Omega}) \\ \Omega^*(\delta-\tilde{\Omega}) & (\delta-\tilde{\Omega})^2 \end{pmatrix}$$

$$|-\rangle\langle-| = \frac{1}{|\Omega|^2+(\delta-\tilde{\Omega})^2}\begin{pmatrix} (\delta-\tilde{\Omega})^2 & -\Omega(\delta-\tilde{\Omega}) \\ -\Omega^*(\delta-\tilde{\Omega}) & |\Omega|^2 \end{pmatrix}$$

$$I = |+\rangle\langle+| + |-\rangle\langle-| \quad ✓$$

$$\exp\left(-\frac{i}{\hbar}\bar{H}t\right) = \frac{1}{|\Omega|^2+(\delta-\tilde{\Omega})^2}\cdot\begin{pmatrix} |\Omega|^2 e^{-i\frac{\tilde{\Omega}}{2}t}+(\delta-\tilde{\Omega})^2 e^{i\frac{\tilde{\Omega}}{2}t} & \Omega(\delta-\tilde{\Omega})\left(e^{-i\frac{\tilde{\Omega}}{2}t}-e^{i\frac{\tilde{\Omega}}{2}t}\right) \\ \Omega^*(\delta-\tilde{\Omega})\left(e^{-i\frac{\tilde{\Omega}}{2}t}-e^{i\frac{\tilde{\Omega}}{2}t}\right) & (\delta-\tilde{\Omega})^2 e^{-i\frac{\tilde{\Omega}}{2}t}+|\Omega|^2 e^{i\frac{\tilde{\Omega}}{2}t} \end{pmatrix}$$

$$\hookrightarrow |\Omega|^2+(\delta-\tilde{\Omega})^2 = |\Omega|^2+\delta^2+\tilde{\Omega}^2-2\delta\tilde{\Omega} = 2\tilde{\Omega}^2-2\delta\tilde{\Omega} = 2\tilde{\Omega}(\tilde{\Omega}-\delta)$$

$$\begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = \frac{1}{2\tilde{\Omega}(\tilde{\Omega}-\delta)}\underbrace{\begin{pmatrix} |\Omega|^2 e^{-i\frac{\tilde{\Omega}}{2}t}+(\tilde{\Omega}-\delta)^2 e^{i\frac{\tilde{\Omega}}{2}t} & \Omega(\tilde{\Omega}-\delta)\cdot 2i\sin\frac{\tilde{\Omega}}{2}t \\ \Omega^*(\tilde{\Omega}-\delta)\cdot 2i\sin\frac{\tilde{\Omega}}{2}t & |\Omega|^2 e^{i\frac{\tilde{\Omega}}{2}t}+(\tilde{\Omega}-\delta)^2 e^{-i\frac{\tilde{\Omega}}{2}t} \end{pmatrix}}_{\bar{U}} \cdot \begin{pmatrix} b_0(0) \\ b_1(0) \end{pmatrix}$$

- Si $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = \frac{1}{2\tilde{\Omega}(\tilde{\Omega}-\delta)}\begin{pmatrix} |\Omega|^2 e^{-i\frac{\tilde{\Omega}}{2}t}+(\tilde{\Omega}-\delta)^2 e^{i\frac{\tilde{\Omega}}{2}t} \\ \Omega^*(\tilde{\Omega}-\delta)\,2i\sin\frac{\tilde{\Omega}}{2}t \end{pmatrix}$$

$$|b_0|^2 = \frac{1}{4\tilde{\Omega}^2(\tilde{\Omega}-\delta)^2}\left(|\Omega|^4+(\tilde{\Omega}-\delta)^4 + |\Omega|^2(\tilde{\Omega}-\delta)^2\cdot 2\cos\tilde{\Omega}t\right)$$

$\pm 2|\Omega|^2(\tilde{\Omega}-\delta)^2 \to$ para sacar un 1

$$= \frac{|\Omega|^4+(\tilde{\Omega}-\delta)^4+2|\Omega|^2(\tilde{\Omega}-\delta)^2}{\underbrace{(|\Omega|^2+(\delta-\tilde{\Omega})^2)^2}_{=[2\tilde{\Omega}(\tilde{\Omega}-\delta)]^2}} + \frac{|\Omega|^2}{\tilde{\Omega}^2}(\cos\tilde{\Omega}t-1)$$

$$= 1 - \frac{|\Omega|^2}{\tilde{\Omega}^2}\sin^2\frac{\tilde{\Omega}}{2}t$$

$$|b_1|^2 = \frac{|\Omega|^2(\tilde{\Omega}-\delta)^2\cdot 4}{4\tilde{\Omega}^2(\tilde{\Omega}-\delta)^2}\cdot\sin^2\frac{\tilde{\Omega}}{2}t = \frac{|\Omega|^2}{\tilde{\Omega}^2}\cdot\sin^2\frac{\tilde{\Omega}}{2}t$$

$$|b_0|^2+|b_1|^2 = 1 \quad ✓$$

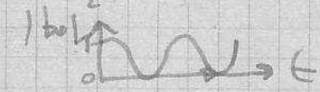
- Si $|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Por simetría de las ecuaciones, es como el resultado anterior pero cambiando 0 por 1.

$$|b_0|^2 = \frac{|\Omega|^2}{\tilde{\Omega}^2}\sin^2\frac{\tilde{\Omega}}{2}t$$

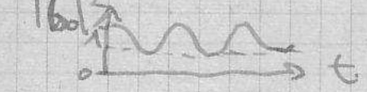
- El átomo se excita y desexcita periódicamente, intercambiando la energía con el campo indefinidamente. Por tanto, el modelo tiene en cuenta (predice) la absorción y emisión estimulada.

- Si $\delta = 0$ la probabilidad oscila armónicamente entre 0 y 1.



(para $b_0(0) = 1$) $\hookrightarrow |\Omega|^2 = \tilde{\Omega}^2$

- Si $\delta \neq 0$, no llega al valor mínimo 0 (desintonía), es decir,



nunca llegamos a excitarlo con probabilidad del (100%), $|b_1|^2 \neq 1 \ \forall t$

- Si inicialmente átomo en estado excitado: $b_1(0) = 1$ y no conectamos el campo ($\Omega = 0$), $|b_1|^2 = 1 \ \forall t$, $|b_0|^2 = 0 \ \forall t$. Es decir, en ausencia de campo, no se desexcita. Por tanto, no predice la emisión espontánea, el modelo falla. Para conseguirlo, se debe cuantizar el campo, de manera que las interacciones con las fluctuaciones del vacío cuántico (según el modo), permitan la emisión espontánea.

Otra manera de resolver el problema sin diagonalizar es resolver el sistema de ecuaciones diferenciales acopladas:

$$\begin{cases} \dot{b}_0 = \dfrac{\delta}{2i} b_0 - \dfrac{\Omega}{2i} b_1 \\[2mm] \dot{b}_1 = -\dfrac{\Omega^*}{2i} b_0 - \dfrac{\delta}{2i} b_1 \end{cases}$$

Derivo 1ª ecuac.: sustituyo $\dot{b}_1$ — sustituyo b_1

$$\ddot{b}_0 = \frac{\delta}{2i}\dot{b}_0 - \frac{\Omega}{2i}\dot{b}_1 = \frac{\delta}{2i}\dot{b}_0 - \frac{\Omega}{2i}\left(-\frac{\Omega^*}{2i} b_0 - \frac{\delta}{2i} b_1\right) =$$

$$= \frac{\delta}{2i}\dot{b}_0 - \frac{\Omega}{4}\left[-\Omega^* b_0 + \delta\left(\dot{b}_0 - \frac{\delta}{2i} b_0\right)\left(\frac{-2i}{-\Omega}\right)\right]$$

$$= \frac{\delta}{2i}\dot{b}_0 - \frac{|\Omega|^2}{4} b_0 + \frac{\cancel{\Omega}}{\cancel{\Omega}}\left(+\frac{i}{2}\right)\delta\dot{b}_0 - \frac{\delta^2}{4} b_0$$

$$\ddot{b}_0 + \left(\frac{\tilde{\Omega}}{2}\right)^2 b_0 = 0$$

$$\ddot{b}_1 = -\frac{\Omega^*}{2i}\dot{b}_0 - \frac{\delta}{2i}\dot{b}_1 = -\frac{\delta}{2i}\dot{b}_1 - \frac{\Omega^*}{2i}\left(\frac{\delta}{2i} b_0 - \frac{\Omega}{2i} b_1\right)$$

$$= -\frac{\delta}{2i}\dot{b}_1 + \frac{\Omega^*}{4}\left(\delta\left(\dot{b}_1 + \frac{\delta}{2i} b_1\right)\frac{2i}{-\Omega^*} - \Omega b_1\right)$$

$$= -\frac{\delta}{2i}\dot{b}_1 - \frac{|\Omega|^2}{4} b_1 - \frac{\delta^2}{4} b_1 + \frac{\cancel{\Omega^*}}{\cancel{\Omega^*}}\frac{i}{2}(-)\delta\dot{b}_1$$

$$\hookrightarrow \ddot{b}_1 + \left(\frac{\tilde{\Omega}}{2}\right)^2 b_1 = 0$$

b_1, b_2 → ecuas del oscilador armónico:

$$b_0(t) = A\cos\frac{\tilde\Omega}{2}t + B\sin\frac{\tilde\Omega}{2}t$$

$$b_1(t) = C\cos\frac{\tilde\Omega}{2}t + D\sin\frac{\tilde\Omega}{2}t$$

$$b_0(0) = A$$

$$b_1(0) = C$$

$$\dot b_0(0) = \frac{\tilde\Omega}{2}B$$

$$\dot b_1(0) = \frac{\tilde\Omega}{2}D$$

$$\rightarrow \quad b_0(t) = b_0(0)\cos\frac{\tilde\Omega}{2}t + \frac{2\dot b_0(0)}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t$$

$$b_1(t) = b_1(0)\cos\frac{\tilde\Omega}{2}t + \frac{2\dot b_1(0)}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t$$

Utilizamos las ecuas del sistema inicial en $t=0$:

$$\begin{cases} \dot b_0(0) = +\frac{i}{2}\left(\Omega b_1(0) - \delta b_0(0)\right) \\ \dot b_1(0) = \frac{i}{2}\left(\Omega^* b_0(0) + \delta b_1(0)\right) \end{cases}$$

↓ Sustituyo:

$$\begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\frac{\tilde\Omega}{2}t - \frac{i\delta}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t & \frac{i\Omega}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t \\ \frac{i\Omega^*}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t & \cos\frac{\tilde\Omega}{2}t + \frac{i\delta}{\tilde\Omega}\sin\frac{\tilde\Omega}{2}t \end{pmatrix}}_{\bar U} \begin{pmatrix} b_0(0) \\ b_1(0) \end{pmatrix}$$

Coincide con el resultado por la vía de diagonalizar, tras algo de álgebra:

$$\bar U_{11} = \left(e^{i\frac{\tilde\Omega}{2}t} + e^{-i\frac{\tilde\Omega}{2}t}\right)\frac{1}{2} - \frac{\delta}{2\tilde\Omega}\left(e^{i\frac{\tilde\Omega}{2}t} - e^{-i\frac{\tilde\Omega}{2}t}\right) =$$

$$= \frac{1}{2\tilde\Omega}\left(e^{-i\frac{\tilde\Omega}{2}t}(\tilde\Omega+\delta) + e^{i\frac{\tilde\Omega}{2}t}(\tilde\Omega-\delta)\right) =$$

$$= \frac{1}{2\tilde\Omega(\tilde\Omega-\delta)}\left(e^{-i\frac{\tilde\Omega}{2}t}\underbrace{(\tilde\Omega+\delta)(\tilde\Omega-\delta)}_{\tilde\Omega^2-\delta^2=\Omega^2} + e^{i\frac{\tilde\Omega}{2}t}(\tilde\Omega-\delta)^2\right) \quad // \checkmark$$

$$\bar U_{11} = \bar U_{22}^* \checkmark$$

$$\bar U_{21} = \frac{2i\,\Omega^*(\tilde\Omega-\delta)}{2\cdot\Omega\,(\tilde\Omega-\delta)}\sin\frac{\tilde\Omega}{2}t \checkmark \quad \text{ídem para } \bar U_{12}$$

Como $\bar U$ coincide según ambos métodos $b_0(t)$, $b_1(t)$ para cada caso es el mismo, por lo que no repito el análisis (ver página anterior).

- $f(\hat{N})\cdot\hat{a} = \hat{a}\, f(\hat{N}-\mathbb{I})$

siendo $f(\hat{N})$ una función continuamente diferenciable del operador número $\hat{N}$, y siendo $\hat{a}$ el operador de bajada, y $\hat{a}^+$ el de subida

Partimos de:

$\hat{N} = \hat{a}^+\hat{a}$

$f(\hat{N}) = \sum_n c_n \hat{N}^n$ (desarrollo en serie de potencias)

$[\hat{a},\hat{a}^+] = \mathbb{I} \equiv 1 = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - \hat{N}$

↳ demostraremos por inducción:

$\hat{N}\cdot\hat{a} = (\hat{a}^+\hat{a})\hat{a} = (\hat{a}\hat{a}^+ - 1)\hat{a} = \hat{a}\hat{a}^+\hat{a} - \hat{a} = \hat{a}(\hat{a}^+\hat{a} - 1)$

$= \hat{a}(\hat{N}-1)$ → Es válido para $n = 1$

Suponemos válido para n:

$\hat{N}^n\hat{a} = \hat{a}(\hat{N}-1)^n$ → Demostramos para $n+1$:

$\hat{N}^{n+1}\hat{a} = \hat{N}(\hat{N}^n\hat{a}) = \hat{N}\hat{a}(\hat{N}-1)^n = \hat{a}(\hat{N}-1)(\hat{N}-1)^n = \hat{a}(\hat{N}-1)^{n+1}$ // ✓

Por tanto:

$f(\hat{N})\cdot\hat{a} = \sum c_n \hat{N}^n\cdot\hat{a} = \sum c_n\,\hat{a}(\hat{N}-1)^n = \hat{a}\cdot f(\hat{N}-1)$ // cqd.

Por ejemplo, si $f(\hat{N}) = e^{i\omega t\hat{N}}$ (desarrollable en serie)

↳ $e^{i\omega t\hat{N}}\cdot\hat{a} = \hat{a}\, e^{i\omega t(\hat{N}-1)}$ //

↳ $e^{i\omega t\hat{N}}\hat{a}\,e^{-i\omega t\hat{N}} = \hat{a}\cdot e^{i\omega t(\hat{N}-1)}\cdot e^{-i\omega t\hat{N}} = \hat{a}\cdot e^{-i\omega t}$ //

- $f(\hat{N})\cdot\hat{a}^+ = \hat{a}^+ f(\hat{N}+\mathbb{I})$

Idem por inducción:

$\hat{N}\hat{a}^+ = \hat{a}^+(\hat{a}\hat{a}^+) = \hat{a}^+(\hat{N}+1)$ ✓

$\hat{N}^n\hat{a}^+ = \hat{a}^+(\hat{N}+1)^n$

↳ $\hat{N}^{n+1}\hat{a}^+ = \hat{N}\cdot\hat{N}^n\hat{a}^+ = \hat{N}\hat{a}^+(\hat{N}+1)^n = \hat{a}^+(\hat{N}+1)^{n+1}$ ✓

- $\hat{a}\cdot f(\hat{N}) = f(\hat{N}+1)\cdot\hat{a}$

Idem:

$\hat{a}\hat{N} = \hat{a}\hat{a}^+\hat{a} = (\hat{N}+1)\hat{a}$

$\hat{a}\hat{N}^n = (\hat{N}+1)^n\hat{a}$

↳ $\hat{a}\hat{N}^n\hat{N} = (\hat{N}+1)^n\hat{a}\hat{N} = (\hat{N}+1)^{n+1}\hat{a}$

- $\hat{a}^+ f(\hat{N}) = f(\hat{N}-1)\hat{a}^+$

$\hat{a}^+\hat{N} = \hat{a}^+\hat{a}^+\hat{a} = \hat{a}^+(\hat{a}\hat{a}^+ - 1) = \hat{a}^+\hat{a}\hat{a}^+ - \hat{a}^+ = (\hat{N}-1)\hat{a}^+$

$\hat{a}^+\hat{N}^n = (\hat{N}-1)^n\hat{a}^+$

$\hat{a}^+\hat{N}^{n+1} = \hat{a}^+\hat{N}^n\hat{N} = (\hat{N}-1)^n\hat{a}^+\hat{N} = (\hat{N}-1)^{n+1}\hat{a}^+$ ✓

- Sea $\hat{A}$ un operador y $f(\hat{A})$ una función cont. diferenciable del mismo:

↳ $\hat{A}\cdot f(\hat{A}) = \hat{A}\sum_n c_n\hat{A}^n = \sum_n c_n\hat{A}^n\cdot\hat{A} = f(\hat{A})\cdot\hat{A}$ // → commutan porque $[\hat{A},\hat{A}]=0$

↳ $[\hat{A}, f(\hat{A})] = 0$

• $(\hat{N}-j)\,\hat{a} = \hat{a}(\hat{N}-(j+1))$

$(\hat{N}-j)\,\hat{a} = (\hat{a}\hat{a}^{+}-1-j)\,\hat{a} = \hat{a}(\hat{N}-(j+1))$

• $(\hat{N}-j)\,\hat{a}^{+} = (\hat{a}^{+}\hat{a})\hat{a}^{+} - j\hat{a}^{+} = \hat{a}^{+}(\hat{N}+1-j) = \hat{a}^{+}(\hat{N}-(j-1))$

• $\hat{a}(\hat{N}-j) = \hat{a}(\hat{a}^{+}\hat{a}-j) = (\hat{N}+1-j)\,\hat{a} = (\hat{N}-(j-1))\,\hat{a}$

• $\hat{a}^{+}(\hat{N}-j) = a^{+}(\hat{a}\hat{a}^{+}-1-j) = (\hat{N}-(j+1))\,\hat{a}^{+}$

• $(\hat{N}-j)^{n}\,\hat{a} = \hat{a}(\hat{N}-(j+1))^{n}$

↳ Cierto para $n=1$

↳ Para $n+1$:

$(N-j)^{n+1}\,\hat{a} = (\hat{N}-j)\cdot\hat{a}(\hat{N}-(j+1))^{n} = \hat{a}(\hat{N}-(j+1))^{n+1}$ ✓

• $\hat{N}^{n}\,\hat{b}^{j} = \hat{b}^{j}(\hat{N}-j)^{n}$ → es cierto para $j=1$ (ver página anterior)

$b\equiv a$ → uso indistintamente

Demuestro para $j+1$:

$\hat{N}^{n}\,\hat{b}^{j+1} = \hat{N}^{n}\,\hat{b}^{j}\hat{b} = \hat{b}^{j}(\hat{N}-j)^{n}\,\hat{b} = \hat{b}^{j}\,\hat{b}(\hat{N}-(j+1))^{n} = \hat{b}^{j+1}(\hat{N}-(j+1))^{n}$ ✓

• $(\hat{N}+j)\,\hat{a} = \hat{a}(\hat{N}+(j-1))$ ··· Cambiar signo de j en expresiones anteriores

• $(\hat{N}+j)\,\hat{a}^{+} = \hat{a}^{+}(\hat{N}+(j+1))$ → $\cdot(\hat{N}-j)$

• $\hat{a}(\hat{N}+j) = (\hat{N}+(j+1))\,\hat{a}$

• $\hat{a}^{+}(\hat{N}+j) = (\hat{N}+(j-1))\,\hat{a}^{+}$ • $(\hat{N}+j)^{n}\,\hat{a} = \hat{a}(\hat{N}+(j-1))^{n} \Rightarrow \hat{a}(\hat{N}+j)^{n} = (\hat{N}+j+1)^{n}\hat{a}$

• $\hat{b}^{j}\,\hat{N}^{n} = (\hat{N}+j)^{n}\,\hat{b}^{j}$ → Cierto para $j=1$ (pág. ant.)

Para $j+1$:

$\hat{b}^{j+1}\hat{N}^{n} = \hat{b}\,\hat{b}^{j}\hat{N}^{n} = \hat{b}\,(N+j)^{n}\,\hat{b}^{j} = (\hat{N}+(j+1))\,\hat{b}^{j+1}$ ✓

• $(\hat{N}+j)^{n}\,\hat{b}^{+} = \hat{b}^{+}(\hat{N}+(j+1))^{n}$ → cierto para $n=1$

Para $n+1$:

$(\hat{N}+j)^{n+1}\,\hat{b}^{+} = (\hat{N}+j)\,\hat{b}^{+}(\hat{N}+(j+1))^{n} = \hat{b}^{+}(\hat{N}+(j+1))^{n+1}$

• $\hat{b}^{+}(\hat{N}-j)^{n} = (\hat{N}-(j+1))^{n}\,\hat{b}^{+}$ → cierto para $n=1$

para $n+1$ →

$\hat{b}^{+}(\hat{N}-j)^{n}(\hat{N}-j) = (\hat{N}-(j+1))^{n}\,\hat{b}^{+}(\hat{N}-j) = (\hat{N}-(j+1))^{n+1}\,\hat{b}^{+}$

• $\hat{N}^{n}\,\hat{b}^{+j} = \hat{b}^{+j}(\hat{N}+j)^{n}$ → Cierto para $j=1$ (pág. anterior)

para $j+1$: $\hat{N}^{n}\,\hat{b}^{+j}\hat{b}^{+} = \hat{b}^{+j}(\hat{N}+j)^{n}\,\hat{b}^{+} = \hat{b}^{+(j+1)}(\hat{N}+(j+1))^{n}$ ✓

• $\hat{b}^{+j}\,\hat{N}^{n} = (\hat{N}-j)^{n}\,\hat{b}^{+j}$ → cierto para $j=1$ (ver pág. anterior)

para $j+1$:

$\hat{b}^{+}\hat{b}^{+j}\hat{N}^{n} = \hat{b}^{+}(\hat{N}-j)^{n}\,\hat{b}^{+j} = (\hat{N}-(j+1))^{n}\,\hat{b}^{+(j+1)}$ ✓

2) $\hat{H} = H_0 + H_{int} = \underbrace{\hbar\omega_r \hat{b}^+\hat{b} + \frac{\hbar\omega_A}{2}\hat{\sigma}_z}_{H_0} + \underbrace{\frac{\hbar}{2}\Omega_s(\hat{\sigma}_+ + \hat{\sigma}_-)\cdot(\hat{F}e^{-i\omega_L t} + \hat{F}^+ e^{+i\omega_L t})}_{H_{int}}$

↳ Enunciado: $U' \equiv U^+$

Cambio de imagen:

$U = e^{\frac{i H_0 t}{\hbar}} \rightarrow |\bar{\psi}\rangle = U|\psi\rangle$; $U^+U = I$; $\frac{d(I)}{dt} = 0 = U^+\dot{U} + \dot{U}^+U$

$\Rightarrow U^+\dot{U} = -\dot{U}^+U$

o bien: $0 = U\dot{U}^+ + \dot{U}U^+$

$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$

$i\hbar\frac{d}{dt}(U^+|\bar{\psi}\rangle) = \hat{H}\overbrace{U^+U}^{I}|\psi\rangle = \hat{H}U^+|\bar{\psi}\rangle \quad \cdot U|$

$\overset{U^+U\equiv I}{↳ i\hbar U(\dot{U}^+|\bar{\psi}\rangle + U^+\frac{d}{dt}|\bar{\psi}\rangle) = i\hbar U\dot{U}^+|\bar{\psi}\rangle + i\hbar\frac{d}{dt}|\bar{\psi}\rangle}$

↳ $i\hbar\frac{d}{dt}|\bar{\psi}\rangle = \underbrace{(U\hat{H}U^+ - i\hbar U\dot{U}^+)}_{\bar{H}}|\bar{\psi}\rangle$

$\bar{H} = U\hat{H}U^+ + i\hbar\dot{U}U^+$

$\dot{U} = \frac{iH_0}{\hbar}U = U\cdot\frac{iH_0}{\hbar} \rightarrow$ como $U(\hat{H}_0)$, conmutan $(\hat{A}\cdot f(\hat{A}) = f(\hat{A})\cdot\hat{A})$

↳ $\dot{U}U^+ = \frac{iH_0}{\hbar}UU^+ = \frac{i}{\hbar}H_0 \rightarrow i\hbar\dot{U}U^+ = -H_0$

$U\hat{H}U^+ = UH_0U^+ + UH_{int}U^+$

Como $U = f(H_0)$, conmuta con H_0, idem para U^+.

↳ $UH_0U^+ = H_0UU^+ = H_0$

↳ $\bar{H} = H_0 + UH_{int}U^+ - H_0 = UH_{int}U^+$

producto tensorial $\mathcal{H}_{rad} \otimes \mathcal{H}_{átomo}$

Como $\hat{N} = \hat{b}^+\hat{b}$ → campo de radiación (cuantos) y $\hat{\sigma}_z$ → átomo de dos niveles actúan sobre espacios distintos, podemos factorizar la exponencial y actuar sólo sobre ~~la~~ espacio, conmutando los operadores de espacios distintos.

$U = e^{i\omega_r\hat{N}}\cdot e^{i\frac{\omega_A}{2}\hat{\sigma}_z}$; $U^+ = e^{-i\omega_r\hat{N}}e^{-i\frac{\omega_A}{2}\hat{\sigma}_z}$

$\bar{H} = \frac{\hbar}{2}\Omega_s e^{\frac{i\omega_A}{2}\hat{\sigma}_z t}(\hat{\sigma}_+ + \hat{\sigma}_-)e^{-i\frac{\omega_A}{2}\hat{\sigma}_z t}\cdot e^{i\omega_r\hat{N}t}(\hat{F}_{\dots} + \hat{F}^+_{\dots})\cdot e^{-i\omega_r\hat{N}t}$

$e^{i\frac{\omega_A}{2}\hat{\sigma}_z t}\hat{\sigma}_\pm e^{-i\frac{\omega_A}{2}\hat{\sigma}_z t}$; $\hat{\sigma}_+ = |e\rangle\langle f|$, $\hat{\sigma}_- = |f\rangle\langle e|$

$\hat{\sigma}_z = |e\rangle\langle e| - |f\rangle\langle f|$

$\hat{\sigma}_z\cdot\hat{\sigma}_+ = |e\rangle\langle f| = \hat{\sigma}_+$; $\hat{\sigma}_+\hat{\sigma}_z = -|e\rangle\langle f| = -\hat{\sigma}_+$

$\hat{\sigma}_z\cdot\hat{\sigma}_- = -|f\rangle\langle e| = -\hat{\sigma}_-$; $\hat{\sigma}_-\hat{\sigma}_z = |f\rangle\langle e| = \hat{\sigma}_-$

↳ $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$

↳ $[\hat{\sigma}_z, [\hat{\sigma}_z, \hat{\sigma}_\pm]] = (\pm 2)^2\hat{\sigma}_\pm$

⋮

→ Útil para aplicar tma. de Hadamard:

$e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{B} + x[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$

$\hat{A} = \hat{\sigma}_z$

$\hat{B} = \hat{\sigma}_\pm$

$x = \frac{i\omega_A t}{2}$

↳ $\underbrace{[\hat{\sigma}_z, [\hat{\sigma}_z, [\hat{\sigma}_z, [\dots, [\sigma_z, \sigma_\pm]]]]\dots]}_{n \text{ veces}} = (\pm 2)^n\hat{\sigma}_\pm$

Cierto para $n=1$.

Para $n+1$: inducción:

$[\hat{\sigma}_z, [\dots]]]\dots] = [\hat{\sigma}_z, (\pm 2)^n\hat{\sigma}_\pm] = (\pm 2)^n[\hat{\sigma}_z, \hat{\sigma}_\pm]$

$= (\pm 2)^{n+1}\hat{\sigma}_\pm$ ✓

$\rightarrow e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{\sigma}_\pm + x(\pm 2)\hat{\sigma}_\pm + \frac{x^2}{2!}(\pm 2)^2\hat{\sigma}_\pm + \ldots \overset{\pm 2x \equiv y_\pm}{=} \hat{\sigma}_\pm + y_\pm \hat{\sigma}_\pm + \frac{y_\pm^2}{2!}\hat{\sigma}_\pm + \ldots$

$$= \hat{\sigma}_\pm \left(1 + y_\pm + \frac{y_\pm^2}{2!} + \ldots\right) = \hat{\sigma}_\pm e^{y_\pm} = \hat{\sigma}_\pm e^{\pm i\omega_A t}$$

$\hookrightarrow \hat{\hat{H}} = \frac{\hbar}{2}\Omega_s(\hat{\sigma}_+ \cdot e^{i\omega_A t} + \hat{\sigma}_- e^{-i\omega_A t}) \ldots (\hat{F}\ldots + \hat{F}^+ \ldots)\ldots$

Analicemos el primer sumando del 2º factor, con $\hat{F} = e^{i\varepsilon(\hat{b}+\hat{b}^+)}$

$e^{i\varepsilon(\hat{b}+\hat{b}^+)} \neq e^{i\varepsilon\hat{b}} \cdot e^{i\varepsilon\hat{b}^+}$

$\downarrow$

$\exp(A+B) = \exp\left(\frac{1}{2}[A,B]\right)\cdot \exp(B)\cdot \exp(A)$ sii $[A,[A,B]] = 0$ con $[A,B]\neq 0$

$\hookrightarrow$ En este caso: $A = \hat{b}$, $B = \hat{b}^+$; $[A,B] = \mathbb{I}$, $[A,[A,B]] = 0$ $\rightarrow$ válido aplicar tma.

$e^{i\varepsilon(\hat{b}+\hat{b}^+)} = \exp\left(\frac{1}{2}(i\varepsilon)^2\right)\cdot \exp(i\varepsilon\hat{b}^+)\exp(i\varepsilon\hat{b})$

Analizamos término parcial: $\rightarrow$ se podría hacer con tma. de Hadamard quizá, pero aprovecho propiedades calculadas 2 pág. antes.

$$e^{i\omega_T\hat{N}t}\cdot e^{i\varepsilon\hat{b}^+} \overset{\text{desarrollo en serie}}{=} \sum_n \frac{(i\omega_T t)^n}{n!}\hat{N}^n \cdot \sum_j \frac{(i\varepsilon)^j}{j!}\cdot \hat{b}^{+j} =$$

$$= \sum_j \frac{(i\varepsilon)^j}{j!}\sum_n \frac{(i\omega_T t)^n}{n!}\cdot \underbrace{\hat{N}^n \hat{b}^{+j}}_{\text{propiedad 2 p. antes}} = \sum_j \frac{(i\varepsilon)^j}{j!}\sum_n \frac{(i\omega_T t)^n}{n!}\hat{b}^{+j}(\hat{N}+j)^n$$

$$= \sum_j \frac{(i\varepsilon)^j}{j!}\cdot \hat{b}^{+j}\cdot e^{i\omega_T t\cdot(\hat{N}+j)} \overset{\text{separo, } [\mathbb{I},\hat{N}]=0}{=} \sum_j \frac{(i\varepsilon)^j}{j!}\left(\hat{b}^+\cdot e^{i\omega_T t}\right)^j \cdot e^{i\omega_T t\hat{N}}$$

$$= \exp\left(i\varepsilon\hat{b}^+ e^{i\omega_T t}\right)\cdot e^{i\omega_T\cdot t\cdot\hat{N}}$$

$\rightarrow$ factor $e^{-i\omega_L t}$ sale fuera, lo añadimos al final

$$e^{i\varepsilon\hat{b}}\cdot e^{-i\omega_T\hat{N}t} = \sum_j \frac{(i\varepsilon)^j}{j!}\hat{b}^j \sum_n \frac{(-i\omega_T t)^n}{n!}\hat{N}^n = \sum_j \frac{(i\varepsilon)^j}{j!}\sum_n \frac{(-i\omega_T t)^n}{n!}\cdot(\hat{N}+j)^n \hat{b}^j$$

$$= \sum_j \frac{(i\varepsilon)^j}{j!}\hat{b}^j \cdot e^{-i\omega_T t(\hat{N}+j)} = \sum_j \frac{(i\varepsilon)^j}{j!}\left(\hat{b}\cdot e^{-i\omega_T t}\right)^j \cdot e^{-i\omega_T t\hat{N}}$$

¡IMPORTA EL ORDEN!

$$= \exp(-i\omega_T t\hat{N})\cdot \exp(i\varepsilon\hat{b}e^{-i\omega_T t})$$

$\hookrightarrow e^{i\omega_T\hat{N}t} e^{i\varepsilon\hat{b}^+}\cdot e^{i\varepsilon\hat{b}}\cdot e^{-i\omega_T\hat{N}t}\cdot e^{\frac{1}{2}(i\varepsilon)^2} = \exp(i\varepsilon\hat{b}^+ e^{i\omega_T t})\cdot \mathbb{I}\cdot \exp(i\varepsilon\hat{b}e^{-i\omega_T t})\cdot e^{\frac{1}{2}(i\varepsilon)^2}$

$\rightarrow$ Reagrupamos exponenciales, válido porque $[A,B]$ sigue siendo $(i\varepsilon)^2$ ya que $e^{-i\omega_T t}e^{i\omega_T t} = \mathbb{I}$

$$= \exp\left(i\varepsilon\left(\hat{b}e^{-i\omega_T t} + \hat{b}^+ e^{i\omega_T t}\right)\right) \equiv \hat{\hat{F}}$$

Idem para el término con $\hat{F}^+$

Si $e^{i\omega_T t\hat{N}}\cdot\hat{F}\cdot e^{-i\omega_T t\hat{N}} = \hat{\hat{F}}$ $\rightarrow$ | Tomamos adjuntos, cambia el orden al trasponer

$\hookrightarrow (e^{-i\omega_T t\hat{N}})^+\cdot\hat{F}^+(e^{i\omega_T\hat{N}})^+ = e^{i\omega_T t\hat{N}}\hat{F}^+ e^{-i\omega_T\hat{N}t} = \hat{\hat{F}}$ que es lo que queríamos calcular.

$$\hookrightarrow \hat{\hat{H}} = \frac{\hbar}{2}\Omega_s(\hat{\sigma}_+ e^{i\omega_A t} + \hat{\sigma}_- e^{-i\omega_A t})\left(\hat{\hat{F}}e^{-i\omega_L t} + \hat{\hat{F}}^+ e^{i\omega_L t}\right)$$

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• Demostrar que $\langle K | a^{+(m+l)} a^{(m+l)} | K \rangle = \frac{K!}{(K-m-l)!}$, con $m+l \leq K$, $K \in \mathbb{N}$

$\quad\quad \rightarrow$ base estados de Fock

Renombro $m+l \equiv n$

$\hookrightarrow \langle K | (a^+)^n a^n | K \rangle$, con $n \leq K$

$a^+ ; a \equiv$ operadores de creac y destrucción.

$a|K\rangle = \sqrt{K}\, |K-1\rangle$

$a^+|K\rangle = \sqrt{K+1}\, |K+1\rangle$

⊛ Calculemos $(a^+)^n |j\rangle$ por inducción.

$n=1: a^+|j\rangle = \sqrt{j+1}\, |j+1\rangle$

$n=2: (a^+)^2 |j\rangle = a^+ (a^+|j\rangle) = \sqrt{j+1}\, a^+ |\underbrace{j+1}_{\equiv j'}\rangle = \sqrt{j+1}\sqrt{j'+1}\, |j'+1\rangle = \sqrt{j+1}\sqrt{j+2}\, |j+2\rangle$

$\downarrow$ Propongo:

$(a^+)^n |j\rangle = \sqrt{\frac{(j+n)!}{j!}}\, |j+n\rangle$, válido para $n = 1,2$

Compruebo para $n+1$:

$(a^+)^{n+1} |j\rangle = \sqrt{\frac{(j+n)!}{j!}}\, a^+ |j+n\rangle = \sqrt{\frac{(j+n)!}{j!}} \cdot \sqrt{j+(n+1)}\, |j+(n+1)\rangle$

$= \sqrt{\frac{(j+(n+1))!}{j!}}\, |j+(n+1)\rangle$ ✓

⊛ Idem para $a^n |K\rangle$

$n=1 \quad a|K\rangle = \sqrt{K}\, |K-1\rangle$

$n=2 \quad a^2|K\rangle = \sqrt{K}\sqrt{K-1}\, |K-2\rangle$

$\downarrow$

$a^n|K\rangle = \sqrt{\frac{K!}{(K-n)!}}\, |K-n\rangle$

$n+1$

$\hookrightarrow a^{n+1}|K\rangle = \sqrt{\frac{K!}{(K-n)!}}\sqrt{K-n}\, |K-(n+1)\rangle = \sqrt{\frac{K!}{(K-(n+1))!}}\, |K-(n+1)\rangle$ ✓

Por tanto

$\langle K | (a^+)^n a^n | K \rangle = \langle K | (a^+)^n \cdot \sqrt{\frac{K!}{(K-n)!}}\, |\overset{j}{\overset{|||}{K-n}}\rangle = \sqrt{\frac{K!}{(K-n)!}} \langle K | (a^+)^n | j \rangle$

$= \sqrt{\frac{K!}{(K-n)!}} \langle K | \sqrt{\frac{((K-n)+n)!}{(K-n)!}}\, |(K-n)+n\rangle = \frac{K!}{(K-n)!} \underbrace{\langle K | K \rangle}_{1} = \frac{K!}{(K-(m+l))!}$

O bien:

$\langle K | (a^+)^n a^n | K \rangle = (a^n|K\rangle)^+ a^n |K\rangle = \frac{K!}{(K-n)!} \langle K-n | K-n \rangle =$ idem

$\hookrightarrow$ porque $(a^n)^+ = (a^+)^n$

c.q.d.

• Sea $P_m(T) = \sum_{K=m}^{\infty} \binom{K}{m} \mu^m (1-\mu)^{K-m} \dfrac{\langle n\rangle^K e^{-\langle n\rangle}}{K!}$ para estados coherentes iniciales

$\dfrac{\langle n\rangle^K e^{-\langle n\rangle}}{K!} = |\langle K|\alpha\rangle|^2$, $K \geq m$

Demostrar:

⊛ $\sum_{m=0}^{\infty} P_m(T) = 1$

$$\sum_{m=0}^{\infty}\sum_{K=m}^{\infty} \frac{K!}{m!(K-m)!} \mu^m (1-\mu)^{K-m} \cdot \frac{\langle n\rangle^{K-m+m} e^{-\langle n\rangle}}{K!}$$

$$= \sum_{m=0}^{\infty} e^{-\langle n\rangle} \frac{\mu^m \langle n\rangle^m}{m!} \sum_{K=m}^{\infty} \frac{(1-\mu)^{K-m}\langle n\rangle^{K-m}}{(K-m)!} = \quad K-m \equiv q$$

$$= e^{-\langle n\rangle} \sum_{m=0}^{\infty} \frac{(\mu\langle n\rangle)^m}{m!} \sum_{q=0}^{\infty} \frac{((1-\mu)\langle n\rangle)^q}{q!}$$

$$= e^{-\langle n\rangle} \sum_{m=0}^{\infty} \frac{(\mu\langle n\rangle)^m}{m!} \cdot e^{(1-\mu)\langle n\rangle} = e^{-\mu\langle n\rangle} \cdot e^{\mu\langle n\rangle} = 1 \checkmark$$

⊛ $\overline{m} = \sum_{m=0}^{\infty} m P_m(T) = \mu\langle n\rangle$) Parto de este valor de P_m directamente

$$\overline{m} = \sum_{m=0}^{\infty} m \cdot e^{-\mu\langle n\rangle} \cdot \frac{(\mu\langle n\rangle)^m}{m!} = 0 + \sum_{m=1}^{\infty} e^{-\mu\langle n\rangle} \frac{(\mu\langle n\rangle)^{m-1}\cdot\mu\langle n\rangle}{(m-1)!} = \quad m-1 \equiv q$$

$$= \sum_{q=0}^{\infty} e^{-\mu\langle n\rangle} \mu\langle n\rangle \cdot \frac{(\mu\langle n\rangle)^q}{q!} = \mu\langle n\rangle \cdot e^{-\mu\langle n\rangle} \cdot e^{\mu\langle n\rangle} = \mu\langle n\rangle \checkmark$$

⊛ $\overline{m^2} = \sum_{m=0}^{\infty} m^2 P_m(T) = \mu^2\langle n^2\rangle + \mu(1-\mu)\langle n\rangle$

$$\overline{m^2} = e^{-\mu\langle n\rangle} \sum_{m=0}^{\infty} m^2 \frac{(\mu\langle n\rangle)^m}{m!} = 0 + e^{-\mu\langle n\rangle}\sum_{m=1}^{\infty} \frac{m(\mu\langle n\rangle)^m}{(m-1)!} = \sum_{m=1}^{\infty} \frac{((m-1)+1)(\mu\langle n\rangle)^m}{(m-1)!} e^{-\mu\langle n\rangle}$$

$$= \left(0 + \sum_{m=2}^{\infty} \frac{(\mu\langle n\rangle)^{m-2+2}}{(m-2)!} + \sum_{m=1}^{\infty} \frac{(\mu\langle n\rangle)^{m-1+1}}{(m-1)!}\right) e^{-\mu\langle n\rangle} = \quad q = m-2, \; r = m-1$$

$$= e^{-\mu\langle n\rangle}\left(\sum_{q=0}^{\infty} (\mu\langle n\rangle)^2 \frac{\mu^q\langle n\rangle^q}{q!} + \sum_{r=0}^{\infty} \mu\langle n\rangle \frac{\mu^r\langle n\rangle^r}{r!}\right)$$

$$= e^{-\mu\langle n\rangle} \cdot e^{\mu\langle n\rangle}\left(\mu^2\langle n\rangle^2 + \mu\langle n\rangle\right) = \mu^2\langle n\rangle^2 + \mu\langle n\rangle \checkmark$$

Aparte: operador número; $[a, a^+] = 1$; $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\hat{a}^+ = \langle\alpha|\alpha^*$

$$\langle \hat{n}^2\rangle_\alpha = \langle (a^+a)^2\rangle_\alpha = \langle a^+(a a^+)a\rangle_\alpha = \langle a^+a\rangle_\alpha + \langle a^{+2}a^2\rangle_\alpha$$

estado coherente

$$= \langle n\rangle_\alpha + \langle\alpha| a^{+2}a^2|\alpha\rangle = \langle n\rangle_\alpha + (\alpha^*)^2\alpha^2 \underbrace{\langle\alpha|\alpha\rangle}_{=1}$$

$$\langle n\rangle_\alpha = \langle\alpha|a^+a|\alpha\rangle = \alpha^*\alpha \longrightarrow \quad = \langle n\rangle_\alpha + (\langle n\rangle_\alpha)^2$$

$$\overline{m^2} = \mu\langle n\rangle + \mu^2(\langle n^2\rangle - \langle n\rangle) = \mu^2\langle n^2\rangle + \mu(1-\mu)\langle n\rangle \checkmark$$

⊛ $\mathrm{Var}(m) = \mu^2\,\mathrm{Var}(n) + \mu(1-\mu)\langle n\rangle$

$$\mathrm{Var}(m) = \overline{m^2} - \overline{m}^2 = \mu^2\langle n^2\rangle + \mu(1-\mu)\langle n\rangle - \mu^2\langle n\rangle^2$$

$$= \mu^2(\langle n^2\rangle - \langle n\rangle^2) + \mu(1-\mu)\langle n\rangle = \mathrm{Var}(n)\cdot\mu^2 + \mu(1-\mu)\langle n\rangle \checkmark$$

⊛ $(\mathrm{Var}(m) - \overline{m})/\overline{m}^2 = (\mathrm{Var}(n) - \langle n\rangle)/\langle n\rangle^2$

$$= \frac{\mu^2\mathrm{Var}(n) + \mu(1-\mu)\langle n\rangle - \mu\langle n\rangle}{\mu^2\langle n\rangle^2} = \frac{\mu^2\mathrm{Var}(n) - \mu^2\langle n\rangle}{\mu^2\langle n\rangle^2} = \frac{\mathrm{Var}(n) - \langle n\rangle}{\langle n\rangle^2} \checkmark$$

c.q.d.

• Dado el hamiltoniano $\hat{H} = \underbrace{\hbar\omega\,\hat{a}^+\hat{a}}_{\hat{H}_0} + \underbrace{\hbar(g\hat{a}^2 e^{2i\omega t} + g^*\hat{a}^{+2} e^{-2i\omega t})}_{\hat{H}_{int}}$, escribirlo en la imagen de interacción. (Transformación $\hat{U} = e^{\frac{i\hat{H}_0 t}{\hbar}}$)

$$\hat{\bar{H}} = \hat{U}\hat{H}\hat{U}^+ + i\hbar\dot{\hat{U}}\hat{U}^+ = \hat{U}\hat{H}_0\hat{U}^+ + \hat{U}\hat{H}_{int}\hat{U}^+ + i\hbar\frac{i}{\hbar}\hat{H}_0\hat{U}\hat{U}^+ \qquad \hat{U}\hat{U}^+ = \mathbb{I}$$

$$= \hat{U}\hat{H}_{int}\hat{U}^+ = e^{i\omega t\,\hat{a}^+\hat{a}}\,\hat{H}_{int}\,e^{-i\omega t\,\hat{a}^+\hat{a}}$$

Función del operador nº $\hat{N} = \hat{a}^+\hat{a}$: $f(\hat{N}) = e^{i\omega t\hat{N}}$, $\hat{N} = \hat{N}^+$

$$* \; e^{i\omega t\hat{a}^+\hat{a}}\,\hbar g\,\hat{a}^2 e^{2i\omega t} = e^{2i\omega t}\,\hbar g \cdot f(\hat{N})\cdot\hat{a}^2$$

En un problema ya entregado vimos que $f(\hat{N})\hat{a} = \hat{a}f(\hat{N}-1)$

$$= \hbar g\,e^{2i\omega t}\,\hat{a}f(\hat{N}-1)\hat{a} = \hbar g\,e^{2i\omega t}\,\hat{a}^2 f(\hat{N}-2)$$

$$\to f(\hat{N})\cdot\hbar g\,\hat{a}^2 e^{2i\omega t}\cdot[f(\hat{N})]^+ = \hbar g\,e^{2i\omega t}\,\hat{a}^2\cdot f(\hat{N}-2)[f(\hat{N})]^+$$

$$= \hbar g\,e^{2i\omega t}\,\hat{a}^2\,e^{i\omega t(\hat{N}-2)}\cdot e^{-i\omega t\hat{N}} = \hbar g\,\hat{a}^2$$

$\to [\hat{N},\hat{N}] = 0$

Idem para el 2º término:

Vimos que $f(\hat{N})\cdot\hat{a}^+ = \hat{a}^+ f(\hat{N}+1)$

$$e^{i\omega t\hat{N}}\,\hbar g^*\,\hat{a}^{+2}e^{-2i\omega t}\,e^{-i\omega t\hat{N}} = \hbar g^* e^{-2i\omega t}\,\hat{a}^{+2}\cdot e^{i\omega t(\hat{N}+2)}\cdot e^{-i\omega t\hat{N}}$$

$$= \hbar g^*\,\hat{a}^{+2}$$

Por tanto, $\hat{\bar{H}} = \hbar(g\hat{a}^2 + g^*\hat{a}^{+2})$ ✓ c.q.d.

• Demostrar que $\hat{D}(\alpha)\,\hat{a}\,\hat{D}^+(\alpha) = \hat{a} - \alpha$, donde $\hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}\cdot e^{-\frac{1}{2}[\hat{A},\hat{B}]} \quad \text{si } [[\hat{A},\hat{B}],\hat{A}/\hat{B}] = 0 \to \text{es el caso ✓}$$

$[A,B] = \alpha(-\alpha^*)\cdot\underbrace{[\hat{a}^+,\hat{a}]}_{-1} = |\alpha|^2$

$$\hat{D}(\alpha)\,\hat{a}\,\hat{D}^+(\alpha) = (e^{-\frac{|\alpha|^2}{2}}\cdot e^{\alpha\hat{a}^+}e^{-\alpha^*\hat{a}})\cdot\hat{a}\cdot(e^{\frac{|\alpha|^2}{2}}e^{\alpha^*\hat{a}}e^{-\alpha\hat{a}^+})$$

$$= e^{\alpha\hat{a}^+}\,\hat{a}\,e^{-\alpha\hat{a}^+} = \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}(\hat{a}^+)^m\,\hat{a}\,e^{-\alpha\hat{a}^+}$$

Ya vimos que $\begin{cases} f(\hat{N})(\hat{a}^+)^k = (\hat{a}^+)^k f(\hat{N}+k) \\ f(\hat{N})\hat{a}^+ = \hat{a}^+ f(\hat{N}+1) \end{cases}$

$$\to (\hat{a}^+)^m\,\hat{a} = (\hat{a}^+)^{m-1}\hat{a}^+\hat{a} = (\hat{a}^+)^{m-1}\hat{N} \Rightarrow \hat{a}^+ f(\hat{N}) = f(\hat{N}-1)\,a^+$$

$(\hat{a}^+)^k f(\hat{N}) = f(\hat{N}-k)(\hat{a}^+)^k$

$$\to (\hat{a}^+)^{m-1}\hat{N} = (\hat{N}-(m-1))(\hat{a}^+)^{m-1} = (\hat{N}+1-m)(\hat{a}^+)^{m-1}$$

$[\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{N} = 1$

$$= (\hat{a}\hat{a}^+ - m)(\hat{a}^+)^{m-1} = \hat{a}(\hat{a}^+)^m - m(\hat{a}^+)^{m-1}$$

$$\to \sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\hat{a}(\hat{a}^+)^m e^{-\alpha\hat{a}^+} + (-)\sum_{m=0}^{\infty}\frac{(\alpha)^m m(\hat{a}^+)^{m-1}}{m!}e^{-\alpha\hat{a}^+}$$

$$= \hat{a}\,e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+} - \alpha\sum_{m=1}^{\infty}\frac{\alpha^{m-1}(\hat{a}^+)^{m-1}}{(m-1)!}e^{-\alpha\hat{a}^+} \overset{m-1\equiv q}{=}$$

$$= \hat{a} - \alpha\,e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+} = \hat{a} - \alpha$$

• $\hat{D}(\alpha)\,\hat{a}^{+}\,\hat{D}^{+}(\alpha) = \hat{a}^{+} - \alpha^{*}$ → relación complejo hermítico de la anterior

$\hookrightarrow = e^{-\alpha^{*}\hat{a}}\, e^{\alpha\hat{a}^{+}}\, e^{\frac{|\alpha|^2}{2}}\, \hat{a}^{+}\, e^{-\alpha\hat{a}^{+}}\, e^{\alpha^{*}\hat{a}}\, e^{-\frac{|\alpha|^2}{2}}$

$= e^{-\alpha^{*}\hat{a}}\, \hat{a}^{+}\, e^{\alpha^{*}\hat{a}} = e^{-\alpha^{*}\hat{a}} \sum_{m=0}^{\infty} \hat{a}^{+}\,\hat{a}\,\hat{a}^{m-1} \frac{(\alpha^{*})^{m}}{m!}$

$= e^{-\alpha^{*}\hat{a}} \sum_{m=0}^{\infty} \hat{N}\,\hat{a}^{m-1} \frac{(\alpha^{*})^{m}}{m!}$

→ Lo demostramos en otro ejercicio entregado

$\hat{N}\,\hat{a}^{m-1} = \hat{a}^{m-1}\left(\hat{N} - (m-1)\right) = \hat{a}^{m-1}\,(\underbrace{\hat{N}+1}_{\hat{a}\hat{a}^{+}} - m) = \hat{a}^{m}\,\hat{a}^{+} - \hat{a}^{m-1}\cdot m$

$\hookrightarrow \hat{D}(\alpha)\,\hat{a}^{+}\,\hat{D}^{+}(\alpha) = e^{-\alpha^{*}\hat{a}} \sum_{m=0}^{\infty} \hat{a}^{m}\,\hat{a}^{+} \frac{(\alpha^{*})^{m}}{m!} - \sum_{m=0}^{\infty} e^{-\alpha^{*}\hat{a}} \frac{(\alpha^{*})(\alpha^{*})^{m-1}}{(m-1)!}\,\hat{a}^{m-1}$

$\swarrow m-1=q$

$= e^{-\alpha^{*}\hat{a}}\, e^{\alpha^{*}\hat{a}}\left(\hat{a}^{+} - \alpha^{*}\right) = \hat{a}^{+} - \alpha^{*}$ //

Las últimas dos propiedades se pueden demostrar más fácilmente utilizando el teorema de Hadamard:

$e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots$

• $e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}\,\hat{a}\,e^{\alpha^{*}\hat{a} - \alpha\hat{a}^{+}}$ → $\hat{A} = \alpha\hat{a}^{+} - \alpha^{*}\hat{a}$, $\hat{B} = \hat{a}$

$[\hat{A},\hat{B}] = \alpha[\hat{a}^{+},\hat{a}] - 0 = -\alpha\cdot\mathbb{I}$

$[A,[A,B]] = 0$ también los sucesivos porque $[A,B] \propto I$ → serie se trunca

$\hookrightarrow e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}\,\hat{a}\,e^{-(\alpha\hat{a}^{+} - \alpha^{*}\hat{a})} = \hat{a} - \alpha$ // ✓

• $e^{\hat{A}}\,\hat{a}^{+}\,e^{-\hat{A}}$ (Ahora $\hat{B} = \hat{a}$)

Mismo procedimiento:

$[\hat{A},\hat{B}] = -\alpha^{*}[\hat{a},\hat{a}^{+}] + 0 = -\alpha^{*}$

$e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{a}^{+} - \alpha^{*}$ // ✓

- Sea $|z;\alpha\rangle$ un estado de Yuen; ...... luz comprimida con
  - $\hat{b}|z;\alpha\rangle = \alpha|z;\alpha\rangle$
  - $\langle z;\alpha|\hat{b}^+ = \alpha^* \langle z;\alpha|$ $\Big\}$ $\langle z;\alpha|z;\alpha\rangle = 1$
  - $\hat{b} = \mu\hat{a} + \nu\hat{a}^+$; $\hat{b}^+ = \mu\hat{a}^+ + \nu^*\hat{a}$
  - $\mu, \nu, \alpha \in \mathbb{R}$; $\mu^2 - \nu^2 = 1$
  - $\hat{X} = \dfrac{\hat{a} + \hat{a}^+}{2}$
  - $\hat{Y} = -i\dfrac{(\hat{a} - \hat{a}^+)}{2}$
  - $\hat{N} = \hat{a}^+\hat{a}$

a) Probar que $\langle\hat{X}\rangle = \alpha(\mu - \nu)$

$\hookrightarrow \hat{b} + \hat{b}^+ = (\mu+\nu)(\hat{a} + \hat{a}^+) \quad \Big| \cdot (\mu-\nu)/2$

$$\frac{(\mu-\nu)}{2}(\hat{b}+\hat{b}^+) = (\mu^2-\nu^2)\frac{(\hat{a}+\hat{a}^+)}{2} = \frac{\hat{a}+\hat{a}^+}{2} = \hat{X}$$

$$\langle\hat{X}\rangle = \langle z;\alpha|\hat{X}|z;\alpha\rangle = \left[\langle z;\alpha|\hat{b}|z;\alpha\rangle + \langle z;\alpha|\hat{b}^+|z;\alpha\rangle\right]\cdot\frac{(\mu-\nu)}{2}$$

$$= \langle z;\alpha|\alpha|z;\alpha\rangle \cdot 2 \cdot \frac{\mu-\nu}{2} = \alpha(\mu-\nu) \;// \checkmark$$

b) $\langle\hat{Y}\rangle = 0$

$$\hat{b} - \hat{b}^+ = (\mu-\nu)(\hat{a}-\hat{a}^+) \rightarrow \left(-\frac{i}{2}\right)(\hat{b}-\hat{b}^+)(\mu+\nu) = \left(-\frac{i}{2}\right)\cdot(\mu^2-\nu^2)(\hat{a}-\hat{a}^+) = \hat{Y}$$

$$\langle\hat{Y}\rangle = -\frac{i}{2}(\mu+\nu)\left(\langle z;\alpha|\hat{b}|z;\alpha\rangle - \langle z;\alpha|\hat{b}^+|z;\alpha\rangle\right) = -\frac{i}{2}(\mu+\nu)(\alpha-\alpha) = 0 \;//$$

c) $\langle\hat{N}\rangle = \alpha^2(\mu-\nu)^2 + \nu^2$

$$\hat{b}\cdot\nu - \mu\hat{b}^+ = \mu\nu\hat{a} + \nu^2\hat{a}^+ - \mu^2\hat{a}^+ - \mu\nu\hat{a} = (\nu^2-\mu^2)\hat{a}^+ = -\hat{a}^+$$

$$\hat{b}\cdot\mu - \nu\hat{b}^+ = \mu^2\hat{a} + \nu\mu\hat{a}^+ - \mu\nu\hat{a}^+ - \nu^2\hat{a} = \hat{a}$$

$$\hat{N} = \hat{a}^+\hat{a} = (\mu\hat{b}^+ - \nu\hat{b})\cdot(\hat{b}\mu - \nu\hat{b}^+) = \mu^2\hat{b}^+\hat{b} - \mu\nu(\hat{b}^+)^2 - \mu\nu(\hat{b})^2 + \nu^2\hat{b}\hat{b}^+$$

$$[\hat{b},\hat{b}^+] = [\mu\hat{a} + \nu\hat{a}^+, \mu\hat{a}^+ + \nu\hat{a}] = \mu^2[\hat{a},\hat{a}^+] + \nu^2[\hat{a}^+,\hat{a}] = \mu^2 - \nu^2 = 1$$

$$= \hat{b}\hat{b}^+ - \hat{b}^+\hat{b} \longrightarrow \hat{b}\hat{b}^+ = \hat{b}^+\hat{b} + 1$$

$$\hat{N} = (\mu^2+\nu^2)\hat{b}^+\hat{b} + \nu^2 - \mu\nu\left((\hat{b}^+)^2 + \hat{b}^2\right)$$

$$\langle\hat{N}\rangle = (\mu^2+\nu^2)\cdot\alpha^2 + \nu^2 - \mu\nu(\alpha^2+\alpha^2) = \alpha^2(\mu^2 - 2\mu\nu + \nu^2) + \nu^2$$

$$= \alpha^2(\mu-\nu)^2 + \nu^2 \;// \checkmark$$

ya que

$$\langle z;\alpha|\hat{b}^+\hat{b}|z;\alpha\rangle = \langle z;\alpha|\alpha\cdot\alpha|z;\alpha\rangle = \alpha^2$$

$$\langle z;\alpha|\hat{b}^2|z;\alpha\rangle = \langle z;\alpha|\hat{b}\cdot\alpha|z;\alpha\rangle = \alpha\langle z;\alpha|\alpha|z;\alpha\rangle = \alpha^2$$

$$\langle z;\alpha|\hat{b}^{+2}|z;\alpha\rangle = \langle z;\alpha|\alpha\hat{b}^+|z;\alpha\rangle = \alpha\langle z;\alpha|\alpha|z;\alpha\rangle = \alpha^2$$

d) $\Delta\hat{x} = \frac{\mu-\nu}{2} = e^{-r}$

↳ $\hat{x} = \frac{\mu-\nu}{2}(\hat{b}+\hat{b}^{+})$ $= \hat{b}^{+}\hat{b}+1$

$$\hat{x}^2 = \left(\frac{\mu-\nu}{2}\right)^2\left(\hat{b}^2+\hat{b}^{+2}+\hat{b}^{+}\hat{b}+\hat{b}\hat{b}^{+}\right) = \left(\frac{\mu-\nu}{2}\right)^2\left(\hat{b}^2+\hat{b}^{+2}+2\hat{b}^{+}\hat{b}+1\right)$$

$$\langle\hat{x}^2\rangle = \left(\frac{\mu-\nu}{2}\right)^2(4\alpha^2+1) = (\mu-\nu)^2\cdot\alpha^2 + \frac{(\mu-\nu)^2}{2^2}$$

$$(\langle\hat{x}\rangle)^2 = \alpha^2(\mu-\nu)^2$$

$$\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \frac{\mu-\nu}{2} \;// \;✓$$

$$\left.\begin{array}{l}\mu = \cosh r = \frac{e^{r}+e^{-r}}{2}\\ \nu = \sinh r = \frac{e^{r}-e^{-r}}{2}\end{array}\right\} \rightarrow \begin{array}{l}\mu-\nu = \frac{e^{r}-e^{r}}{2} + \frac{e^{-r}+e^{-r}}{2} = e^{-r}\;✓\\ \mu+\nu = \frac{e^{r}+e^{r}}{2} + \frac{e^{-r}-e^{-r}}{2} = e^{r}\;✓\end{array}$$

e) $\Delta\hat{y} = \frac{\mu+\nu}{2} = e^{r}$

$$\hat{y}^2 = \left(-\frac{i}{2}\right)^2(\mu+\nu)^2\left(\hat{b}^2+\hat{b}^{+2}-\hat{b}^{+}\hat{b}-\hat{b}\hat{b}^{+}\right) = -\left(\frac{\mu+\nu}{2}\right)^2\left(\hat{b}^2+\hat{b}^{+2}-2\hat{b}^{+}\hat{b}-1\right)$$

$$\langle\hat{y}^2\rangle = -\left(\frac{\mu+\nu}{2}\right)^2(\alpha^2+\alpha^2-2\alpha^2-1) = \left(\frac{\mu+\nu}{2}\right)^2$$

$$\Delta\hat{y} = \sqrt{\langle\hat{y}^2\rangle - \underset{0}{\langle\hat{y}\rangle}^2} = \frac{\mu+\nu}{2} = e^{r} \;//\; ✓$$

f) $\Delta\hat{N} = \sqrt{\alpha^2(\mu-\nu)^4 + 2\mu^2\nu^2}$

Previamente, calculamos (N-ordenamos) los siguientes productos de operadores que nos saldrán al elevar $\hat{N}^2$:

- $\hat{b}\hat{b}^{+} = \hat{b}^{+}\hat{b} + 1$ // ; $[b, b^{+}] = 1$

$$[b,b^{+}] = [\mu\hat{a}+\nu\hat{a}^{+}, \mu\hat{a}^{+}+\nu^*\hat{a}] = \mu^2\overbrace{[\hat{a},\hat{a}^{+}]}^{1} + \mu\nu^*\cancel{[\hat{a},\hat{a}]} + \nu\mu\overbrace{[\hat{a}^{+},\hat{a}^{+}]}^{0} + |\nu|^2\underbrace{[\hat{a}^{+},\hat{a}]}_{-1} = \mu^2-|\nu|^2 = 1 \;✓$$

→ Omito sombreros por simplificar notación, pero recordando que son operadores y no conmutan

- $b^{+}bb^{+}b = b^{+}(b^{+}b+1)b = b^{+2}b^2 + b^{+}b$ //
- $b^{+}b\,b^{+2} = b^{+}(b^{+}b+1)b^{+} = b^{+2}bb^{+} + b^{+2} = b^{+2}(b^{+}b+1) + b^{+2} = b^{+3}b + 2b^{+2}$ //
- $b^2b^{+}b = (b^{+}bb^{+2})^{+} \stackrel{\downarrow}{=} b^{+}b^3 + 2b^2$ //
- $b^2b^{+2} = b(b^{+}b+1)b^{+} = bb^{+}bb^{+} + bb^{+} = bb^{+}(bb^{+}+1) = (b^{+}b+1)(b^{+}b+2)$

$$\stackrel{\wedge}{=} b^{+}bb^{+}b + 3b^{+}b + 2 = b^{+2}b^2 + b^{+}b + 3b^{+}b + 2 = b^{+2}b^2 + 4b^{+}b + 2 \;//$$

↓

Estos operadores están N-ordenados (b^{+} a la izqda., b a la dcha.) $\alpha^*\in\mathbb{R}$

$:O: = K\cdot(b^{+})^{\gamma}(b)^{\delta}$ → Como $b|z;\alpha\rangle = \alpha|z;\alpha\rangle$; $\langle z;\alpha|b^{+} = \alpha^*\langle z;\alpha|$

↳ $\langle z;\alpha|:O:|z;\alpha\rangle = \langle z;\alpha|\alpha^{\gamma}\alpha^{\delta}|z;\alpha\rangle = \alpha^{\gamma+\delta}$ //

$$\hat N^2 = \hat N\cdot\hat N \overset{\text{ver c)}}{=} \left[(\mu^2+\nu^2)(b^+b) - \mu\nu(b^2+b^{+2})+\nu^2\right]\cdot\left[(\mu^2+\nu^2)(b^+b)-\mu\nu(b^2+b^{+2})+\nu^2\right]$$

$$= (\mu^2+\nu^2)^2(\underline{b^+b\,b^+b}) - \mu\nu(\mu^2+\nu^2)(b^+b^3 + \underline{b^+b\,b^{+2}}) + (\mu^2+\nu^2)\nu^2 b^+b$$

$$-\mu\nu(\mu^2+\nu^2)(\underline{b^2b^+b} + b^{+3}b) + \mu^2\nu^2(b^4 + \underline{b^2b^{+2}} + b^{+2}b^2 + b^{+4}) - \mu\nu^3(b^2+b^{+2})$$

$$+\nu^2(\mu^2+\nu^2)(b^+b) - \mu\nu^3(b^2+b^{+2}) + \nu^4$$ ← Sustituyo factores subrayados, calculados en pág. anterior para N-ordenar

$$= (\mu^2+\nu^2)^2(b^{+2}b^2+b^+b) - (\mu^2+\nu^2)\mu\nu(b^+b^3 + b^{+3}b + 2b^{+2}) + (\mu^2+\nu^2)\nu^2 b^+b$$

$$-\mu\nu(\mu^2+\nu^2)\cdot(b^+b^3+2b^2+b^{+3}b) + \mu^2\nu^2(b^4+b^{+2}b^2+4b^+b+2+b^{+2}b^2+b^{+4}) - \mu\nu^3(b^2+b^{+2})$$

$$+\nu^2(\mu^2+\nu^2)b^+b - \mu\nu^3(b^2+b^{+2})+\nu^4$$ → Todos N-ordenados → $\langle\ \rangle$ implica cambiar $b, b^+ \to \alpha$

$$\langle\hat N^2\rangle = (\mu^2+\nu^2)^2(\alpha^4+\alpha^2) - 2\mu\nu(\mu^2+\nu^2)\cdot(\alpha^4+\alpha^2) + (\mu^2+\nu^2)\nu^2\alpha^2$$

$$-2\mu\nu(\mu^2+\nu^2)(\alpha^4+\alpha^2) + 2\mu^2\nu^2(2\alpha^4+2\alpha^2+1) - 2\mu\nu^3\alpha^2 + \nu^2(\mu^2+\nu^2)\alpha^2 - 2\mu\nu^3\alpha^2 + \nu^4$$

$$= (\mu^2+\nu^2)^2(\alpha^2+1)\alpha^2 - 4\mu\nu(\mu^2+\nu^2)\alpha^2(\alpha^2+1) - 4\mu\nu^3\alpha^2 + 2(\mu^2+\nu^2)\alpha^2\nu^2 + 2\mu^2\nu^2(2\alpha^2(\alpha^2+1)+1)+\nu^4$$

$$= (\mu^2+\nu^2)\alpha^2(\alpha^2+1)\cdot\left[\mu^2+\nu^2-4\mu\nu\right] - 4\mu\nu^3\alpha^2 + 2\alpha^2\mu^2\nu^2 + 2\alpha^2\nu^4 + 2\mu^2\nu^2(2\alpha^2(\alpha^2+1)+1)+\nu^4$$

$$= \alpha^2(\alpha^2+1)\cdot(\mu^4+2\nu^2\mu^2-4\mu^3\nu+\nu^4-4\mu\nu^3) - 4\mu\nu^3\alpha^2 + 2\mu^2\nu^2(2\alpha^4+3\alpha^2+1)+\nu^4(1+2\alpha^2)$$

$$= (\alpha^4+\alpha^2)\mu^4 - \mu^3\nu\, 4(\alpha^4+\alpha^2) - 4\mu\nu^3(\alpha^4+2\alpha^2) + 2\mu^2\nu^2(3\alpha^4+4\alpha^2+1) + \nu^4(\alpha^4+3\alpha^2+1) \;//$$

$$(\langle\hat N\rangle)^2 \overset{\text{ver c)}}{=} (\alpha^2(\mu-\nu)^2+\nu^2)^2 = \alpha^4(\mu-\nu)^4 + 2\alpha^2\nu^2(\mu-\nu)^2+\nu^4 = \alpha^4(\mu^4-4\mu^3\nu+6\mu^2\nu^2-4\mu\nu^3+\nu^4)$$

$$+2\alpha^2\nu^2(\mu^2-2\mu\nu+\nu^2)+\nu^4 = \mu^4(\alpha^4) - \mu\nu^3(4\alpha^4+4\alpha^2) - \mu^3\nu(4\alpha^4) + \mu^2\nu^2(6\alpha^4+2\alpha^2) + \nu^4(\alpha^4+2\alpha^2+1)$$

$$= \mu^4\alpha^4 - \mu^3\nu\cdot4\cdot(\alpha^4) - 4\mu\nu^3(\alpha^4+\alpha^2) + 2\mu^2\nu^2(3\alpha^4+\alpha^2) + \nu^4(\alpha^4+2\alpha^2+1) \;//$$

$$(\Delta\hat N)^2 = \langle\hat N^2\rangle - \langle\hat N\rangle = \mu^4\alpha^2 - \mu^3\nu 4\alpha^2 + 2\mu^2\nu^2(3\alpha^2+1) + \nu^4\cdot\alpha^2 - 4\mu\nu^3\alpha^2$$

$$= \alpha^2(\mu^4-4\mu^3\nu+6\mu^2\nu^2-4\mu\nu^3+\nu^4)+2\mu^2\nu^2 = \alpha^2(\mu-\nu)^4+2\mu^2\nu^2$$

$$\Rightarrow \Delta\hat N = \sqrt{\alpha^2(\mu-\nu)^4+2\mu^2\nu^2} \;// \;✓$$

Si $\alpha=0$ (vacío comprimido)

↳ $\Delta\hat N = \sqrt{2}\,\mu\nu \neq 0$ → No es vacío puro

$\langle\hat N\rangle = \nu \neq 0$ → Hay cuantos (fotones) excitados y fluctuaciones.