CROMODINÁMICA CUÁNTICA

Problemas propuestos (2011-2012)

---

1. Calcular, en función de las variables de Mandelstam s, t y u, la sección eficaz diferencial de alguna de las siguientes colisiones partónicas (tomar $m_q = 0$):

   a) $q + \bar{q} \to q + \bar{q}$
   b) $q + q \to q + q$
   c) $q + g \to q + g$
   d) $q + \bar{q} \to g + g$
   e) $g + g \to q + \bar{q}$

2. Calcular la parte divergente de alguna de las siguientes funciones de Green y deducir el (los) correspondiente(s) factor(es) Z de renormalización:

   a) Autoenergía del quark
   b) Autoenergía del gluón
   c) Autoenergía del fantasma
   d) Vértice quark-gluón
   e) Vértice fantasma-gluón
   f) Vértice de 3 gluones
   g) Vértice de 4 gluones

3. Considerar el proceso $e^+e^- \to q\bar{q}g$, al orden más bajo en teoría de perturbaciones. Calcular la fracción de sucesos de 3 jets con $s_{ij} > y\, s$, $R_3(y)$.

4. Considerar el Lagrangiano de QCD $\mathcal{L}_{\mathrm{QCD}}^{(n_F)}$, con $n_F - 1$ sabores de quarks ligeros ($m_q \approx 0$) y un quark pesado de masa M. Para $\mu < M$ podemos eliminar el quark pesado de la acción ("integrate out"); la teoría efectiva resultante es $\mathcal{L}_{\mathrm{QCD}}^{(n_F-1)}$ (más operadores con dimensión mayor que 4, suprimidos por potencias de $1/M$, que vamos a despreciar). Dado que la función β depende del número de sabores, $\beta_1 = (2n_F - 11N_C)/6$, los dos Lagrangianos tienen acoplamientos distintos, relacionados por la condición $[L \equiv \log(\mu/M)]$:

$$\alpha_s^{(n_F)}(\mu^2) \;=\; \alpha_s^{(n_F-1)}(\mu^2) \left\{ 1 + \sum_{k=1} C_k(L) \left(\frac{\alpha_s^{(n_F-1)}(\mu^2)}{\pi} \right)^k \right\} ,$$

   a) Sabiendo que $C_1(0) = 0$, determinar $C_1(L)$.

   b) Calcular en las dos teorías la dependencia en momento de $\alpha_s(Q^2)$ a dos loops (NLO), en función de β_1 y β_2.

   c) Particularizarlo a $n_F = 5$ y representar gráficamente $\alpha_s(Q^2)$ entre 3 y 100 GeV.

<u>Nota:</u> Es necesario entregar un mínimo de 2 problemas. La fecha límite para entregarlos es el miércoles 22 de febrero.

Fernando Hueso González
Interacciones fuertes
Máster de Física Avanzada 2011/12

①

→ seguiré notación: y un signo global en los vértices parecida al Pascual y Tarrach (p.35) pero cambiando el signo de g_s, como sugiere Pich.

0 • Deducción de las reglas de Feynman de QCD

Partimos del lagrangiano de QCD incluyendo los ghosts (Faddeev-Popov)

$$\mathcal{L}_{QCD} = -\frac{1}{4}(\partial^\mu G^\nu{}_a - \partial^\nu G^\mu{}_a)\cdot(\partial_\mu G_\nu{}^a - \partial_\nu G_\mu{}^a) + \sum_q \bar{q}_\alpha[i\gamma^\mu\partial_\mu - m_q]\cdot q^\alpha$$

$$-\frac{1}{2}\sum_q g_s(\bar{q}_\alpha(\lambda^a)_{\alpha\beta}\gamma^\mu q_\beta)G^a_\mu \to \mathcal{L}_1$$

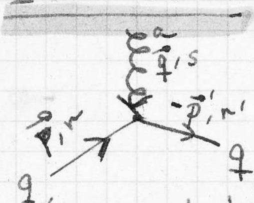
$$+\frac{1}{2}g_s f_{abc}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)G^\mu_b G^\nu_c \to \mathcal{L}_2$$

$$-\frac{1}{4}g_s^2 f_{abc}f_{ade}G^\mu_b G^\nu_c G^d_\mu G^e_\nu \to \mathcal{L}_3$$

$$-\partial_\mu\bar{\phi}_a\partial^\mu\phi^a \overbrace{-\partial_\mu\bar{\phi}_a\, g_s f_{abc}\phi^b G_c^\mu}^{\to\mathcal{L}_4}$$

Tenemos 4 partes del lagrangiano que darán lugar a vértices. Omitiremos por simplicidad el índice de sabor A, que se conserva en la int. fuerte.

1er vértice: (asociado a $\mathcal{L}_1$) → $q g q$: Vértice fermiónico



Nótese que todos los momentos son entrantes.
r, r', s : polarización
α, β : color
a : tipo de gluón

Por el tma. de Gellmann-Löw, se tiene que:

$$S = \overset{\text{producto T-ordenado}}{T}\left(\exp\left\{i\int d^4x\cdot\underset{\text{lagrangiano de interacción}}{\mathcal{L}_I^I(x)}\right\}\right)$$

y a 1er orden en tª de perturbaciones:

$$S - I = A \simeq i\int d^4x\ T(\mathcal{L}_I(x))$$

La amplitud de probabilidad de transición de un estado inicial a final:

$$A_{i\to f} = A_{fi} = \langle f|A|i\rangle = S_{fi}$$

Para este vértice $|i\rangle = a^+_\alpha(\vec{p}, r)\, a^+_a(\vec{q}, s)\,|0\rangle$

$$|f\rangle = \langle 0|\, a_\beta(-\vec{p}', r')$$

De todos los términos de $\mathcal{L}$, el único que contribuye a este vértice es $\mathcal{L}_1$ (las contracciones de las patas con los restantes dan 0).

Además, utilizaremos que:

$$q_\alpha(x) = \sum_{\vec{k},\sigma}\left(a_\alpha(\vec{k},\sigma)u_\alpha(\vec{k},\sigma)e^{-ikx} + b^+_\alpha(\vec{k},\sigma)v_\alpha(\vec{k},\sigma)e^{ikx}\right) \qquad \alpha = 1,2,3 \text{ (color)}$$

$$G^\mu{}_a(x) = \sum_{\vec{k},\sigma}\left(a_a(\vec{k},\sigma)\cdot\epsilon^\mu{}_a(\vec{k},\sigma)e^{-ikx} + a^+_a(\vec{k},\sigma)\epsilon^{*\mu}{}_a(\vec{k},\sigma)e^{ikx}\right) \qquad a = 1,\ldots,8 \text{ (tipo gluón)}$$

y que $\overbrace{a_\alpha(\vec{k},\sigma)a^+_\beta(\vec{p},s)}\equiv\underset{\text{conmutador/anticonmutador según si es bosón o fermión}}{[a_\alpha(\vec{k},\sigma), a^+_\beta(\vec{p},s)]} = \Delta_{\vec{k}\vec{p}}\,\delta_{\sigma s}\cdot\delta_{\alpha\beta}$
idem con b b^+

y 0 para las combinaciones restantes. $\bar{q} = q^+\gamma^0$

Además, $\sum_{\vec{k},\sigma} f(\vec{k},\sigma)\cdot\Delta_{\vec{k},\vec{p}}\cdot\delta_{\sigma s} = f(\vec{p}, s)$ $\|$ $\int d^4x\, e^{ikx} = \delta^{(4)}(k)\cdot(2\pi)^4$

todas posibles

Una vez reunidos estos ingredientes: . . para no repetir índices

$$A_{fi} = \int < 0 | \underbrace{a_\beta(-\vec{p}\,', r') \cdot \bar{q}_j} \lambda^b{}_{jk} \gamma^\mu \underbrace{q_k G^b_\mu | a^+_\alpha(\vec{p}, r)}\, \underbrace{ a^+_a(\vec{q}, s)} | 0 > d^4x \cdot \left(-i\frac{g_s}{2}\right)$$

. . . no hay signo − porque el cruce es con línea bosónica

$$= \int d^4x \cdot \sum_{\vec{k},\sigma} \bar{u}_j(\vec{k},\sigma) \cdot e^{+ikx} \cdot \Delta_{\vec{k},-\vec{p}\,'} \delta_{j\beta} \delta_{r',\sigma} \cdot \lambda^b{}_{jk} \cdot \gamma^\mu \cdot \sum_{\vec{q}\,',\sigma'} \epsilon^b_\mu(\vec{q}\,',\sigma') \cdot e^{-iq'x} \Delta_{\vec{q}\,'\vec{q}} \cdot \overset{\delta_{s,\sigma'}}{\checkmark}\, \delta_{ba}$$

$$\cdot \sum_{\vec{k}'',\sigma''} u_k(\vec{k}'',\sigma'') e^{-ik''x} \Delta_{\vec{k}'',\vec{p}} \delta_{k\alpha} \delta_{\sigma'',r} \cdot \left(-i\frac{g_s}{2}\right)$$

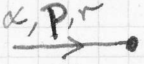
$$= \int d^4x \cdot \bar{u}_\beta(-\vec{p}\,', r') e^{-ip'x} \lambda^a{}_{\beta\alpha} \gamma^\mu \epsilon^a_\mu(\vec{q}, s) \cdot e^{-iqx} \cdot u_\alpha(\vec{p}, r) e^{-ipx} \cdot \left(-i\frac{g_s}{2}\right)$$

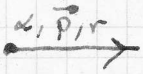
$$= -i \cdot (2\pi)^4 \cdot \delta^{(4)}(p + p' + q) \cdot \bar{u}_\beta(-\vec{p}\,', r') \, g_s \frac{\lambda^a{}_{\beta\alpha}}{2} \gamma^\mu u_\alpha(\vec{p}, r) \cdot \epsilon^a_\mu(\vec{q}, s)$$

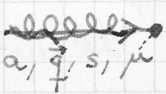
$$= -i \cdot (2\pi)^4 \cdot \delta^{(4)}(p + p' + q) \cdot \mathcal{M}$$

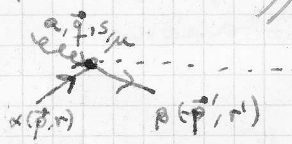
↳ Conservación del cuadrimomento (todos entrantes)

De donde se deducen estas reglas de Feynman:

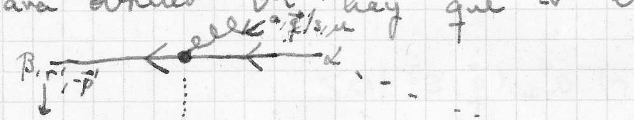
Quark in:  $u_\alpha(\vec{p}, r)$

" out:  $\bar{u}_\alpha(\vec{p}, r)$

Gluón in:  $\epsilon^a_\mu(\vec{q}, s)$

Vértice fermiónico  $g_s \frac{\lambda^a{}_{\beta\alpha}}{2} \gamma^\mu$

Para obtener $\mathcal{M}$ hay que ir en sentido contrario a las flechas

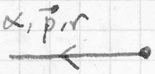


$$\bar{u}_\beta(-\vec{p}\,', r') \, g_s \gamma^\mu \left(\frac{\lambda^a_{\beta\alpha}}{2}\right) \cdot \epsilon^a_\mu(\vec{q}, s) \cdot u_\alpha(\vec{p}, r)$$

Análogamente para antiquarks, sólo que cambiará $\underset{\bar{u}}{u}$ por $\underset{v}{\bar{v}}$

Ya que:

$$\underbrace{\bar{q}_\beta(x)\, d^+_\alpha}(\vec{p}, r) = \bar{v}_\alpha(\vec{p}, r) e^{-ipx} \; ; \quad b_\alpha \underbrace{(\vec{p}, r)\, q(x)} = v_{\alpha(\vec{p},r)} e^{ipx}$$

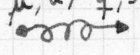
Antiquark in:  $\bar{v}_\alpha(\vec{p}, r)$

" out:  $v_\alpha(\vec{p}, r)$

↳ Saldría un signo − global, que asociamos a una regla extra de Feynman: cruce de líneas fermiónicas da un signo − (sólo importan signos relativos entre diagramas).

Idem para gluón out, sale conjugado:

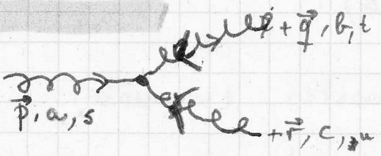
$$\underbrace{a_a(\vec{q}, s) \; G_\mu^b}(x) = \epsilon^*{}_\mu{}^a(\vec{q}, s) \, e^{iqx}$$

Gluón out: $\epsilon^*{}_\mu{}^a(\vec{q}, s)$ 

NOTA: La diferencia con P y T es que hay un cambio de signo en g_s, y un signo global (−) en todos los vértices. ②

↳ porque lo engloba en regla ii) (i^{n+1})

2º vértice: $(\mathcal{L}_2) \rightarrow ggg$ (triple gluón)



Momentos y gluones todos entrantes con $\vec{p}_i > 0$ (o bien cambiar el signo del momento a los salientes)

evito repetir índices

$$A_{fi} = i\frac{g_s}{2}\int \langle 0 | a^+_b(+\vec{q},t)\, a^+_c(+\vec{r},u)\, T\Big[f_{def}\,(\partial_\alpha G_\beta{}^d - \partial_\beta G_\alpha{}^d)\, G^\alpha{}_e\, G^\beta{}_f\Big] | a^+_a(\vec{p},s) | 0\rangle\, d^4x$$

$$= A_1 + A_2 + A_3$$

↗ 6 combinaciones posibles para contraer (3!)

↳ Supongamos que a^+_a se contrae con el término de las derivadas. (A_1) los a_b, a_c pueden ir a G_e G_f o viceversa. Por tanto, y ahorrando pasos intermedios:

$$A_1 = \frac{ig_s}{2}\int\Big[\Big(\partial_\alpha\big(\epsilon_\beta{}^a{}_{(\vec{p},s)}e^{-ipx}\big) - \partial_\beta\big(\epsilon_\alpha{}^a{}_{(\vec{p},s)}e^{-ipx}\big)\Big)\cdot \epsilon^{\alpha(+\vec{q},t)}{}_b\,\epsilon^{\beta(+\vec{r},u)}{}_c\cdot f_{abc}\cdot e^{i(-q)x}\cdot e^{i(-r)x}$$

$$+\big(\;''\;\big)\cdot \epsilon^{\alpha(+\vec{r},u)}{}_c\,\epsilon^{\beta(+\vec{q},t)}{}_b\, f_{acb}\quad ''\;\Big]\cdot d^4x$$

para poder juntar términos y simplificar.

↗ uso antisimetría de f_{abc} y lo saco factor común, y además intercambio $\alpha \leftrightarrow \beta$ en el 2º término (mudos)

$$= (-i)\cdot(2\pi)^4\cdot\delta^{(4)}(p+q+r)\cdot\left(-\frac{g_s}{2}\right)\cdot f_{abc}\,\cdot$$

$$\Big\{\Big[(-ip_\alpha)\,\epsilon_\beta{}^a(\vec{p},s) - (-ip_\beta)\,\epsilon_\alpha{}^a(\vec{p},s)\Big]\cdot\epsilon^\alpha{}_b(\vec{q},t)\cdot\epsilon^p{}_c(\vec{r},u)$$

cambio $\alpha \leftrightarrow \beta$

$f_{acb} = -f_{abc}$

$$-\Big[(-ip_\beta)\,\epsilon_\alpha{}^a(\vec{p},s) - (-ip_\alpha)\,\epsilon_\beta{}^a(\vec{p},s)\Big]\cdot\epsilon^\beta{}_c(\vec{r},u)\cdot\epsilon^\alpha{}_b(\vec{q},t)\Big\}$$

↗ sumando, obtengo un factor 2, y quitando $(-i)(2\pi)^4\cdot\delta^{(4)}\cdot M_1 = A_1$

$$M_1 = -\frac{g_s}{\cancel{2}}\cdot\cancel{2}\cdot f_{abc}\cdot\epsilon^\alpha{}_b(\vec{q},t)\,\epsilon^\beta{}_c(-\vec{r},u)\cdot\Big[ip_\beta\,\epsilon_\alpha{}^a(\vec{p},s) - ip_\alpha\,\epsilon^a_p(\vec{p},s)\Big]$$

→ Quiero dejarlo como $\epsilon^\mu{}_a\,\epsilon^\nu{}_b\,\epsilon^\sigma{}_c\cdot V^{abc}_{\mu\nu\sigma}$ para sacar regla Feynman del vértice V

$$= -ig_s f_{abc}\cdot\epsilon^\alpha{}_b\,\epsilon^\beta{}_c\cdot\big[p_\beta\cdot\epsilon^\mu{}_a\cdot g_{\mu\alpha} - p_\alpha\,\epsilon^\mu{}_a\, g_{\mu\beta}\big]$$

→ cambio $\alpha \rightarrow \nu$, $\beta \rightarrow \sigma$

$$= \underbrace{\epsilon^\mu{}_a\cdot\epsilon^\nu{}_b\,\epsilon^\sigma{}_c}_{\substack{\text{reglas gluones}\\ \text{in}\quad\text{out}\quad\text{out}}}\cdot\underbrace{(-ig_s)\cdot f_{abc}\cdot(p_\sigma g_{\mu\nu} - p_\nu g_{\mu\sigma})}_{V_{1\,\mu\nu\sigma}^{abc}}$$

Para obtener M_2, que corresponde a b, $\vec{q}$, t yendo al término con derivadas, en lugar de a, $\vec{p}$, s basta con cambiar $\vec{p} \leftrightarrow q$, $s \leftrightarrow t$, $a \leftrightarrow b$. Como μ y ν me quedarán al revés en las ϵ, los intercambio también (mudos), para poder juntarlo con M_1.

respecto

$$M_2 = \epsilon^{\nu(\vec{q},t)}{}_b\,\epsilon^{\mu(\vec{p},s)}{}_a\,\epsilon^\sigma{}_c\,(-ig_s)\,(f_{bac})\,(q_\sigma\, g_{\nu\mu} - q_\mu\, g_{\nu\sigma}))$$

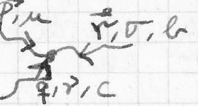
$f_{bac} = -f_{abc}$, $g_{\nu\mu} = g_{\mu\nu}$

$$= \epsilon^\mu{}_a\,\epsilon^\nu{}_b\,\epsilon^\sigma{}_c\,(-ig_s)\,f_{abc}\,(-)\cdot\,(q_\sigma\, g_{\mu\nu} - q_\mu\, g_{\nu\sigma})$$

Análogo con M_3: $q \leftrightarrow r$, $t \leftrightarrow u$, $\nu \leftrightarrow \sigma$, $b \leftrightarrow c$

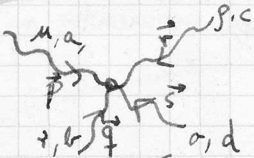
$$M_3 = \epsilon^\mu{}_a\,\epsilon^\nu{}_b\,\epsilon^\sigma{}_c\,(-ig_s)\,f_{acb}(-)(r_\nu\, g_{\mu\sigma} - r_\mu\, g_{\sigma\nu})$$

Sumando las 3 contribuciones, queda que:



$$V^{abc}_{\mu\nu\sigma} = -ig_s f^{abc}\cdot\left(g_{\mu\nu}(p-q)_\sigma + g_{\nu\sigma}(q-r)_\mu + g_{\sigma\mu}(r-p)_\nu\right)$$

3^er vértice : (α_3) gggg (gluón cuártico)



Supongo 4 gluones entrantes (momento,...)

Agí $= (-i)\,\frac{g_s^2}{4}\Big[\langle 0|T\int \cancel{f_{ghk}}\, f_{gij}\, G^{\alpha}{}_{g} G^{\beta}{}_{h} G_{\alpha}{}^{i} G_{\beta}{}^{j}\Big]\, a^+_a(\vec{p})\, a^+_b(\vec{q})\, a^+_c(\vec{r})\, a^+_d(\vec{s})\,|0\rangle$

↳ 4! combinaciones $= (4\cdot 3)\cdot 2 \Rightarrow (m_1 + m_2 + \cancel{m_3}\ldots + m_{12} = M_1 + M_2 + M_3 + M_4$

Si a se contrae con g, y b con h, c y d pueden ir a i y j o viceversa (sacando δ conservación)

↓

$$m_1 = \frac{g_s^2}{4}\cdot f_{fab}\left(f_{fcd}\, \epsilon^{\alpha}{}_{a} \epsilon^{\beta}{}_{b} \epsilon_{\alpha}{}^{c} \epsilon_{\beta}{}^{d} + f_{fdc}\, \epsilon^{\alpha}{}_{a} \epsilon^{\beta}{}_{b} \epsilon_{\alpha}{}^{d} \epsilon_{\beta}{}^{c}\right)$$

$$= \frac{g_s^2}{4} f_{fab} f_{fcd}\, \epsilon^{\alpha}{}_{a} \epsilon^{\beta}{}_{b} \left(\epsilon^{\gamma}{}_{c}\, g_{\alpha\gamma}\, \epsilon^{\delta}{}_{d}\, g_{\beta\delta} - \epsilon^{\delta}{}_{d}\, g_{\delta\alpha}\, \epsilon^{\gamma}{}_{c}\, g_{\beta\gamma}\right)$$

$$= \frac{g_s^2}{4}\cdot f_{fab} f_{fcd}\, \epsilon^{\alpha}{}_{a} \epsilon^{\beta}{}_{b} \epsilon^{\gamma}{}_{c} \epsilon^{\delta}{}_{d} \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right)$$

renombro $\alpha, \beta, \gamma, \delta = \mu, \nu, \rho, \sigma$; $f = e$

$$= \underbrace{\epsilon^{\mu}{}_{a} \epsilon^{\nu}{}_{b} \epsilon^{\rho}{}_{c} \epsilon^{\sigma}{}_{d}}_{g.\ \text{entrantes}} \cdot \underbrace{f_{eab} f_{ecd} \cdot \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) g_s^2/4}_{v^{abcd}_{\mu\nu\rho\sigma}}$$

También puede suceder que sea c quien se contraiga con h, y b y d tengan 2 opciones

~~m_2~~ Intercambiar $b \leftrightarrow c$, $\nu \leftrightarrow \rho$: (conmuto momentos...)

$m_2 = \epsilon^{\mu}{}_{a} \epsilon^{\nu}{}_{b} \epsilon^{\rho}{}_{c} \epsilon^{\sigma}{}_{d}\, f_{eac} f_{ebd} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\rho\nu}\right) g_s^2/4$

↳ Idem con d : (respecto a m_1 $b \leftrightarrow d$, $\nu \leftrightarrow \sigma$)

$m_3 =$ " $f_{ead} f_{ecb} \cdot \left(g_{\mu\rho} g_{\sigma\nu} - g_{\mu\nu} g_{\sigma\rho}\right) g_s^2/4$

Sumando y omitiendo las ϵ :

$$V_1 = v_1 + v_2 + v_3 = \frac{g_s^2}{4}\Big(\underline{f_{eab}\cdot f_{ecd}}_{A} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) + \underline{f_{eac} f_{ebd}}_{B} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\rho\nu}\right)\Big)$$

$$+ \underline{f_{ead} f_{ecb}}_{C} \left(g_{\mu\rho} g_{\sigma\nu} - g_{\mu\nu} g_{\sigma\rho}\right)$$

Ahora, hay 3 términos más intercambiando $a \leftrightarrow b, c, d$

$a \leftrightarrow b$, $\mu \leftrightarrow \nu$ $\quad V_2 = \frac{g_s^2}{4}\cdot\left(\underline{f_{eba} f_{ecd}}_{A} \left(g_{\nu\rho} g_{\mu\sigma} - g_{\nu\sigma} g_{\mu\rho}\right) + \underline{f_{ebc} f_{ead}}_{C} \left(g_{\nu\mu} g_{\rho\sigma} - g_{\nu\sigma} g_{\rho\mu}\right) + \underline{f_{ebd} f_{eca}}_{B} \left(g_{\nu\rho} g_{\sigma\mu} - g_{\nu\mu} g_{\sigma\rho}\right)\right)$

$a \leftrightarrow c$, $\mu \leftrightarrow \rho$ $\quad V_3 = \frac{g_s^2}{4}\cdot\left(\underline{f_{ecb} f_{ead}}_{C} \left(g_{\rho\mu} g_{\nu\sigma} - g_{\rho\sigma} g_{\nu\mu}\right) + \underline{f_{eca} f_{ebd}}_{B} \left(g_{\rho\nu} g_{\mu\sigma} - g_{\rho\sigma} g_{\mu\nu}\right) + \underline{f_{ecd} f_{eab}}_{A} \left(g_{\rho\mu} g_{\nu\sigma} - g_{\rho\nu} g_{\sigma\mu}\right)\right)$

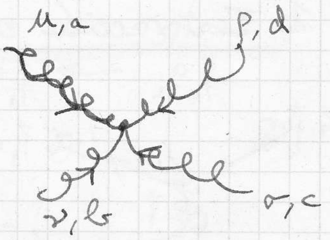
$a \leftrightarrow d$, $\nu \leftrightarrow \sigma$ $\quad V_4 = \frac{g_s^2}{4}\cdot\left(f_{edb} f_{eca}\cdot\left(g_{\sigma\rho} g_{\nu\mu} - g_{\sigma\mu} g_{\nu\rho}\right) + \underline{f_{edc} f_{eba}}_{A} \left(g_{\sigma\nu} g_{\rho\mu} - g_{\sigma\mu} g_{\rho\nu}\right) + f_{eda} f_{ecb} \left(g_{\sigma\rho} g_{\mu\nu} - g_{\sigma\nu} g_{\mu\rho}\right)\right)$

y sumando: $V = V_1 + V_2 + V_3 + V_4$, y utilizando antisimetría f_{abc}:

$$V = \frac{g_s^2}{4}\Big\{ \cdot f_{eab} f_{ecd} \cdot \big(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} - \left(g_{\nu\rho} g_{\mu\sigma} - g_{\nu\sigma} g_{\mu\rho}\right) + g_{\rho\mu} g_{\nu\sigma} - g_{\rho\nu} g_{\sigma\mu} + \left(g_{\sigma\nu} g_{\sigma\mu} - g_{\sigma\mu} g_{\rho\nu}\right)$$

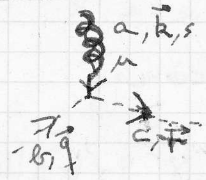
Cambio ρ por σ para parecerse al P y Tarrach (asigna ρ a d, σ a c y yo lo ~~hab~~ he hecho al revés)

$$V = g_s^2 \{ f_{abe} f_{cde} \cdot (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})$$
$$+ f_{ace} f_{bde} \cdot (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\nu\sigma})$$
$$+ \overset{\text{cambio}}{f_{ade} f_{cbe}} \cdot (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\sigma\rho}) \}$$



→ Idéntico a Pich cambiando
$e \to a$
$a \to b$
$b \to c$
$c \to d$
$d \to e$

4º vértice : $(\mathcal{L}_4)$ GgG (ghosts)



(Momentos entrantes)

$$A_{i\to f} = -i g_s f_{def} \int d^4x \langle 0| a_c(\vec{r})\, \partial_\mu \bar{\phi}_d\; \phi_e\, G_f^{\mu}\, a^{\dagger}_a(\vec{k},s)\; a^{\dagger}_b(\vec{q})|0\rangle$$

$$= -i(2\pi)^4 \cdot \delta^{(4)}(k+q+r) \cdot (-i r_\mu) \cdot \epsilon^{\mu}{}_a(\vec{k},s) \cdot g_s \cdot f_{cba}$$

$$\Rightarrow \mathcal{M} = i r_\mu\, \epsilon^{\mu}{}_a(\vec{k},s)\, f_{abc}\, g_s$$

Reglas Feynman:

Ghost in - out → 1

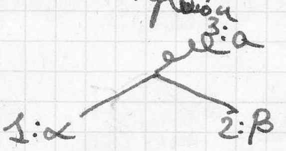
Vértice : → $i g_s f_{abc} \overset{\text{del ghost out}}{r^\mu}$ → Si el momento es saliente, cambia signo r: → Pich; P y T

→ $-i g_s f_{abc} r^\mu$

Los propagadores se demuestran por integración en el plano complejo.
↳ Ver p. 35 P y T o notas Pich

Comentario: En general, los índices de color (¡los Lorentz sí!) no están sumados (si son patas externas). Por ejemplo, en el primer vértice,

$$\mathcal{M} = \bar{u}_\beta\, g_s\, \gamma^\mu \frac{\lambda^a_{\beta\alpha}}{2} \epsilon_\mu{}^a(\vec{q}, s) \cdot u_\alpha$$



los índices β, α, a NO están sumados aunque estén repetidos. Simplemente recuerdan a qué partícula corresponden. Sólo si hacemos promedio sobre colores, se sumarán:

$$\overline{\sum_{a,\alpha,\beta}} |\mathcal{M}|^2$$

Para evitar esta ambigüedad por la convención de Einstein, podemos asignar números a las partículas y dejar los índices sólo en la λ:

$$\mathcal{M}(a, \alpha, \beta) = \bar{u}_2\, g_s \frac{\lambda^a{}_{\beta\alpha}}{2} \epsilon_{\mu,3}\, u_1$$

⇓
LIBRES, no mudos

→ Y análogamente para el resto de vértices, los índices de color que aparezcan especificados

En los propagadores o loops, al no ser patas externas, sí estarán sumados los índices.

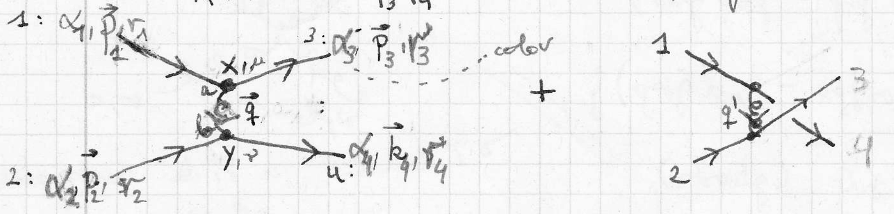
/ número

I. PROCESOS ELEMENTALES

Omito índice sabor, se conserva en int. fuertes.

1) ~~I~~ Interacción entre quarks del mismo sabor (A).

$q_1 q_2 \longrightarrow q_3 q_4$ 2 topologías



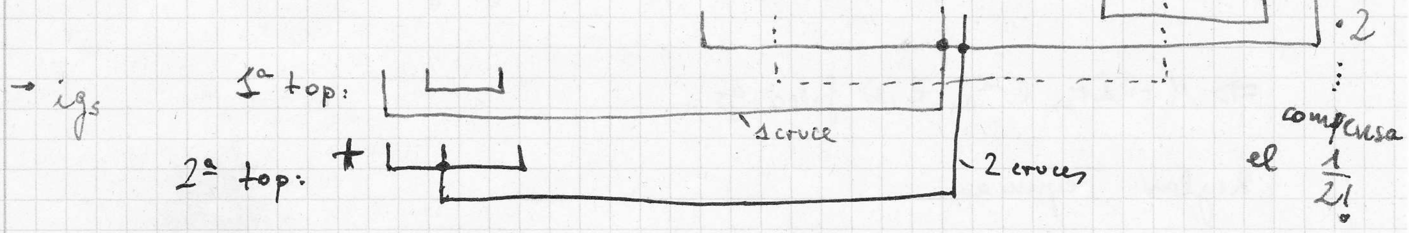
y factor 2 de simetría ya que hemos fijado que 1 va a x.

p_i, r_i, α_i : libres ; q : dependiente ; a, b, μ, ν : auxiliares, sumados, no libres

Más formalmente, se vería con las contracciones:

$$A_{i\to f} = \frac{i^2}{2!} g_s^2 \int d^4x \int d^4y \, T \langle 0 | a_3 a_4 \left[\bar{q}_{\alpha'} \frac{(\lambda^{a'})_{\alpha'\beta'}}{2} \gamma^{\mu'} q_{\beta'} G^{a'}_{\mu'} \right](y) \cdot \left[\bar{q}_{\tilde{\alpha}} \frac{(\lambda^{\tilde{a}})_{\tilde{\alpha}\tilde{\beta}}}{2} \gamma^{\tilde{\mu}} q_{\tilde{\beta}} G^{\tilde{a}}_{\tilde{\mu}} \right](x) \, a_1^+ a_2^+ |0\rangle$$

Hay 4 combinaciones posibles, pero dos son repetidas. Fijo 1 a x y saco un factor 2. Luego, a_2 debe ir a y, y a_3 y a_4 tienen 2 posibilidades.



→ signo − relativo entre topologías. O bien aplicando reglas de Feynman y añadiendo un − por cruce de líneas fermiónicas, se tiene que: $\mathcal{M} \equiv (\mathcal{M}_1 + \mathcal{M}_2)$.

usaré $\xi = 1$

ya sustituye a los de color

$$\mathcal{M}_1 = \bar{u}_3(\vec{p}_3, r_3) \frac{g_s}{2} (\lambda^a)_{\alpha_3\alpha_1} \cdot \gamma^\mu u_1(\vec{p}_1, r_1) \cdot \left(-g_{\mu\nu} + \frac{(1-\xi) q_\mu q_\nu}{q^2 + i\eta}\right) \cdot \frac{\delta^{ab}}{q^2 + i\eta} \cdot$$

$q = p_1 - p_3$

a, b sí sumados, porque es propagador, no gluón externo

$$\cdot \bar{u}_4(\vec{p}_4, r_4) \cdot \frac{g_s}{2} (\lambda^b)_{\alpha_4\alpha_2} \cdot \gamma^\nu \cdot u_2(\vec{p}_2, r_2)$$

$q' = p_1 - p_4$

cruce 3-4

$$\mathcal{M}_2 = -\bar{u}_4(\vec{p}_4, r_4) \frac{g_s}{2} (\lambda^a)_{\alpha_4\alpha_1} \gamma^\mu u_1(\vec{p}_1, r_1) \cdot \left(-g_{\mu\nu} + \frac{(1-\xi) q'_\mu q'_\nu}{q'^2 + i\eta}\right) \cdot \frac{\delta^{ab}}{q'^2 + i\eta} \cdot$$

$$\cdot \bar{u}_3(\vec{p}_3, r_3) \frac{g_s}{2} (\lambda^b)_{\alpha_3\alpha_2} \gamma^\nu u_2(\vec{p}_2, r_2)$$

$\lambda^a \delta^{ab} \lambda^b = \lambda^a \lambda^a$

Ojo, $\mathcal{M}$ tiene índices sumados a, NO repetir los mismos en $\mathcal{M}^+$

$$|\mathcal{M}|^2 = (|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1^+ \mathcal{M}_2 + (\mathcal{M}_2)^+ \mathcal{M}_1) = A_1 + A_2 + A_3 + A_4$$

$\xi = 1$

$$A_1 = \frac{g_s^4}{4} \cdot \frac{1}{4} \bar{u}_2 \gamma^\nu (\lambda^{\tilde{a}})^*_{\alpha_4\alpha_2} \cdot u_4 \frac{g_{\mu\nu}}{q^2 + i\eta} \cdot \frac{g_{\mu'\nu'}}{q^2 + i\eta} \cdot (-1) \, \bar{u}_1 \gamma^\mu (\lambda^{\tilde{a}})^*_{\alpha_3\alpha_1} u_3 \cdot$$

$$\cdot \bar{u}_3 (\lambda^a)_{\alpha_3\alpha_1} \gamma^{\mu'} u_1 \, \bar{u}_4 (\lambda^a)_{\alpha_4\alpha_2} \gamma^{\nu'} u_2 \cdot (-1)$$

factor de color ($\lambda_i \ldots$)

$$= \frac{g_s^4}{4^2} \cdot \left(\frac{1}{q^2 + i\eta}\right)^2 \cdot (\bar{u}_2 \gamma_\mu u_4)(\bar{u}_1 \gamma^\mu u_3)(\bar{u}_3 \gamma_{\nu'} u_1)(\bar{u}_4 \gamma^{\nu'} u_2) \cdot f$$

promedio sobre iniciales suma para finales

salen trazas

$\ldots \eta \sim 0$; $\nu' \leftrightarrow \nu$, y bajo - subo entre trazas

$$= \frac{1}{2^{6}} \frac{g_s^4}{(p_1-p_3)^4} \cdot \underbrace{Tr\left[(\not{p}_2 + m_2) \gamma_\mu (\not{p}_4 + m_4) \gamma_\nu \right]}_{T_1} \cdot \underbrace{Tr\left[(\not{p}_1 + m_1) \gamma^\mu (\not{p}_3 + m_3) \gamma^\nu \right]}_{T_2} \cdot \underbrace{\overline{\sum_{\alpha_i}} f}_{T_3}$$

↗ traza de nº impar de γ's es 0.

espacio color, separado porque m_f independiente de color (simetría SU(3))

$$T_2 = Tr(\not{p}_1 \gamma^\mu \not{p}_3 \gamma^\nu) + Tr(\gamma^\mu \gamma^\nu) \cdot m_1 m_3$$

$$= Tr(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) \cdot p_{1\alpha} p_{3\beta} + m_1 m_3 \cdot 4 g^{\mu\nu}$$

$$= p_{1\alpha} p_{3\beta} \cdot 4 (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\mu\beta}) + 4 m_1 m_3 g^{\mu\nu}$$

$$= 4 \cdot \left[p_1^\mu p_3^\nu - (p_1 p_3) g^{\mu\nu} + p_1^\nu p_3^\mu + m_1 m_3 g^{\mu\nu} \right]$$

Análogamente, T_1 será:

$$T_1 = 4 \cdot \left[p_{2\mu} p_{4\nu} - (p_2 p_4) g_{\mu\nu} + p_{2\nu} p_{4\mu} + m_2 m_4 g_{\mu\nu} \right]$$

$$\hookrightarrow T_1 T_2 = 16 \cdot \{ (p_1 p_2)(p_3 p_4) - (p_2 p_4)(p_1 p_3) + (p_1 p_4)(p_2 p_3) + m_2 m_4 (p_1 p_3)$$
$$- (p_1 p_3)(p_2 p_4) + (p_1 p_3)(p_2 p_4) \cdot \underbrace{g^{\mu\nu} g_{\mu\nu}}_{4} - (p_1 p_3)(p_2 p_4)$$
$$- m_2 m_4 \cdot 4 (p_1 p_3) + (p_2 p_3)(p_1 p_4) - (p_2 p_4)(p_1 p_3) + (p_1 p_2)(p_3 p_4)$$
$$+ m_2 m_4 (p_1 p_3) + m_1 m_3 (p_2 p_4) - 4 m_1 m_3 (p_2 p_4) + m_1 m_3 (p_2 p_4)$$
$$+ m_1 m_2 m_3 m_4 \cdot 4 \}$$

$$= 16 \cdot \{ 2 (p_1 p_2)(p_3 p_4) + 2 (p_1 p_4)(p_2 p_3)$$
$$- 2 m_2 m_4 (p_1 p_3) - 2 m_1 m_3 (p_2 p_4) + 4 m_1 m_2 m_3 m_4 \}$$

$$= \underset{2^5}{\overset{\shortparallel}{32}} \cdot \{ (p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3) - m_1 m_3 (p_2 p_4)$$
$$- m_2 m_4 (p_1 p_3) + 2 m_1 m_2 m_3 m_4 \} \rightarrow m_i = m_j$$ porque todos quarks mismo sabor

↗ promedio sobre iniciales (α_1, α_2), suma sobre finales (α_3, α_4)

($a, \tilde{a}$ sumados)

$$T_3 = \overline{\sum_{\alpha_i}} f = \frac{1}{3} \cdot \frac{1}{3} \cdot \sum_{\substack{\alpha_1, \alpha_2 \\ \alpha_3, \alpha_4}} \cdot (\lambda^{\tilde{a}})^*_{\alpha_4, \alpha_2} \cdot (\lambda^{\tilde{a}})^*_{\alpha_3 \alpha_1} \cdot (\lambda^a)_{\alpha_3 \alpha_1} (\lambda^a)_{\alpha_4 \alpha_2} = \frac{1}{9} \sum_{\alpha_2, \alpha_4} \lambda^{\tilde{a}}_{\alpha_2 \alpha_4} \lambda^a_{\alpha_4 \alpha_2} \cdot \sum_{\alpha_3, \alpha_1} \lambda^{\tilde{a}}_{\alpha_1 \alpha_3} \lambda^a_{\alpha_3 \alpha_1}$$

$$(\lambda)^*_{ij} = \lambda_{ji}$$

$$= \frac{1}{9} \cdot Tr(\lambda^{\tilde{a}} \lambda^a) \cdot Tr(\lambda^{\tilde{a}} \lambda^a) = \frac{1}{9} \cdot 2 \delta^{\tilde{a}a} \cdot 2 \delta^{\tilde{a}a} = \frac{4}{9} \sum_a (\delta^{aa})^2 = 4 \cdot \frac{8}{9} = \frac{2^5}{9}$$

$$\overline{\sum} A_1 = \frac{2^5 \cdot 2^5}{2^{6}} \cdot \frac{1}{3^2} \cdot \frac{g_s^4}{(p_1 - p_3)^4} \cdot \{ (p_1 p_2) \cdot (p_3 p_4) + (p_1 p_4)(p_2 p_3) - m^2 [(p_1 p_3) + (p_2 p_4)] + 2 m^4 \}$$

El término A_2 es análogo cambiando 3 por 4. El (-), al estar al cuadrado, no contribuye

$$\overline{\sum} A_2 = \frac{2^4}{3^2} \cdot \frac{g_s^4}{(p_1 - p_4)^4} \cdot \{ (p_1 p_2)(p_3 p_4) + (p_1 p_3)(p_2 p_4)$$
$$- m^2 \cdot [(p_1 p_4) + (p_2 p_3)] + 2 m^4 \}$$

$A_3 = (\mathcal{M}_1^+ \mathcal{M}_2)$

a, ã sumados

$$= - \cdot \left(\frac{g_s^2}{4}\right)^2 \cdot \frac{1}{(p_1-p_3)^2} \cdot \frac{1}{(p_1-p_4)^2} \cdot \cdot \overline{u_2}\gamma_\mu \overset{(\lambda^{\tilde a})_{\alpha_2\alpha_4}}{\overset{\shortparallel}{(\lambda^{\tilde a})^*_{\alpha_4\alpha_2}}} u_4 \cdot \overline{v_1}\gamma^\mu \overset{\lambda^{\tilde a}_{\alpha_1\alpha_3}}{\overset{\shortparallel}{(\lambda^{\tilde a})^*_{\alpha_3,\alpha_1}}} v_3 \cdot$$

$$\cdot\, \overline{u_4}(\lambda^a)_{\alpha_4\alpha_1}\gamma_\nu u_1 \overline{u_3}(\lambda^a)_{\alpha_3\alpha_2}\gamma^\nu u_2$$

$$= -\frac{g_s^4}{2^4}\cdot\frac{1}{(p_1-p_3)^2(p_1-p_4)^2}\, \overline{v_2}\gamma_\mu u_4\, \overline{u_1}\gamma^\mu u_3\, \overline{u_4}\gamma_\nu u_1 \overline{u_3}\gamma^\nu u_2 \cdot (\lambda^{\tilde a})_{\alpha_2\alpha_4}(\lambda^{\tilde a})_{\alpha_1\alpha_3} \cdot (\lambda^a)_{\alpha_4\alpha_1}\cdot(\lambda^a)_{\alpha_3\alpha_2}$$

↓ salen trazas ; como quarks son de mismo sabor, $m_i = m_j \equiv m$

$$\overline{\sum_{\lambda_i, r_i}} A_3 = -\left(\frac{1}{2}\right)^2 \cdot \frac{g_s^4}{2^4}\cdot \frac{1}{(p_1-p_3)^2(p_1-p_4)^2}\cdot Tr\underbrace{[(\not p_2+m)\gamma_\mu(\not p_4+m)\gamma_\nu(\not p_1+m)\gamma^\mu\cdot(\not p_3+m)\gamma^\nu]}_{\tilde T}$$

a, ã sumados

$$\cdot \frac{1}{3^2} Tr_c[(\lambda^{\tilde a})\lambda^a\lambda^{\tilde a}\lambda^a]$$

... promedio

$$\tilde T = Tr\{[\overset{4}{\not p_2\gamma_\mu\not p_4\gamma_\nu} + \overset{2}{m^2\gamma_\mu\gamma_\nu} + \overset{3}{m\not p_2\gamma_\mu\gamma_\nu} + \overset{3}{m\gamma_\mu\not p_4\gamma_\nu}]\cdot$$

$$\cdot[\overset{4}{\not p_1\gamma^\mu\not p_3\gamma^\nu} + \overset{2}{m^2\gamma^\mu\gamma^\nu} + \overset{3}{m\not p_1\gamma^\mu\gamma^\nu} + \overset{3}{m\gamma^\mu\not p_3\gamma^\nu}]\}$$

nº impar γs → Tr = 0

$$\underset{\tilde T_1}{=} Tr[\not p_2\gamma_\mu\not p_4\gamma_\nu\not p_1\gamma^\mu\not p_3\gamma^\nu]^{\cdots 8} + \underset{\tilde T_2}{m^2} Tr[\overset{\not p_4}{\cancel{\not p_2}}\gamma_\mu\gamma_\nu\gamma^\mu\gamma^\nu]^{\cdots 6} + 0 + 0$$

$$\underset{\tilde T_3}{+} m^2 Tr[\gamma_\mu\gamma_\nu\not p_1\gamma^\mu\not p_3\gamma^\nu]^{\cdots 6} + \underset{\tilde T_4}{m^4} Tr[\gamma_\mu\gamma_\nu\gamma^\mu\gamma^\nu]^{\cdots 4} + 0 + 0$$

$$+ 0 + 0 + \underset{\tilde T_5}{m^2} Tr[\not p_2\gamma_\mu\gamma_\nu\not p_1\gamma^\mu\gamma^\nu]^{\cdots 6} + \underset{\tilde T_6}{m^2} Tr[\not p_2\gamma_\mu\gamma_\nu\not p_3\gamma^\mu\gamma^\nu]^{\cdots 6}$$

$$+ 0 + 0 + \underset{\tilde T_7}{m^2} Tr[\gamma_\mu\not p_4\gamma_\nu\not p_1\gamma^\mu\gamma^\nu]^{\cdots 6} + m^2 \underset{\to T_8}{Tr}[\gamma_\mu\not p_4\gamma_\nu\gamma^\mu\not p_3\gamma^\nu]^{\cdots 6}$$

⇒ Intento juntar $\gamma^\mu\gamma_\mu$, $\gamma^\nu\gamma_\nu$ usando que:

PROPIEDADES

$\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$

$\not p\gamma^\mu = p_\nu\gamma^\nu\gamma^\mu = p_\nu(2g^{\mu\nu}-\gamma^\mu\gamma^\nu) = 2p^\mu - \gamma^\mu\not p$

$\not a\not b = 2ab - \not b\not a$; $\gamma_\alpha\gamma^\alpha = 4$; $Tr[\not a\not b] = 4ab$

~~$\tilde T_1 = Tr[\not p_2\cdot(2p_{4\mu} - \not p_4\gamma_\mu)\gamma_\nu\cdot(2p_1^\mu - \gamma^\mu\not p_1)(2p_3^\nu - \gamma^\nu\not p_3)]$~~

~~$= Tr[\not p_2\cdot 2p_{4\mu}\gamma_\nu 2p_1^\mu\cdot 2p_3^\nu] + Tr[\not p_2\cdot 2p_{4\mu}\cdot\gamma_\nu\cdot 2p_1^\mu\cdot(-\gamma^\nu\not p_3)]$~~

~~$+ Tr[\not p_2 2p_{4\mu}\gamma_\nu\cdot(-\gamma^\mu\not p_1)\cdot 2p_3^\nu] + Tr\cdot[\not p_2 2p_{4\mu}\gamma_\nu(-\gamma^\mu\not p_1)\cdot(-\gamma^\nu\not p_3)]$~~

~~$+ Tr[\not p_2(-\not p_4\gamma_\mu)\gamma_\nu\cdot 2p_1^\mu\cdot 2p_3^\nu] + Tr[\not p_2(-\not p_4\gamma_\mu)\cdot\gamma_\nu\cdot 2p_1^\mu\cdot(-\gamma^\nu\not p_3)]$~~

~~$+ Tr[\not p_2(-\not p_4\gamma_\mu)\gamma_\nu\cdot(-\gamma^\mu\not p_1)\cdot 2p_3^\nu] + Tr[\not p_2(-\not p_4\gamma_\mu)\gamma_\nu\cdot(-\gamma^\mu\not p_1)(-\gamma^\nu\not p_3)]$~~

$= Tr[\not p_2\not p_3]\cdot 4(p_1p_4) = Tr[\not p_2\gamma_\nu\gamma^\nu\not p_3]\cdot 4(p_1p_4)$

prop. cíclica

$Tr[\not a\not b] = 4ab$

Más fácil: (utilizando las propiedades de la página anterior

$$\tilde{T}_1 = Tr[\not{p}_2 \gamma_\mu \not{p}_4 \cdot (2p_{1\nu} - \not{p}_1 \gamma_\nu)\gamma^\mu \not{p}_3 \gamma^\nu]$$

$$= 2Tr[\not{p}_2 \gamma_\mu \underbrace{\not{p}_4 \gamma^\mu}_{(2p_4^\mu - \gamma^\mu \not{p}_4)} \not{p}_3 \not{p}_1] - Tr[\not{p}_2 \gamma_\mu \not{p}_4 \not{p}_1 \gamma_\nu \gamma^\mu \underbrace{\not{p}_3 \gamma^\nu}_{(2p_3^\nu - \gamma^\nu \not{p}_3)}]$$

$$= 4Tr[\not{p}_2 \not{p}_4 \not{p}_3 \not{p}_1] - 2\cdot Tr[\not{p}_2 \cdot 4 \cdot \not{p}_4 \not{p}_3 \not{p}_1] - 2Tr[\underbrace{\not{p}_2 \gamma_\mu}_{2p_{2\mu} - \gamma_\mu \not{p}_2} \not{p}_4 \not{p}_1 \not{p}_3 \gamma^\mu]$$

$$+ Tr[\not{p}_2 \gamma_\mu \not{p}_4 \not{p}_1 \gamma_\nu \underbrace{\gamma^\mu \gamma^\nu}_{2g^{\mu\nu} - \gamma^\nu \gamma^\mu} \not{p}_3]$$

pr cíclica

$$= -4Tr[\not{p}_2 \not{p}_4 \not{p}_3 \not{p}_1] - 4Tr[\not{p}_4 \not{p}_1 \not{p}_3 \not{p}_2] + 2[\gamma_\mu \not{p}_2 \not{p}_4 \not{p}_1 \not{p}_3 \gamma^\mu]$$

$$+ 2Tr[\not{p}_2 \gamma_\mu \not{p}_4 \not{p}_1 \gamma^\mu \not{p}_3] - Tr[\not{p}_2 \gamma_\mu \not{p}_4 \not{p}_1 \underbrace{\gamma_\nu \gamma^\nu}_{4} \gamma^\mu \not{p}_3]$$

$$= -4Tr[\not{p}_2 \not{p}_4 \not{p}_3 \not{p}_1] - 4Tr[\not{p}_4 \not{p}_1 \not{p}_3 \not{p}_2] + 8Tr[\not{p}_2 \not{p}_4 \not{p}_1 \not{p}_3]$$

$$- 2Tr[\not{p}_2 \underbrace{\gamma_\mu \not{p}_4}_{2p_{4\mu} - \not{p}_4 \gamma_\mu} \not{p}_1 \gamma^\mu \not{p}_3]$$

$$= 4Tr[\not{p}_2 \not{p}_4 \not{p}_1 \not{p}_3] - 4Tr[\not{p}_2 \not{p}_4 \not{p}_3 \not{p}_1] - 4Tr[\not{p}_2 \not{p}_1 \not{p}_4 \not{p}_3]$$

$$+ 2Tr[\not{p}_2 \not{p}_4 \underbrace{\gamma_\mu \not{p}_1}_{2p_{1\mu} - \not{p}_1 \gamma_\mu} \gamma^\mu \not{p}_3]$$

$$= 4T[2,4,1,3] - 4\cdot T[2,4,3,1] - 4T[2,1,4,3]$$

$$+ 4T[2,4,1,3] - 8T[2,4,1,3]$$

$$= -4\cdot T[\not{p}_2 \not{p}_4 \not{p}_3 \not{p}_1] - 4T[\not{p}_2 \not{p}_1 \not{p}_4 \not{p}_3]$$

$$= -4\cdot\{\cancel{(p_2p_4)(p_3p_1)} - \cancel{(p_1p_4)(p_2p_3)} + (p_1p_2)(p_3p_4) + (p_2p_1)(p_4p_3) - \cancel{(p_2p_4)(p_1p_3)}$$
$$+ \cancel{(p_2p_3)(p_1p_4)}]\cdot 4$$

$$= -16\cdot 2(p_1p_2)(p_3p_4) = -32(p_1p_2)(p_3p_4)$$

$$\tilde{T}_2 = Tr[\not{p}_2 \gamma_\mu \not{p}_4 \gamma_\nu \cdot (2g^{\mu\nu} - \gamma^\nu \gamma^\mu)]\cdot m^2$$

$$= 2m^2 Tr[\not{p}_2 \gamma_\mu \not{p}_4 \gamma^\mu] - 4\cdot Tr[\not{p}_2 \gamma_\mu \not{p}_4 \gamma^\mu]\cdot m^2$$

$$= -2m^2 \cdot Tr[\not{p}_2 (2p_{4\mu} - \not{p}_4 \gamma_\mu)\gamma^\mu] = -4m^2 Tr[\not{p}_2 \not{p}_4] + 2m^2 \cdot 4 \cdot Tr[\not{p}_2 \not{p}_4]$$

$$= 4m^2 Tr[\not{p}_2 \not{p}_4] = 16m^2 (p_2p_4)$$

análogo a $\tilde{T}_2$ cambiando $\begin{matrix} 2 \to 1 \\ 4 \to 3 \end{matrix}$

$$\tilde{T}_3 \overset{\text{cíclica}}{=} m^2 Tr[\not{p}_1 \gamma^\mu \not{p}_3 \gamma^\nu \gamma_\mu \gamma_\nu] \equiv 16m^2 (p_1p_3)$$

$$\tilde{T}_4 = m^4 \cdot Tr[\gamma_\mu \gamma_\nu (2g^{\mu\nu} - \gamma^\nu \gamma^\mu)] = 2m^4 Tr[\overbrace{\gamma_\mu \gamma^\mu}^{4\cdot I}] - m^4 Tr[16\cdot I]$$

$$= -8m^4 Tr[I] = -32m^4$$

$$\tilde{T}_5 = m^2\,\mathrm{Tr}[\not{p}_2\gamma_\mu\underbrace{\gamma_\nu\not{p}_1}_{2p_{1\nu}-\not{p}_1\gamma_\nu}\gamma^\mu\gamma^\nu] = 2m^2\,\mathrm{Tr}[\not{p}_2\underbrace{\gamma_\mu\gamma^\mu}_{4}\not{p}_1] - \mathrm{Tr}[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\gamma^\mu\gamma^\nu]$$

analogía con $\tilde{T}_2$ cambiando 4 por 1

$$= 32m^2(p_1p_2) - 16m^2(p_1p_2) = 16m^2(p_1p_2)$$ //

analogía con T_5 cambiando $1\to3$

$$\tilde{T}_6 = 16m^2(p_2p_3)$$

análogo a $\tilde{T}_2$ cambiando $\mu\leftrightarrow\nu$, $2\to4$, $4\to1$ y bajando subiendo índices.

$$\tilde{T}_7 = m^2\,\mathrm{Tr}[\not{p}_4\gamma_\nu\not{p}_1\gamma^\mu\gamma^\nu\gamma_\mu] = 16m^2(p_1p_4)$$

análogo a $\tilde{T}_5$ cambiando $\mu\leftrightarrow\nu$, $2\to4$, $1\to3$

$$\tilde{T}_8 = m^2\,\mathrm{Tr}[\not{p}_4\gamma_\nu\gamma_\mu\not{p}_3\gamma^\nu\gamma^\mu] = 16m^2(p_3p_4)$$

$\Downarrow$

$$\tilde{T} = -32(p_1p_2)(p_3p_4) + 16m^2(p_2p_4) + 16m^2(p_1p_3) - 32m^4$$
$$+ 16m^2(p_1p_2) + 16m^2(p_2p_3) + 16m^2(p_1p_4) + 16m^2(p_3p_4)$$

$$= +\underset{=2^5}{32}\left[m^2\cdot\left((p_1p_2)+(p_1p_3)+(p_1p_4)+(p_2p_3)+(p_2p_4)+(p_3p_4)-m^2\right) - (p_1p_2)(p_3p_4)\right]$$ //

→ Nota, el cálculo se hubiese simplificado más si no hubiese separado los 8 términos y hubiese hecho los cambios $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}$ ahí directamente.

$$\mathrm{Tr}_c[(\lambda^{\tilde{a}})\lambda^a\lambda^{\tilde{a}}\lambda^a] \overset{\text{pr. cíclica}}{=} \mathrm{Tr}[\lambda^a\lambda^{\tilde{a}}\lambda^a\lambda^{\tilde{a}}] \overset{\text{renombro}}{\equiv} \mathrm{Tr}[\lambda^a\lambda^b\lambda^a\lambda^b]$$

Usamos que: $[\lambda^a,\lambda^b] = \lambda^a\lambda^b - \lambda^b\lambda^a = 2i f^{abc}\lambda_c$

↳ $\lambda^b\lambda^a = \lambda^a\lambda^b - 2i f^{abc}\lambda_c$

casimir cuadrático $4C_2(R) = 4\frac{N^2-1}{2N}\delta_{ab}$

↳ $\lambda^a\lambda^b\lambda^a = (\lambda^a)^2\lambda^b - 2i f^{abc}\lambda^a\lambda^c = 4C_2(R)\mathbb{1}\lambda^b - 2i f^{abc}\lambda^a\lambda^c$

Aparte, $\lambda^a\lambda^c = \frac{2}{N}\delta_{ac}I + d_{ace}\lambda_e + i f_{ace}\lambda_e$

$$\lambda^a\lambda^b\lambda^a = 4C_2(R)\lambda^b - \underbrace{2i f^{abc}\cdot\frac{2}{N}\cdot\delta_{ac}}_{0 \text{ porque } f^{aba}=0 \text{ antisim.}} - \underbrace{2i \underbrace{f^{abc}}_{\text{antis.}}\underbrace{d_{ace}}_{\text{sim}}\lambda_e}_{=0} - 2i f^{abc}\, i f_{ace}\lambda_e$$

Ver Peskin p.º 530

$$= 4C_2(R)\lambda^b - 2(-if^{abc})(-if^{ace})\lambda^e = 4C_2(R)\lambda^b - 2\underbrace{(t_A^a)_{bc}(t_A^a)_{ce}}_{=N\cdot\delta_{be} = C_2(G)\delta_{be}}\cdot\lambda^e$$

$$= [4C_2(R) - 2C_2(G)]\cdot\lambda^b = 4\cdot[C_2(R) - \tfrac{1}{2}C_2(G)]\lambda^b$$

↳ $\lambda^a\lambda^b\lambda^a\lambda^b = 4\cdot[C_2(R) - \frac{1}{2}C_2(G)]\cdot\lambda^b\lambda^b = 4\cdot(C_2(R) - \frac{1}{2}C_2(G))\cdot 4C_2(R)\cdot\underset{\text{identidad}}{\mathbb{1}}$

$$= 16\cdot\left(\frac{9-1}{2\cdot3} - \frac{3}{2}\right)\cdot\frac{9-1}{2\cdot3} = \frac{16}{9\cdot4}\cdot(9-1-9)\cdot8 = -\frac{2^5}{9}\cdot\mathbb{1}$$

↳ $\mathrm{Tr}[\lambda^a\lambda^b\lambda^a\lambda^b] = -\frac{2^5}{9}\cdot\underbrace{\mathrm{Tr}(\mathbb{1})}_{3} = -\frac{2^5}{3}$ //

→ Factores globales de A_3:

$$\underbrace{\frac{1}{2^2}}_{\text{promedio spin}}\cdot\underbrace{\frac{1}{3^2}}_{\text{promedio color}}\cdot\underbrace{\frac{g_s^4}{2^4}}_{\text{acoplamiento}}\cdot\underbrace{2^5}_{\text{traza en Dirac }(T)}\cdot\underbrace{\left(-\frac{2^5}{3}\right)}_{\text{traza en color}} = \frac{g_s^4\cdot2^4}{3^3}$$

Como $A_4 = A_3^+$ y $\overline{\sum} A_3 \in \mathbb{R}$, entonces $\overline{\sum} A_3 + A_4 = \overline{\sum} 2 A_3$

$$\overline{\sum}(A_3 + A_4) = -2 \cdot \frac{1}{2^2} \cdot \frac{1}{3^2} \left(-\frac{2^5}{3}\right) \frac{g_s^4}{2^4} \cdot \frac{1}{(p_1-p_3)^2 (p_1-p_4)^2} \cdot \tilde{T}.$$

$$= + \frac{g_s^4}{\cdot 27} \cdot \frac{1}{(p_1-p_3)^2 (p_1-p_4)^2} \cdot T$$

$\lambda \equiv \lambda(s, m^2, m^2) = s^2 - 4m^2$

$$\left.\frac{d\sigma}{dt}\right|_{SCM} = \frac{1}{16\pi \lambda} \overline{\sum} |M|^2 = \frac{g_s^4}{16\pi \cdot s^2} \cdot \left\{ \frac{1 (2^4)}{32 \cdot (p_1-p_3)^4} \cdot \left[(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) + 2m^4 - m^2((p_1p_3) + (p_2p_4))\right]\right.$$

$$+ \frac{2^4}{3^2 \cdot (p_1-p_4)^4} \cdot \left[(p_1p_2)(p_3p_4) + (p_1p_3)(p_2p_4) + 2m^4 - m^2((p_1p_4) + (p_2p_3))\right]$$

$$+ \frac{2^5}{3^3 \cdot (p_1-p_3)^2 (p_1-p_4)^2} \cdot \left[m^2((p_1p_2) + (p_1p_3) + (p_1p_4) + (p_2p_3) + (p_2p_4) + (p_3p_4)) - m^4 - (p_1p_2)(p_3p_4)\right]$$

↓

$$\frac{d\sigma}{dt} = \frac{g_s^4}{\underset{\overset{\shortparallel}{9}}{3^2} \cdot \pi \cdot \lambda} \cdot \left\{ \frac{(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) + 2m^4 - m^2[(p_1p_3) + (p_2p_4)]}{(p_1-p_3)^4} \right. \rightarrow \text{Canal } t$$

$$+ \frac{(p_1p_2)(p_3p_4) + (p_1p_3)(p_2p_4) + 2m^4 - m^2[(p_1p_4) + (p_2p_3)]}{(p_1-p_4)^4} \rightarrow \text{Canal}$$

$$\left. -2 \cdot \frac{1}{3} \cdot \frac{((p_1p_2)(p_3p_4) + m^4 - m^2[(p_1p_2) + (p_1p_3) + (p_1p_4) + (p_2p_3) + (p_2p_4) + (p_3p_4)])}{(p_1-p_3)^2 (p_1-p_4)^2} \right\} \rightarrow \text{Interferencia } tu$$

En función de las variables de Mandelstam:

$s = (p_1+p_2)^2 = (p_3+p_4)^2 = 2m^2 + 2p_1p_2 = 2m^2 + 2p_3p_4 \rightarrow p_1p_2 = p_3p_4 = \frac{s}{2} - m^2$

$t = (p_1-p_3)^2 = (p_2-p_4)^2 = 2m^2 - 2p_1p_3 = 2m^2 - 2p_2p_4 \rightarrow p_1p_3 = p_2p_4 = m^2 - \frac{t}{2}$

$u = (p_1-p_4)^2 = (p_2-p_3)^2 = 2m^2 - 2p_1p_4 = 2m^2 - 2p_2p_3 \rightarrow p_1p_4 = p_2p_3 = m^2 - \frac{u}{2}$

$\Downarrow \quad \{s + t + u = 4m^2\}$

$$\frac{d\sigma}{dt} = \frac{g_s^4}{3^2 \pi \lambda} \cdot \left\{ \frac{1}{t^2} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 + \left(m^2 - \frac{u}{2}\right)^2 + 2m^4 - m^2\left(m^2 - \frac{t}{2}\right) \cdot 2\right]\right.$$

$$+ \frac{1}{u^2} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 + \left(m^2 - \frac{t}{2}\right)^2 + 2m^4 - m^2\left(m^2 - \frac{u}{2}\right) \cdot 2\right]$$

$$- \frac{2}{3} \cdot \frac{1}{tu} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 + m^4 - m^2 \cdot (s - t - u + 2m^2)\right]$$

$$\boxed{\frac{d\sigma}{dt} = \frac{g_s^4}{3^2 \pi \lambda} \cdot \left\{ \frac{1}{t^2} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 + \left(m^2 - \frac{u}{2}\right)^2 + m^2 t\right] + \frac{1}{u^2} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 + \left(m^2 - \frac{t}{2}\right)^2 + m^2 u\right] - \frac{2}{3} \cdot \frac{1}{tu} \cdot \left[\left(\frac{s}{2} - m^2\right)^2 - m^4 - m^2(s - t - u)\right]\right\}}$$

Secc. eficaz
$qq \rightarrow qq$
at tree level

En el límite $s \gg m^2$ $(m \to 0)$: $\lambda \to s^2$

$$\frac{d\sigma}{dt}(m \to 0) = \frac{g_s^4}{\cdot 3^2 \cdot \pi \, s^2} \cdot \left\{ \frac{\frac{s^2}{4} + \frac{u^2}{4}}{t^2} + \frac{\frac{s^2}{4} + \frac{t^2}{4}}{u^2} - \frac{2}{3} \cdot \frac{1}{tu} \cdot \frac{s^2}{4} \right\}$$

$$= \frac{g_s^4}{4 \cdot 3^2 \pi \, s^2 t^2 u^2} \cdot \left\{ s^2 t^2 + u^4 + t^4 + s^2 t^2 - \frac{2}{3} s^2 tu \right\}$$

Arreglando el resultado para comparar con el Peskin (p. 571).
con $\alpha_s = g_s^2/4\pi$

$$\boxed{\frac{d\sigma}{dt}(uu \to uu) = \frac{4\pi \alpha_s^2}{9s^2} \cdot \left\{ \frac{u^2+s^2}{t^2} + \frac{t^2+s^2}{u^2} - \frac{2}{3} \frac{s^2}{ut} \right\}}$$

Este es un resultado de orden árbol. A 2º orden, habría autoenergía del gluón, fantasmas, ... y se complica.

1b) Interacción entre antiquarks del mismo sabor $(\bar{A})$

Si repetimos el cálculo de $\mathcal{M}$, lo único que cambiarán son $u \underset{y\ \mathcal{M} \leftrightarrow \mathcal{M}^+}{\longrightarrow} v$. Al nivel de trazas, en lugar de $(\not p + m)$ obtendremos $(\not p - m)$. Es decir, podemos "reciclar" los resultados de 1a) sustituyendo $m \to -m$. Pero como m siempre aparece al cuadrado (o a la cuarta) el resultado NO cambia.

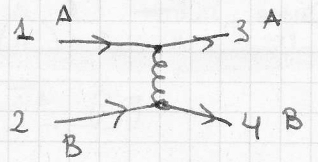
→ debido a que los términos con m elevado a potencia impar van acompañados de $\not p$ ó $\not p \not p \not p$, y la traza de un número de γs impar es cero.

Es lógico que la interacción entre quarks sea la misma que entre antiquarks, pues en ambas partículas (1 y 2) se cambian todas las cargas internas y no hay ninguna diferencia relativa entre ellas.

2) Interacción entre quarks de distinto sabor (A, B)

$q^A q^B \to q^A q^B$

A diferencia del caso 1), sólo hay un diagrama, pues los quarks son distinguibles y el sabor debe ser conservado en int. fuertes:



$m_A \neq m_B$

Aplicando las reglas de Feynman, nos sale exactamento lo mismo que el valor M_1 del caso 1, pues es el mismo diagrama.

$$\mathcal{M} = \bar{u}_3 \frac{g_s}{2} (\lambda^a)_{\alpha_3\alpha_1} \gamma^\mu u_1 \left(-g^{\mu\nu}\right) \frac{\delta^{ab}}{q^2+i\eta} \bar{u}_4 \frac{g_s}{2} (\lambda^b)_{\alpha_4\alpha_2} \gamma^\nu u_2$$

$$= \frac{g_s^2}{4} (\bar{u}_3 \gamma^\mu u_1) \cdot (\bar{u}_4 \gamma_\mu u_2) \cdot \frac{1}{(p_1-p_3)^2} \cdot (\lambda^a)_{\alpha_3\alpha_1} (\lambda^a)_{\alpha_4\alpha_2}$$

$\downarrow$

$$A = \sum_{r_i,\alpha_i} |M|^2$$

$$|M|^2 = \frac{g_s^4}{2^4 t^2} \cdot (\bar{u}_3\gamma^\mu u_1)(\bar{u}_4\gamma_\mu u_2)(\bar{u}_2\gamma_\nu u_4)(\bar{u}_1\gamma^\nu u_3) \cdot (\lambda^a)_{\alpha_3\alpha_1}(\lambda^a)_{\alpha_4\alpha_2}\overset{\overset{(\lambda^b)^*_{\alpha_3\alpha_1}}{\shortparallel}}{(\lambda^b)_{\alpha_1\alpha_3}}(\lambda^b)_{\alpha_2\alpha_4}$$

$$A = \overline{\sum} |M|^2 = \frac{g_s^4}{2^4 \cdot t^2} \cdot \frac{1}{2} \cdot \overbrace{\frac{1}{2} \cdot \frac{1}{3}}^{\text{promedio iniciales}} \cdot \frac{1}{3} \cdot \mathrm{Tr}\left[(\not{p}_3 + m_A)\gamma^\mu(\not{p}_1+m_A)\gamma^\nu\right] \cdot \mathrm{Tr}\left[(\not{p}_4+m_B)\gamma_\mu(\not{p}_2+m_B)\gamma_\nu\right]$$

μ, ν
a, b
sumados

$$\cdot \mathrm{Tr}[\lambda^a\lambda^b] \cdot \mathrm{Tr}[\lambda^a\lambda^b]$$

Las trazas de Dirac están calculadas en el apartado 1), son $T_1 \cdot T_2$ sustituyendo $m_1 = m_3 \equiv m_A$; $m_2 = m_4 \equiv m_B$

Las trazas de color son $2 \cdot \delta^{ab} \cdot 2\delta^{ab} = 4\,\delta^{aa} \overset{\text{a sumado de 1 a 8}}{=} 4 \cdot 8 = 32 = 2^5$

Queda que:

$$A = \frac{g_s^4}{2^6 3^2 t^2} \cdot \overset{\text{color}}{2^5} \cdot \overset{\text{Dirac}}{2^5} \cdot \left\{(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) - m_A^2(p_2p_4) - m_B^2(p_1p_3) + 2m_A^2m_B^2\right)$$

$\lambda \equiv \lambda(s, m_A^2, m_B^2)$

$$\boxed{\frac{d\sigma}{dt}} = \frac{A}{16\pi\lambda} = \frac{g_s^4}{9\pi\lambda t^2} \cdot \left\{\left(\frac{s}{2} - \frac{m_A^2}{2} - \frac{m_B^2}{2}\right)^2 + \left(\frac{u}{2} - \frac{m_A^2}{2} - \frac{m_B^2}{2}\right)^2 - m_A^2 \cdot \left(\frac{t}{2} - m_B^2\right) - m_B^2\left(\frac{t}{2} - m_A^2\right) + 2m_A^2m_B^2\right\}$$

$$= \frac{g_s^4}{\pi\lambda t^2 9} \cdot \left\{\frac{1}{4}(s - m_A^2 - m_B^2)^2 + \frac{1}{4}(u - m_A^2 - m_B^2)^2 - \frac{1}{2} \cdot t(m_A^2 + m_B^2) + 4m_A^2m_B^2\right\}\Bigg]$$

Si $s \gg m_A^2; m_B^2 (\to 0)$

Coincide con Peskin p. 569

$$\frac{d\sigma}{} \quad g_s^4 \quad \{s^2+u^2\} \quad 4\pi\alpha_s^2\{ \quad 4\pi\alpha_s^2 \cdot \{s^2+u^2\}$$

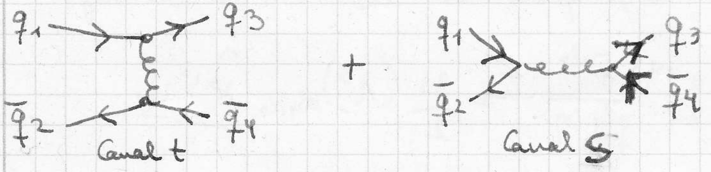
2b) Interacción entre antiquarks de distinto sabor $(\bar{A},\bar{B})$, o entre quark y antiquark de distinto sabor $(A,\bar{B})$; $(\bar{A},B)$.

Análogamente al caso 1b, el resultado NO cambia respecto al 2), al estar las masas al cuadrado. $q^A\bar{q}^{\bar{B}}$ no se aniquilarán pues violarían sabor (que se conserva en las int. fuertes).

3) Interacción entre quark-antiquark del mismo sabor $(A,\bar{A})$ que van a " " " " $(" \; ")$.

$q\bar{q} \to q\bar{q}$ p.ej: $u\bar{u} \to u\bar{u}$

Al ser una aniquilación o una interacción, habrá dos diagramas:

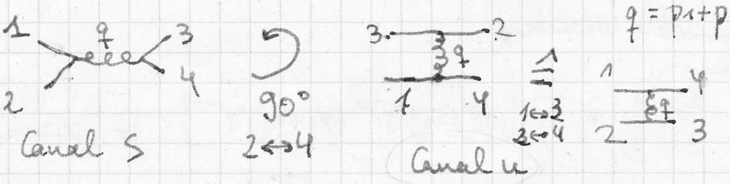


Canal t Canal S

Podríamos repetir el cálculo entero, pero saldría algo muy parecido a 1), por lo que aplicaremos la simetría de cruce.

- El primer diagrama es idéntico (sólo habría que cambiar el signo a las masas 2 y 4 antip., pero están al cuadrado). Canal t también. $\begin{pmatrix} p_2 \to -p_2 \\ p_4 \to -p_4 \end{pmatrix}$ $(m_3, m_4 \to -m_3, -m_4) \to$ queda $=$
- El segundo diagrama hay que rotarlo 90° ↻, y cambiar 2 por 4. ~~[illegible]~~ Además de rotar 90°, estás cambiando el canal s por el canal u.

$q = p_1 + p_2 \longleftarrow q = p_1 - p_4$



(Masas todas iguales).

Por tanto: cambiar en apartado 1) el M_2 con la regla $\begin{cases} u \to s \\ 2 \leftrightarrow 4 \end{cases}$.

Como M_1 es simétrico bajo intercambio de 2 y 4 (aunque en realidad no hay que cambiarle nada), podemos aplicar la regla $s \leftrightarrow u$ sobre los tres factores A_1, A_2, A_3.

Conclusión: En la expresión $\overline{\sum}|M|^2$: intercambiar $s \leftrightarrow u$ (y $m_{3,4} \to -m_{3,4}$ no influye) y queda:

$$\boxed{\frac{d\sigma}{dt} = \frac{g_s^4}{3^2\pi\lambda}\cdot\left\{\frac{1}{t^2}\left[\left(\frac{u}{2}-m^2\right)^2+\left(m^2-\frac{s}{2}\right)^2+m^2t\right]+\frac{1}{s^2}\cdot\left[\left(\frac{u}{2}-m^2\right)^2+\left(m^2-\frac{t}{2}\right)^2+m^2s\right] -\frac{2}{3}\cdot\frac{1}{ts}\cdot\left[\left(\frac{u}{2}-m^2\right)^2-m^4-m^2(u-t-s)\right]\right\}}$$

Y si despreciamos las masas:

$$\boxed{\frac{d\sigma}{dt} = \frac{4\pi\alpha_s^2}{9s^2}\cdot\left\{\frac{u^2+s^2}{t^2}+\frac{u^2+t^2}{s^2}-\frac{2}{3}\frac{u^2}{ts}\right\}}$$

→ coincide con Peskin (p. 571) ↳ 17.70

Otra manera de verlo es que $1+2 \to 3+4$: canal s $\Rightarrow 1+\bar{4} \to 3+\bar{2}$

4) Interacción entre quark-antiquark del mismo sabor $(A, \bar{A})$ que van a " " de DISTINTO " $(B, \bar{B})$ con $B \neq A$

p. ej $u\bar{u} \to d\bar{d}$

Como se conserva el sabor, no habrá un diagrama como el primero del apartado 3). Digamos que estaríamos en un análogo al apartado 2) respecto al 1).

- Es decir, sólo contribuye un término $\mathcal{M}_2$, pero las masas son distintas $(m_A \neq m_B)$. Lo más fácil es volver a usar simetría de cruce como en 3) respecto a 1), pero ahora respecto a 2).

∘ Rotamos diagrama 90°, con lo que canal $t \Leftrightarrow s$. (Cambio $2 \Leftrightarrow 3$, $q^2 \Rightarrow s$)
en $|\mathcal{M}|^2$

Queda que:

$$\boxed{\frac{d\sigma}{dt} = \frac{g_s^4}{\pi 29 s^2} \cdot \left\{ \frac{1}{4}(t - m_A^2 - m_B^2)^2 + \frac{1}{4}(u - m_A^2 - m_B^2)^2 - \frac{s}{2}(m_A^2 + m_B^2) + 4 m_A^2 m_B^2 \right\}}$$

⇒ Si despreciamos masas:

$$\boxed{\frac{d\sigma}{dt} = \frac{4\pi \alpha_s^2}{9 s^2} \cdot \left\{ \frac{t^2 + u^2}{s^2} \right\}}$$

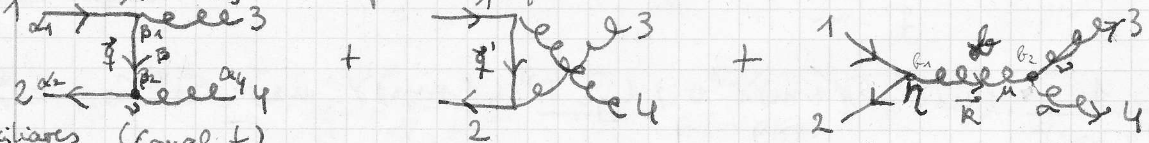
→ Coincide con Peskin (p. 569)
[17.65]

5) (Aniquilación) Interacción quark-antiquark del mismo sabor $(A, \bar{A})$ yendo a gluón - gluón

$m_A = m_{\bar{A}} \equiv m$

$q_1 \bar{q}_2 \to g_3 g_4$

Tenemos 3 diagramas posibles: (a orden árbol)



(Canal t) (Canal u) (Canal s)

q, β: auxiliares
μ, ν, σ, η

$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$

$k = p_1 + p_2$
$q' = p_1 - p_4$
$q = p_1 - p_3$

Aplico reglas de Feynman QCD:

hemos impuesto ya mismo sabor

$$\mathcal{M}_1 = \bar{v}_2 \; g_s \frac{(\lambda^{a_4})_{\alpha_2 \beta_2}}{2} \gamma^\nu \cdot \epsilon^*_{4\nu} \cdot \frac{\not{q} + m}{q^2 - m^2 + i\epsilon} \cdot \delta_{\beta_1 \beta_2} \cdot 1 \cdot g_s \left(\frac{\lambda^{a_3}}{2}\right)_{\beta_1 \alpha_1} \cdot \gamma^\mu \epsilon^*_{3\mu} u_1$$

↳ $\beta_1 = \beta_2 \equiv \beta$

β sumado
μ "
ν "

$$= \frac{g_s^2}{2^2} \left(\bar{v}_2 \gamma^\nu \frac{\not{q} + m}{q^2 - m^2} \cdot \gamma^\mu u_1 \right) \cdot \epsilon^*_{4\nu} \epsilon^*_{3\mu} \cdot (\lambda^{a_4})_{\alpha_2 \beta} (\lambda^{a_3})_{\beta \alpha_1}$$

$$= \frac{g_s^2}{2^2} \cdot \left(\bar{v}_2 \gamma^\nu \frac{(\not{q} + m)}{q^2 - m^2} \gamma^\mu u_1 \right) \epsilon^*_{3\mu} \epsilon^*_{4\nu} \cdot (\lambda^{a_4} \cdot \lambda^{a_3})_{\alpha_2 \alpha_1}$$

Análogamente: (se cruzan bosones, no hay signo negativo)

$$\mathcal{M}_2 = \frac{g_s^2}{} \cdot \left(\bar{v}_2 \gamma^\nu \frac{(\not{q}' + m)}{} \gamma^\mu u_1 \right) \cdot \epsilon^*_{3\nu} \epsilon^*_{4\mu} \cdot (\lambda^{a_3} \cdot \lambda^{a_4})_{\alpha_2 \alpha_1}$$

Y el tercer diagrama:

$$\mathcal{M}_3 = \overline{v_2} \quad g_s \left(\frac{\lambda^{b_1}}{2}\right)_{\alpha_2\alpha_1} \gamma_\eta \, u_1 \cdot \delta_{b_1 b_2} \cdot \frac{-g^{\eta\mu}}{k^2+i\epsilon} \cdot \epsilon^{*\nu}_3 \, \epsilon^{*\sigma}_4 \cdot$$

$$\cdot \, (-i) g_s \, f^{b_2 a_3 a_4} \cdot \left[g_{\mu\nu}(k+p_3)_\sigma + g_{\nu\sigma}(p_4-p_3)_\mu + g_{\sigma\mu}(-p_4-k)_\nu \right]$$

p_3, p_4

Vértice triple gluón: cambiar signo a momentos salientes en regla Feynman

$$= +i \frac{g_s^2}{2} (\overline{v_2} \gamma^\mu u_1) \cdot \frac{1}{k^2+i\epsilon} \cdot (\lambda^b)_{\alpha_2\alpha_1} f^{b a_3 a_4} \cdot [g_{\mu\nu}(\ldots)+\ldots+\ldots] \cdot \epsilon^{*\nu}_3 \, \epsilon^{*\sigma}_4$$

$$= \frac{i g_s^2}{2} \frac{1}{k^2+i\epsilon} (\lambda^b)_{\alpha_2\alpha_1} f^{b a_3 a_4} \cdot \Big\{ \overline{v_2} \gamma_\nu \epsilon^{*\nu}_3 u_1 (k+p_3)_\sigma \cdot \epsilon^{*\sigma}_4 + \overline{v_2} (\not{p}_4 - \not{p}_3) u_1 (\epsilon^{*\nu}_3 \epsilon^*_{4\nu})$$

$$- \overline{v_2} \gamma^\mu \epsilon^*_{4\mu} \cdot u_1 \, (p_4+k)_\nu \cdot \epsilon^{*\nu}_3 \Big\}$$

$$= \frac{i g_s^2}{2} \cdot \frac{1}{k^2+i\epsilon} \cdot (\lambda^b)_{\alpha_2\alpha_1} f^{b a_3 a_4} \cdot \Big\{ (\overline{v_2} \not{\epsilon}^*_3 u_1) \cdot [(k+p_3) \cdot \epsilon^*_4] + \overline{v_2} (\not{p}_4 - \not{p}_3) u_1 \cdot (\epsilon^*_3 \cdot \epsilon^*_4)$$

$$- (\overline{v_2} \not{\epsilon}^*_4 u_1) \cdot [(k+p_4) \cdot \epsilon^*_3] \Big\}$$

$$|\mathcal{M}|^2 = (\mathcal{M}_1^+ + \mathcal{M}_2^+ + \mathcal{M}_3^+) \cdot (\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3) = \underbrace{\mathcal{M}_1^+ \mathcal{M}_1}_{|\mathcal{M}_1|^2} + \underbrace{\mathcal{M}_2^+ \mathcal{M}_2}_{|\mathcal{M}_2|^2} + \underbrace{\mathcal{M}_3^+ \mathcal{M}_3}_{|\mathcal{M}_3|^2}$$

$$+ \mathcal{M}_1^+ \mathcal{M}_2 + \mathcal{M}_2^+ \mathcal{M}_1 + \mathcal{M}_1^+ \mathcal{M}_3 + \mathcal{M}_2^+ \mathcal{M}_3 + \mathcal{M}_3^+ \mathcal{M}_1 + \mathcal{M}_3^+ \mathcal{M}_2$$

$$\mathcal{M}_1^+ \mathcal{M}_1 = \frac{g_s^4}{2^4} \left(\overline{u_1} \gamma^{\tilde{\mu}} \frac{\not{q}+m}{q^2-m^2} \gamma^{\tilde{\nu}} v_2 \right) \epsilon_{3\tilde{\mu}} \epsilon_{4\tilde{\nu}} \; \underbrace{\overset{\shortparallel}{(\lambda^{a_4})^*_{\alpha_2\tilde{\beta}}} \; \overset{\shortparallel}{(\lambda^{a_3})^*_{\tilde{\beta}\alpha_1}}}$$

$$(\lambda^{a_4})_{\tilde{\beta}\alpha_2} \quad (\lambda^{a_3})_{\alpha_1\tilde{\beta}}$$

$$= (\lambda^{a_3} \lambda^{a_4})_{\alpha_1\alpha_2}$$

$$\cdot \left(\overline{v_2} \gamma^\nu \frac{\not{q}+m}{q^2-m^2} \cdot \gamma^\mu u_1 \right) \cdot \epsilon^*_{3\mu} \epsilon^*_{4\nu} \quad (\lambda^{a_4} \lambda^{a_3})_{\alpha_2\alpha_1}$$

No se suman aunque estén repetidos

$$= \frac{g_s^4}{2^4} \cdot \frac{1}{(q^2-m^2)^2} \cdot (\overline{u_1} \gamma^{\tilde{\mu}} (\not{q}+m) \gamma^{\tilde{\nu}} v_2) \cdot (\overline{v_2} \gamma^\nu (\not{q}+m) \gamma^\mu u_1) \cdot \epsilon^*_{3\mu} \epsilon_{3\tilde{\mu}} \cdot \epsilon^*_{4\nu} \epsilon_{4\tilde{\nu}} \cdot$$

$$\cdot \, (\lambda^{a_3} \lambda^{a_4})_{\alpha_1\alpha_2} \cdot (\lambda^{a_4} \lambda^{a_3})_{\alpha_2\alpha_1}$$

$\alpha_1, \alpha_2, a_3, a_4, r_3, r_4, r_1, r_2$ — sobre espines iniciales (r_1, r_2) y colores iniciales (α_1, α_2)

Sumamos y promediamos. Usamos que $\sum_{r_i} \epsilon^*_{i\mu} \epsilon^*_{i\nu} = -g_{\mu\nu}$ y luego corregiremos las polarizaciones no físicas con la introducción de fantasmas.

↓

spin colores $\sum_{r_i}$ → salen trazas

$$\sum \mathcal{M}_1^+ \mathcal{M}_1 = \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{g_s^4}{2^4} \frac{1}{(q^2-m^2)^2} \cdot \mathrm{Tr}\left[(\not{p}_1+m) \gamma^{\tilde{\mu}} (\not{q}+m) \gamma^{\tilde{\nu}} (\not{p}_2-m) \gamma^\nu (\not{q}+m) \gamma^\mu \right]$$

$$\cdot \, (-g_{\mu\tilde{\mu}}) \cdot (-g_{\nu\tilde{\nu}}) \cdot \sum_{a_3,a_4} \mathrm{Tr}_c [\lambda^{a_3} \lambda^{a_4} \lambda^{a_4} \lambda^{a_3}]$$

a_3, a_4 sumados

prop. cíclica

$$= \frac{g_s^4}{2^6 \cdot 3^2} \cdot \frac{1}{(q^2-m^2)^2} \cdot \underbrace{\mathrm{Tr}\left[(\not{p}_1+m) \gamma_\mu (\not{q}+m) \gamma_\nu (\not{p}_2-m) \gamma^\nu (\not{q}+m) \gamma^\mu \right]} \sum_{\substack{a_3 \\ a_4}} \mathrm{Tr}[(\lambda^{a_3})^2 (\lambda^{a_4})^2]$$

⑨

En la traza de color aparece directamente el Casimir cuadrático: $\sum_a \lambda^a \lambda^a = 4 \cdot C_2(R) = 4 \frac{N_c^2 - 1}{2 N_c} \cdot \mathbb{1} = 2 \cdot \frac{9-1}{3} \mathbb{1} = \frac{16}{3} \cdot \underset{3\times 3}{\mathbb{1}}$

$\hookrightarrow \mathrm{Tr}_c\left[\sum_{a_3} (\lambda^{a_3})^2 \sum_{a_4} (\lambda^{a_4})^2 \right] = \mathrm{Tr}_c\left[\frac{16}{3} \cdot \frac{16}{3} \cdot \mathbb{1} \right] = \frac{2^8}{3^2} \cdot 3 = \frac{2^8}{3}$

La traza de Dirac es parecida a la del apartado 1 $(\tilde{T})$.

La obtendré utilizando la misma estrategia, juntar $\gamma^\mu \gamma_\mu = 4 \cdot \mathbb{1}$ con la propiedad $\not p \gamma^\mu = 2 p^\mu - \gamma^\mu \not p$, pero esta vez sin descomponer los 8 factores.

$T' = \mathrm{Tr}[\ldots (\not q + m)(2 p_{2\nu} - \not p_2 \gamma_\nu - m \gamma_\nu) \gamma^\nu \ldots]$

$= 2\mathrm{Tr}[(\not p_1 + m) \gamma_\mu (\not q + m) \not p_2 (\not q + m) \gamma^\mu] - 4 \mathrm{Tr}[(\not p_1 + m) \gamma_\mu (\not q + m)(\not p_2 + m)(\not q + m) \gamma^\mu]$

$= 2\mathrm{Tr}[(2 p_1^\mu - \not p_1 \gamma^\mu + m \gamma^\mu) \gamma_\mu (\not q + m) \not p_2 (\not q + m)] - 4 \mathrm{Tr}[(2 p_1^\mu - \gamma^\mu \not p_1 + m \gamma^\mu) \gamma_\mu (\not q + m)(\not p_2 + m) \cdot$
$(\not q + m)]$

$= 4 \mathrm{Tr}[\not p_1 (\not q + m) \not p_2 (\not q + m)] - 8 \mathrm{Tr}[(\not p_1 - m)(\not q + m) \not p_2 (\not q + m)] - 8 \mathrm{Tr}[\not p_1 (\not q + m)(\not p_2 + m)(\not q + m)]$

$+ 16 \mathrm{Tr}[(\not p_1 - m)(\not q + m)(\not p_2 + m)(\not q + m)]$

$\rightarrow$ Quito trazas de nº impar de γ_5

$= [\mathrm{Tr}[\not p_1 \not q \not p_2 \not q] + m^2 \mathrm{Tr}[\not p_1 \not p_2]] \cdot (4-8) + 8m(\mathrm{Tr}'[\not q \not p_2 m]) \cdot 2$

$+ (16-8)(\mathrm{Tr}[\not p_1 \cdot \not q \not p_2 \not q] + 2\mathrm{Tr}[\not p_1 \not q] m^2 + \mathrm{Tr}[\not p_1 \not p_2] m^2])$

$- 16 m^2 (2\mathrm{Tr}[\not q \not p_2] + \mathrm{Tr}[\not q \not q] + 4 m^2)$

$= \mathrm{Tr}[\not p_1 \not q \not p_2 \not q] \cdot (8-4) + m^2 \mathrm{Tr}[\not p_1 \not p_2] \cdot (8-4)$

$+ m^2 \mathrm{Tr}[\not q \not p_2] \cdot (16 - 32) + m^2 \mathrm{Tr}[\not p_1 \not q] \cdot 16 - 16 m^2 \mathrm{Tr}[\not q \not q] - 16 m^4$

$= 4 \mathrm{Tr}[\not p_1 \not q \not p_2 \not q] + 16 m^2 (p_1 p_2) - 64 m^2 (q p_2) + 64 m^2 (q p_1)$
$- 64 m^2 q^2 - 64 m^4$

$= 16 (p_1 q)(q_2 q) - 16 q^2 (p_1 p_2) + 16 (p_1 q)(p_2 q) + 16 m^2 (p_1 p_2)$
$+ 64 m^2 ((q p_1) - (q p_2) - q^2) - 64 m^4$

$= 32 (p_1 q)(p_2 q) - 16 (p_1 p_2) \cdot (q^2 - m^2) + 64 m^2 [(q p_1) - (q p_2) - q^2 - m^2]$

✓ $= 2^4 \cdot \{ 2 (p_1 q)(p_2 q) - (p_1 p_2) \cdot (q^2 - m^2) + 4 m^2 [(q p_1) - (q p_2) - q^2 - m^2]$

$\hookrightarrow q = p_1 - p_3 \ ; \ q^2 = t \ ; \ p_1 q = p_1^2 - p_3 p_1 = m^2 - (p_3 p_1)$
$p_2 q = p_1 p_2 - p_2 p_3$

↓ Paso a variables de Mandelstam (ver apartado 1) página 6

$s = (p_1 + p_2)^2 = 2m^2 + 2 p_1 p_2 = (p_3 + p_4)^2 = 2 p_3 p_4 \rightarrow p_3 p_4 = \frac{s}{2} \ ; p_1 p_2 = \frac{s}{2} - m^2$

$t = (p_1 - p_3)^2 = m^2 - 2 p_1 p_3 = (p_4 - p_2)^2 = m^2 - 2 p_2 p_4 \rightarrow p_1 p_3 = p_2 p_4 = \frac{m^2 - t}{2}$

$u = (p_1 - p_4)^2 = m^2 - 2 p_1 p_4 = (p_3 - p_2)^2 = m^2 - 2 p_2 p_3 \rightarrow p_2 p_3 = p_1 p_4 = \frac{m^2 - u}{2}$

$$T' = 2^4 \cdot \left\{ \frac{1}{2}(t+m^2)\cdot(s+u-3m^2) - (q^2-m^2)\cdot\left(\frac{s}{2}-m^2\right) + 4m^2\cdot\left[\frac{t}{2}+\frac{m^2}{2}-\frac{s}{2}-\frac{u}{2}+\frac{3m^2}{2} - q^2 - m^2\right]\right.$$

$$= 2^3\{(t+m^2)(s+u-3m^2) - (t-m^2)(s-2m^2) + 4m^2\cdot[t-s-u-2(t-m^2)]$$

$$= 2^3\{t(s+u) - 3m^2t + m^2(s+u) - 3m^4 - ts + 2m^2t + m^2s - 2m^4 + 4m^2(\overbrace{-s-u-t}^{-2m^2}) + 8m^4\}$$

$$= 2^3\cdot\{tu + m^2(s+u-3t+2b-5m^2+s)\}$$

$$= 2^3\cdot\{tu + m^2\cdot(2s+u-t-5m^2)\} = 2^3\{tu + m^2(2s+u-t-\overset{s+t+u}{2m^2}) - 3m^4\}$$

$$= 2^3\cdot\{tu + m^2(s-2t-3m^2)\} //$$

El factor $\mathcal{M}_2^+\mathcal{M}_2$ se obtiene cambiando $3 \leftrightarrow 4$ respecto al anterior.

↳ La traza de color no cambia nada.

↳ En la de Dirac, $q \to p_1 - p_4 = u \Rightarrow$ Cambiar $t \leftrightarrow u$ (s no cambia)

Por tanto:

$$D_{11} \equiv \overline{\sum} \mathcal{M}_1^+\mathcal{M}_1 = \frac{g_s^4}{2^6\cdot 3^3}\cdot\frac{2^8\cdot 2^3}{(t-m^2)^2}\cdot\{tu + m^2(s-2t-3m^2)\}$$

↗ La parte de color da un factor $\frac{16}{27}$

$$D_{22} \equiv \overline{\sum} \mathcal{M}_2^+\mathcal{M}_2 = \frac{2^5 g_s^4}{3^3}\cdot\frac{1}{(u-m^2)^2}\cdot\{ut + m^2(s-2u-3m^2)\}$$

verlo en $\mathcal{M}_1^+\mathcal{M}_1$

$$\mathcal{M}_1^+\mathcal{M}_2 = \frac{g_s^4}{2^4}\cdot\left(\bar{u}_1\gamma^{\tilde{\mu}}\frac{(\not q+m)}{q^2-m^2}\gamma^{\tilde{\nu}}v_2\right)\cdot\left(\bar{v}_2\gamma^{\nu}\frac{(\not q'+m)}{q'^2-m^2}\gamma^{\mu}u_1\right)$$

$$\cdot\, \epsilon_{3\tilde{\mu}}\cdot\epsilon_{4\tilde{\nu}}\cdot(\lambda^{a_3}\cdot\lambda^{a_4})_{\alpha_1\alpha_2}\cdot\epsilon^*_{3\nu}\,\epsilon^*_{4\mu}\cdot(\lambda^{a_3}\lambda^{a_4})_{\alpha_2\alpha_1}$$

$$= \frac{g_s^4}{2^4\cdot(q^2-m^2)(q'^2-m^2)}\cdot(\bar{u}_1\gamma^{\tilde{\mu}}(\not q+m)\gamma^{\tilde{\nu}}v_2)\cdot(\bar{v}_2\gamma^{\nu}(\not q'+m)\gamma^{\mu}u_1)$$

$$\cdot\,\epsilon_{3\tilde{\mu}}\,\epsilon^*_{3\nu}\,\epsilon_{4\tilde{\nu}}\,\epsilon^*_{4\mu}\cdot(\lambda^{a_3}\lambda^{a_4})_{\alpha_1\alpha_2}(\lambda^{a_3}\lambda^{a_4})_{\alpha_2\alpha_1}$$

Usar truco: ecuación de movimiento del quark (esp. de momentos):

↑ polarización: $(\not p_1 - m)\, u_1 = 0$; $\bar{u}_1(\vec{p}_1)(\not p_1 - m) = 0$

$\overset{\shortparallel}{u_{r_1}(\vec{p}_1)}$

$(\not p_2 + m)\, v_2(\vec{p}_2) = 0$; $\bar{v}_2(\vec{p}_2)(\not p_2 + m) = 0$

↓ 1, 2, el libre, NO el propagador

↘ También lo podría haber utilizado en $\mathcal{M}_1^+\mathcal{M}_1$ para simplificar

↳ $\bar{u}_1\gamma^\mu(\not q+m) = \bar{u}_1\gamma^\mu(\not p_1 - \not p_3 + m) = -\bar{u}_1\gamma^\mu\not p_3 + \bar{u}_1(\gamma^\mu\not p_1 + \gamma^\mu m)$

$= -\bar{u}_1\gamma^\mu\not p_3 + \bar{u}_1(2p_1^\mu - \not p_1\gamma^\mu + m\gamma^\mu) = -\bar{u}_1\gamma^\mu\not p_3 + \bar{u}_1 2p_1^\mu$

Lo Análogamente:

$$(\not{q}'+m)\gamma^\mu u_1 = (\not{p}_1 - \not{p}_4 + m)\gamma^\mu u_1) = (2p_1^\mu - \not{p}_4\gamma^\mu)u_1$$

$$-\gamma^\mu \underbrace{(\not{p}_1 - m)u_1}_{=0} = (2p_1^\mu - \not{p}_4\gamma^\mu)u_1$$

$$\overline{\sum} \mathcal{M}_1^+ \mathcal{M}_2 = \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{g_s^4}{2^4 (t-m^2)\cdot(u-m^2)} \cdot \mathrm{Tr}\left[(\not{p}_1+m)(2p_1^{\tilde{\mu}} - \gamma^{\tilde{\mu}}\not{p}_3)\gamma^{\tilde{\nu}}(\not{p}_2-m)\gamma^\nu(2p_1^\mu - \not{p}_4\gamma^\mu)\right]$$

$$\cdot(-g_{\tilde{\mu}\nu})\cdot(-g_{\tilde{\nu}\mu}) \sum_{a_3,a_4} \mathrm{Tr}_c\left[\lambda^{a_3} \underbrace{\lambda^{a_4}\lambda^{a_3}}\lambda^{a_4}\right]$$

calculada en apartado 1) p. 5 b)
$= -\frac{2^5}{3}$

$$= -\frac{g_s^4}{2\cdot 3^3 (t-m^2)(u-m^2)} \cdot \underbrace{\mathrm{Tr}\left[(\not{p}_1+m)\cdot(2p_{1\nu} - \gamma_\nu \not{p}_3)\gamma_\mu (\not{p}_2-m)\gamma^\nu \cdot (2p_1^\mu - \not{p}_4\gamma^\mu)\right]}_{Z}$$

$$Z = 4\mathrm{Tr}[(\not{p}_1+m)\cdot\not{p}_1 (\not{p}_2-m)\not{p}_1]$$
$$-2\mathrm{Tr}[(\not{p}_1+m)\gamma_\mu(\not{p}_2-m)\not{p}_1\not{p}_4\gamma^\mu]$$
$$-2\mathrm{Tr}[(\not{p}_1+m)\gamma_\nu \not{p}_3 \not{p}_1 (\not{p}_2-m)\gamma^\nu]$$
$$+\mathrm{Tr}[(\not{p}_1+m)\gamma_\nu \not{p}_3 \gamma_\mu (\not{p}_2-m)\gamma^\nu \not{p}_4 \gamma^\mu]$$

$$= -4m^2\mathrm{Tr}[\not{p}_1\not{p}_1] + 4\mathrm{Tr}[\not{p}_1\not{p}_1\not{p}_2\not{p}_1]$$
$$-2\mathrm{Tr}[(2p_1^\mu + m\gamma^\mu - \not{p}_1\gamma^\mu)\gamma_\mu(\not{p}_2-m)\not{p}_1\not{p}_4]$$
$$-2\mathrm{Tr}[(2p_1^\nu + m\gamma^\nu - \not{p}_1\gamma^\nu)\gamma_\nu\not{p}_3\not{p}_1(\not{p}_2-m)]$$
$$-m^2\mathrm{Tr}[\gamma_\nu\not{p}_3\gamma_\mu\gamma^\nu\not{p}_4\gamma^\mu] + \mathrm{Tr}[\not{p}_1\gamma_\nu\not{p}_3\gamma_\mu\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu] \rightarrow \text{ver } \tilde{T}_1 \text{ p. 5}$$

↳ ver $\tilde{T}_8$ de p. 5, cambiando $\nu \leftrightarrow \mu$, $3 \leftrightarrow 4$

$$= -16m^4 + 4\mathrm{Tr}[(p_1^2)\not{p}_2\not{p}_1] - 4\mathrm{Tr}[\not{p}_1(\not{p}_2 - \cancel{m})\not{p}_1\not{p}_4] + 8\,\mathrm{Tr}[(\not{p}_1-m)(\not{p}_2-m)\not{p}_1\not{p}_4]$$

← nº impar γs

$$-4\mathrm{Tr}[\not{p}_1\not{p}_3\not{p}_1(\not{p}_2 - \cancel{m})] + 8\mathrm{Tr}[(\not{p}_1-m)\not{p}_3\not{p}_1(\not{p}_2-m)]$$

impar γs

$$-\tilde{T}_8(3\leftrightarrow 4, \mu\leftrightarrow\nu) + \tilde{T}_1(\mu\leftrightarrow\nu, \text{arriba} \updownarrow \text{abajo})$$

$$= -16m^4 + 16m^2(p_1p_2) - 4\mathrm{Tr}[\not{p}_1\not{p}_2\not{p}_1\not{p}_4] + 8m^2\mathrm{Tr}[\not{p}_1\not{p}_4] + 8\mathrm{Tr}[\not{p}_1\not{p}_2\not{p}_1\not{p}_4]$$
$$-4\mathrm{Tr}[\not{p}_1\not{p}_3\not{p}_1\not{p}_2] + 8m^2\mathrm{Tr}[\not{p}_3\not{p}_1] + 8\mathrm{Tr}[\not{p}_1\not{p}_3\not{p}_1\not{p}_2]$$
$$-16m^2(p_3p_4) - 32(p_1p_2)(p_3p_4)$$

$$= -16m^4 + 16m^2((p_1p_2)+(p_3p_4)) - 32(p_1p_2)(p_3p_4) - 32m^2(p_1p_4) - 32m^2(p_1p_3)$$
$$+4\mathrm{Tr}[\not{p}_1\not{p}_2\not{p}_1\not{p}_4] + 4\mathrm{Tr}[\not{p}_1\not{p}_3\not{p}_1\not{p}_2]$$

$$= -16m^4 - 32(p_1p_2)(p_3p_4) + 16m^2((p_1p_2) - (p_3p_4) + 2(p_1p_4) + 2(p_1p_3))$$
$$+16\cdot(2(p_1p_2)(p_1p_4) - m^2(p_2p_4) + 2(p_1p_3)(p_1p_2) - m^2(p_2p_3))$$

$$= -16m^4 + 32(p_1p_2)\cdot[(p_1p_3) + (p_1p_4) - (p_3p_4)]$$

En función de las variables de Mandelstam:

$$z = -16m^4 + 32\left(\frac{s}{2} - m^2\right)\cdot\left[\frac{m^2-t}{2} + \frac{m^2-u}{2} - \frac{s}{2}\right]$$

$$+ 16m^2\left\{\cancel{\frac{s}{2}} - \cancel{m^2} - \cancel{\frac{s}{2}} - \frac{\cancel{m^2}}{\cancel{2}} + \frac{u}{2} - \frac{\cancel{m^2}}{\cancel{2}} + \frac{t}{2} + \cancel{2}m^2 - u + \cancel{m^2} - t\right\}$$

$$= -16m^4 + 16(s-2m^2)\cdot\left[m^2 - \frac{\overbrace{(s+t+u)}^{2m^2}}{2}\right]$$

$$+16m^2\cdot\left\{-\frac{u}{2} - \frac{t}{2}\right\} = -16m^4 - \frac{16}{2}m^2(u+t)$$

$$= -8m^2\cdot(2m^2 + u + t) = -8m^2(2m^2 + (2m^2 - s))$$

$$= 8m^2\cdot(s-4m^2)$$

$$D_{12} \equiv \overline{\sum} \mathcal{M}_1^+ \mathcal{M}_2 = -\frac{g_s^4 \cdot 2^3 \cdot m^2 (s-4m^2)}{2\cdot 3^3 (t-m^2)\cdot(u-m^2)} = -\frac{2^2 g_s^4 m^2 (s-4m^2)}{3^3 (t-m^2)(u-m^2)}$$

$\hookrightarrow$ Como $\overline{\sum} \mathcal{M}_1^+ \mathcal{M}_2 \in \mathbb{R}$, $\rightarrow \overline{\sum} \mathcal{M}_2^+ \mathcal{M}_1 = \overline{\sum} \mathcal{M}_1^+ \mathcal{M}_2$

$$D_{12}+D_{21} \overset{\hookrightarrow}{\equiv} \overline{\sum}(\mathcal{M}_1^+\mathcal{M}_2 + \mathcal{M}_2^+\mathcal{M}_1) = 2\underset{\overset{\shortparallel}{2 D_{12}}}{\overline{\sum} \mathcal{M}_1^+\mathcal{M}_2} = -\frac{2^3}{3^3}\cdot\frac{g_s^4 m^2 (s-4m^2)}{(t-m^2)(u-m^2)}$$

$$\mathcal{M}_3^+ \mathcal{M}_3 = \frac{g_s^4}{4}\cdot\frac{1}{k^4}\cdot f^{\tilde{b}a_3a_4} f^{ba_3a_4} (\lambda^{\tilde{b}})_{\alpha_1\alpha_2}(\lambda^b)_{\alpha_2\alpha_1}\cdot \epsilon_3^{\tilde{\nu}} \epsilon_4^{\tilde{\sigma}} \cdot \epsilon_3^{*\nu} \epsilon_4^{*\sigma}$$

$$\cdot\left[g_{\tilde{\mu}\tilde{\nu}}(k+p_3)_{\tilde{\sigma}} + g_{\tilde{\nu}\tilde{\sigma}}(p_4-p_3)_{\tilde{\mu}} - g_{\tilde{\sigma}\tilde{\mu}}(p_4+k)_{\tilde{\nu}}\right]\cdot\left[g_{\mu\nu}(k+p_3)_\sigma + g_{\nu\sigma}(p_4-p_3)_\mu - g_{\sigma\mu}(p_4+k)_\nu\right]$$

$$\cdot(\overline{u_1}\gamma^{\tilde{\mu}} v_2)\cdot(\overline{v_2}\gamma^\mu u_1)$$

$$D_{33} \equiv \overline{\sum} \mathcal{M}_3^+\mathcal{M}_3 \overset{\cdot\,\text{trazas}}{=} \left(\frac{1}{2}\right)^2\cdot\left(\frac{1}{3}\right)^2\cdot\frac{g_s^4}{4k^4}\sum_{a_3,a_4} f^{\tilde{b}a_4a_3} f^{ba_4a_3}\,^{\cdot(-1)^2 \text{ al permutar 3-4}} \cdot \underbrace{Tr_c[\lambda^{\tilde{b}}\lambda^b]}_{2\delta^{\tilde{b}b}}(-)g^{\tilde{\nu}\nu}(-)g^{\tilde{\sigma}\sigma}\cdot[\ldots]\cdot[\ldots]$$

$$\cdot Tr[(\not{p}_1 + m)\gamma^{\tilde{\mu}}(\not{p}_2 - m)\gamma^\mu]$$

$$= \frac{g_s^4}{2^3\cdot 3^2 k^4}\sum_{a_3,a_4} f^{ba_4a_3} f^{ba_4a_3}\cdot\left[g_{\tilde{\mu}}^{\ \nu}(k+p_3)^\sigma + g^{\nu\sigma}(p_4-p_3)_{\tilde{\mu}} - g^\sigma_{\ \tilde{\mu}}(p_4+k)^\nu\right]\cdot[g_{\mu\nu}\ldots]$$

$$\cdot\left[\underbrace{Tr[\not{p}_1\gamma^{\tilde{\mu}}\not{p}_2\gamma^\mu]}_{4\cdot(p_1^{\tilde{\mu}}p_2^\mu - (p_1p_2)\cdot g^{\tilde{\mu}\mu} + p_1^\mu p_2^{\tilde{\mu}})} - m^2\, Tr\underbrace{[\gamma^{\tilde{\mu}}\gamma^\mu]}_{=4g^{\tilde{\mu}\mu}}\right]$$

$$= \frac{g_s^4}{2\cdot 3^2\cdot k^4}\overbrace{\sum_{b,a_3,a_4}(f^{ba_4a_3})^2}^{F}\cdot\Big[p_1^\nu p_2^\mu (k+p_3)^\sigma + g^{\nu\sigma}((p_4-p_3)\cdot p_1)p_2^\mu - p_1^\sigma p_2^\mu (k+p_4)^\nu - (p_1p_2)\cdot$$

$$\cdot g^{\mu\nu}(k+p_3)^\sigma - g^{\nu\sigma}(p_4-p_3)^\mu(p_1p_2) + g^{\sigma\mu}(p_4+k)^\nu(p_1p_2) + p_2^\nu p_1^\mu (k+p_3)^\sigma + g^{\nu\sigma}((p_4-p_3)\cdot p_2)\cdot p_1^\mu$$

$$- p_1^\mu p_2^\sigma (p_4+k)^\nu - m^2 g^{\mu\nu}(k+p_3)^\sigma - m^2 g^{\nu\sigma}(p_4-p_3)^\mu + m^2 g^{\sigma\mu}(p_4+k)^\nu\Big]\cdot[g_{\mu\nu}\ldots]$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[\overset{a}{(p_1p_2)\cdot(k+p_3)^2} + \overset{b}{(p_1\cdot(k+p_3))\cdot(p_2\cdot(p_4-p_3))} - \overset{d}{(p_2\cdot(k+p_3))(p_1(p_4+k))}$$

$$+ \overset{c}{((p_4-p_3)\cdot p_1)\cdot(p_2\cdot(k+p_3))} + \overset{b}{((p_4-p_3)\cdot p_1)\cdot(p_2\cdot(p_4-p_3))\cdot 4} - \overset{c}{((p_4-p_3)\cdot p_1)\cdot(p_2\cdot(p_4+k))}$$

$$- (p_2\cdot(k+p_4))\overset{e}{(p_1\cdot(k+p_3))} - (p_1\cdot(k+p_4))\overset{b}{(p_2\cdot(p_4-p_3))} + \overset{a}{(p_1p_2)\cdot(k+p_4)^2}$$

$$- \overset{a}{(p_1p_2)(k+p_3)^2\cdot 4} - \overset{a}{(p_1p_2)\cdot((k+p_3)\cdot(p_4-p_3))} + \overset{a}{(p_1p_2)((k+p_3)\cdot(p_4+k))}$$

$$- \overset{a}{(p_1p_2)((k+p_3)\cdot(p_4-p_3))} - \overset{a}{4\cdot(p_4-p_3)^2\cdot(p_1p_2)} + \overset{a}{(p_1p_2)\cdot((p_4-p_3)\cdot(p_4+k))}$$

$$+ \overset{a}{((p_4+k)\cdot(k+p_3))\cdot(p_1p_2)} + \overset{a}{(p_1p_2)\cdot((p_4+k)\cdot(p_4-p_3))} - \overset{a}{4\cdot(p_1p_2)\cdot(p_4+k)^2}$$

$$+ \overset{a}{(p_1p_2)(k+p_3)^2} + \overset{c}{(p_2(k+p_3))(\cdot p_1(p_4-p_3))} - (p_2(p_4+k))\cdot(p_1(k+p_3))^{e}$$

$$+ \overset{b}{(p_2(p_4-p_3))((k+p_3)\cdot p_1)} + \overset{b}{4\cdot(p_2(p_4-p_3))\cdot(p_1(p_4-p_3))} - \overset{b}{(p_2(p_4-p_3))(p_1\cdot(p_4+k))}$$

$$- \overset{d}{(p_1(p_4+k))(p_2(k+p_3))} - \overset{c}{(p_1\cdot(p_4-p_3))\cdot(p_2(p_4+k))} + \overset{a}{(p_1p_2)(p_4+k)^2}$$

$$- m^2\cdot 4(k+p_3)^2 - m^2\cdot((k+p_3)(p_4-p_3)) + m^2((k+p_3)\cdot(k+p_4)) - m^2\cdot(\cdot(p_4-p_3)\cdot(k+p_3))$$

$$- m^2\cdot 4(p_4-p_3)^2 + m^2((p_4-p_3)(p_4+k)) + m^2((p_4+k)(k+p_3)) + m^2(p_4+k)(p_4-p_3) - 4m^2(p_4+k)^2\Big]$$

$$= \frac{g_s^4\cdot F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\cdot\{(k+p_3)^2 + (k+p_4)^2 - 4(k+p_3)^2 - (k+p_3)(p_4-p_3) + (k+p_3)(p_4+k) - (k+p_3)\cdot(p_4-p_3)$$

$$- 4(p_4-p_3)^2 + (p_4-p_3)(p_4+k) + (p_4+k)(p_3+k) + (p_4+k)(p_4-p_3) - 4(p_4+k)^2$$

$$+ (k+p_3)^2 + (p_4+k)^2\}^{13\,a)}$$

$$+ (p_2\cdot(p_4-p_3))\cdot\{(p_1\cdot(k+p_3)) + 4(p_1\cdot(p_4-p_3)) - (p_1(k+p_4)) + (p_1(k+p_3)) + 4(p_1(p_4-p_3)) - (p_1(p_4+k))\}^{6}$$

$$+ (p_1\cdot(p_4-p_3))\cdot\{(p_2\cdot(k+p_3)) - (p_2(k+p_4)) + (p_2(k+p_3)) - (p_2(k+p_4))\}_4$$

$$+ (p_1(p_4+k))\cdot\{-(p_2(k+p_3)) - (p_2(k+p_3))\}^2 + (p_2(k+p_4))\cdot\{-p_1(k+p_3) - p_1(k+p_3)\}^2$$

$$+ m^2\cdot\{(p_4+k)(p_4-p_3) + (p_4+k)(k+p_3) + (p_4+k)(p_4-p_3) + (p_4+k)(k+p_3)$$

$$-4(k+p_3)^2 - (k+p_3)(p_4-p_3) - (p_4-p_3)(k+p_3) - 4(p_4-p_3)^2 - 4(p_4+k)^2\}_9\Big]$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\cdot\{-2(k+p_3)^2 - 2(k+p_4)^2 - 2(k+p_3)(p_4-p_3) + 2(k+p_3)(k+p_4)$$

$$- 4(p_4-p_3)^2 + 2(p_4-p_3)(k+p_4)\}$$

$$+ (p_2\cdot(p_4-p_3))\{(p_1\cdot(k+p_3+4p_4-4p_3-k-p_4+k+p_3+4p_4-4p_3-p_4-k))\}$$

$$+ (p_1\cdot(p_4-p_3))\{(p_2\cdot(k+p_3-k-p_4)\cdot 2)\}$$

$$- 2(p_1(p_4+k))\cdot(p_2(k+p_3)) - 2(p_2(k+p_4))(p_1(k+p_3))$$

$$+ m^2\{(p_4+k)\cdot(p_4-p_3+k+p_3+p_4-p_3+k+p_3-4p_4-4k) - (k+p_3)\cdot(4k+4p_3+p_4-p_3+p_4-p_3)$$

$$- 4(p_4-p_3)^2\}$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\cdot\{(k+p_3)(-2k-2p_3\overbrace{-2p_4+2p_3}^{=0}+2k+2p_4) + 2(k+p_4)\cdot(p_4-p_3-k-p_4) - 4(p_4-p_3)^2$$

$$+ (p_2(p_4-p_3))\cdot\{\cdot(p_1\cdot(-6p_3+6p_4))\} + 2(p_1(p_4-p_3))(p_2\cdot(p_3-p_4)) - 2(p_1(p_4+k))(p_2(k+p_3))$$

$$- 2(p_2(k+p_4))(p_1(k+p_3)) + m^2\cdot\{(-2p_4-2k)\cdot(p_4+k) - (k+p_3)\cdot(4k+2p_3+2p_4) - 4(p_4-p_3)^2\Big]$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\{-2(k+p_4)(k+p_3)-4(p_4-p_3)^2\}$$

$$+6(p_2(p_4-p_3))\cdot(p_1(p_4-p_3))-2(p_2(p_4-p_3))(p_1(p_4-p_3))$$

$$-2(p_1(k+p_4))\cdot(p_2(k+p_3))-2(p_1(k+p_3))\cdot(p_2(k+p_4))$$

$$+m^2\{-2(p_4+k)^2-4(p_4-p_3)^2-2(k+p_3)(2k+p_4+p_3)\}\Big]$$

$p_3^2=p_4^2=0$; $k=p_1+p_2=p_3+p_4$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\{-2(2p_4+p_3)\cdot(2p_3+p_4)+8p_3p_4\}$$

$$+4(p_2(p_4-p_3))\cdot(p_1(p_4-p_3))-2(p_1(2p_4+p_3))(p_2(2p_3+p_4))$$

$$-2(p_1(2p_3+p_4))\cdot(p_2(2p_4+p_3))$$

$$+m^2\{-2(2p_4+p_3)^2+8p_3p_4-6(2p_3+p_4)\cdot(p_3+p_4)\}\Big]$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[(p_1p_2)\cdot\{-\cancel{8p_4p_3}-2p_3p_4+\cancel{8p_3p_4}\}$$

$$+4((p_2p_4)-(p_2p_3))\cdot((p_1p_4)-(p_1p_3))-2(2(p_1p_4)+(p_1p_3))\cdot(2(p_2p_3)+(p_2p_4))$$

$$-2(2(p_1p_3)+(p_1p_4))\cdot(2(p_2p_4)+(p_2p_3))$$

$$+m^2\{-\cancel{8p_3p_4}+\cancel{8p_3p_4}-12p_3p_4-6p_3p_4\}$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[-2(p_1p_2)(p_3p_4)+4(p_2p_4)(p_1p_4)-4(p_2p_4)(p_1p_3)-4(p_2p_3)(p_1p_4)$$

$$+4(p_2p_3)(p_1p_3)-8(p_1p_4)(p_2p_3)-4(p_1p_4)(p_2p_4)-4(p_1p_3)(p_2p_3)-2(p_1p_3)(p_2p_4)$$

$$-8(p_1p_3)(p_2p_4)-4(p_1p_3)(p_2p_3)-4(p_1p_4)(p_2p_4)-2(p_1p_4)(p_2p_3)$$

$$-18m^2(p_3p_4)\Big]_{14}$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[-2(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_4)\cdot(4-4-4)+(p_1p_3)(p_2p_4)\cdot(-4-2-8)$$

$$+(p_1p_4)(p_2p_3)(-4-8-2)+(p_1p_3)(p_2p_3)\cdot(4-4-4)-18m^2(p_3p_4)\Big]$$

$$= \frac{g_s^4 F}{2\cdot 3^2 k^4}\cdot\Big[-2(p_1p_2)(p_3p_4)-4(p_1p_4)(p_2p_4)-4(p_1p_3)(p_2p_3)-14(p_1p_3)(p_2p_4)$$

$$-14(p_1p_4)(p_2p_3)-18m^2(p_3p_4)\Big]$$

↳ Sostituimos Mandelstam (ver p. 9)

$$= \frac{Fg_s^4(-1)}{2\cdot 3^2 s^2}\cdot\Big[(s-2m^2)\cdot\frac{s}{2}+(m^2-t)(m^2-u)+(m^2-t)(m^2-u)+\frac{7}{2}(m^2-t)^2$$

$$+\frac{7}{2}(m^2-u)^2+9m^2s\Big]$$

$$= \frac{-g_s^4 F}{2^2\cdot 3^2\cdot s^2}\cdot\Big[s(s-2m^2)+4(m^2-t)(m^2-u)+7(m^2-t)^2+7(m^2-u)^2+18m^2s)\Big]$$

$$= \frac{-g_s^4 F}{2^2 3^2 s^2}\cdot\Big[s^2+16m^2s+4m^4-4m^2t-4m^2u+4tu+7m^4+7t^2-14m^2t$$

$$+7m^4+7u^2-14m^2u\Big]$$

$$= \frac{-g_s^4 F}{2^2 3^2 s^2}\cdot\Big[s^2+7(t^2+u^2)+4tu+18m^4-18m^2(\overset{=2m^2-s}{\widetilde{t+u}})+16m^2s\Big]$$

En cuanto al factor de color,

$$F=\sum_{a_3,a_4,b} f^{ba_4a_3}\, f^{ba_4a_3}$$

Usamos que: $f_{abc} = i\,(t_A^{a})_{bc} = -i(t_A^{a})_{cb}$; $(t^a_A)_{bc}\,(t^a_A)_{cd} = C_2(G)\delta_{bd}$

$\downarrow$

$$F = \sum_{a_3 a_4, b} (-1)\cdot(t_A^{b})_{a_4a_3}\,\underset{-(t_A^{b})_{a_3a_4}}{\underset{\shortparallel}{(t_A^{b})_{a_4a_3}}} = +\sum_{a_4,b} C_2(G)\cdot\delta_{a_4,a_4} = 8\,\underset{3}{\underset{\shortparallel}{C_2}}(G) = 24 = 3\cdot 2^3$$

$\downarrow$

$$D_{33}\equiv\overline{\sum} M_3^+M_3 = -\frac{2g_s^4}{3s^2}\cdot\left[s^2+7(t^2+u^2)+4tu+34m^2s-18m^4\right]$$

$$\mathcal{M}_1^+\mathcal{M}_3 = \frac{ig_s^4}{2^3}\cdot\frac{1}{q^2-m^2}\cdot\frac{1}{k^2}\,(\bar u_1\gamma^{\tilde\mu}(\not{q}+m)\gamma^{\tilde\nu}v_2)\cdot\epsilon_{3\tilde\mu}\,\epsilon_{4\tilde\nu}\cdot(\lambda^{a_3}\lambda^{a_4})_{\alpha_1\alpha_2}\cdot(\lambda^{b})_{\alpha_2\alpha_1}\overset{\text{permuto}}{(-)f^{ba_4a_3}}$$

$$\cdot\,\epsilon^{*\nu}_3\,\epsilon^{*\sigma}_4\cdot\left[g_{\mu\nu}(k+p_3)_\sigma + g_{\nu\sigma}(p_4-p_3)_\mu - g_{\sigma\mu}(p_4+k)\right]\cdot(\bar v_2\gamma^\mu u_1)$$

$$D_{13}\equiv\overline{\sum} M_1^+M_3 = \left(\frac{1}{2}\right)^2\cdot\left(\frac{1}{3}\right)^2\cdot\frac{ig_s^4}{2^3}\cdot\frac{1}{q^2-m^2}\cdot\frac{1}{k^2}\cdot\mathrm{Tr}[(\not{p}_1+m)\gamma^{\tilde\mu}(\not{q}+m)\gamma^{\tilde\nu}(\not{p}_2-m)\gamma^{\mu}]\,(-g_{\tilde\mu}{}^{\nu})\cdot(-g_{\tilde\nu}{}^{\sigma})\underbrace{\sum_{a_3,a_4,b}\mathrm{Tr}[\lambda^{a_3}\lambda^{a_4}\lambda^{b}]}_{\text{Ver prop. matrices Gell-Mann}}$$

$$\cdot\, f^{ba_4a_3}\cdot[g_{\mu\nu}\dots]$$

$$= -\frac{ig_s^4}{2^5\,3^2}\cdot\frac{1}{q^2-m^2}\cdot\frac{1}{k^2}\cdot\underbrace{\mathrm{Tr}[(\not{p}_1+m)\gamma^{\nu}(\not{q}+m)\gamma^{\sigma}(\not{p}_2-m)\gamma^{\mu}]}_{z'}\cdot 2\sum_{a_i,b}(\underset{\downarrow\ \text{simétrico}}{d_{a_3a_4b}} + i f_{a_3a_4b})\cdot\underset{\downarrow\ \text{antisimét}}{f^{ba_4a_3}}$$

$$\cdots\underset{0}{\underset{\shortparallel}{\times}}\cdots$$

$$\cdot\,[g_{\mu\nu}\dots]$$

$$= \frac{-(i)^2g_s^4}{2^4\cdot3^2}\cdot\frac{1}{k^2}\cdot\frac{1}{q^2-m^2}\cdot z'\underbrace{\sum f^{ba_3a_4}\overset{-f^{ba_3a_4}}{\overset{\shortparallel}{f^{ba_4a_3}}}}_{=F}\cdot[g_{\mu\nu}\dots] = \frac{g_s^4}{2^4\cdot3^2}\cdot\frac{1}{k^2}\cdot\frac{1}{q^2-m^2}\cdot z'\cdot3\cdot2^3$$

$$= -\frac{g_s^4}{2\cdot3}\cdot\frac{1}{k^2}\cdot\frac{1}{q^2-m^2}\cdot z'\cdot\underbrace{[g_{\mu\nu}\dots]}_{\tilde z}$$

$z' \Rightarrow$ Para $\not{q}$ usamos el truco $\bar u_1(\not{p}_1-m)=0$ (ver p. 9b) $^{q=p_1-p_3}$

$\hookrightarrow \bar u_1\gamma^\mu(\not{q}+m) = \bar u_1(2p_1^{\mu}-\gamma^\mu\not{p}_3)$

$\hookrightarrow z' = \mathrm{Tr}[(\not{p}_1+m)\cdot(2p_1^{\nu}-\gamma^{\nu}\not{p}_3)\gamma^{\sigma}(\not{p}_2-m)\gamma^{\mu}]$

$= 2p_1^{\nu}\,\mathrm{Tr}[(\not{p}_1+m)\gamma^{\sigma}(\not{p}_2-m)\gamma^{\mu}]$

$\quad -\mathrm{Tr}[(\not{p}_1+m)\gamma^{\nu}\not{p}_3\gamma^{\sigma}(\not{p}_2-m)\gamma^{\mu}]$

$= -2m^2p_1^{\nu}\cdot\mathrm{Tr}[\gamma^{\sigma}\gamma^{\mu}] + 2p_1^{\nu}\,\mathrm{Tr}[\not{p}_1\gamma^{\sigma}\not{p}_2\gamma^{\mu}]$

$\quad -\mathrm{Tr}[\not{p}_1\gamma^{\nu}\not{p}_3\gamma^{\sigma}\not{p}_2\gamma^{\mu}] + m^2\,\mathrm{Tr}[\gamma^{\nu}\not{p}_3\gamma^{\sigma}\gamma^{\mu}]$

$= -8m^2p_1^{\nu}g^{\sigma\mu} + 8p_1^{\nu}\cdot(p_1^{\sigma}p_2^{\mu}-(p_1p_2)g^{\sigma\mu}+p_1^{\mu}p_2^{\sigma}) + 4m^2[p_3^{\nu}g^{\sigma\mu}-p_3^{\mu}g^{\nu\sigma}+p_3^{\sigma}g^{\mu\nu}]$

$\quad -\mathrm{Tr}[\not{p}_1\gamma^{\nu}\not{p}_3\gamma^{\sigma}\not{p}_2\gamma^{\mu}]$

$= -8m^2p_1^{\nu}g^{\mu\sigma} + 8p_1^{\nu}p_1^{\sigma}p_2^{\mu} - 8p_1^{\nu}(p_1p_2)g^{\mu\sigma} + 8p_1^{\mu}p_1^{\nu}p_2^{\sigma} + 4m^2p_3^{\nu}g^{\mu\sigma} - 4m^2p_3^{\mu}g^{\nu\sigma}$

$$= -8p_1^\nu g^{\mu\sigma}((p_1p_2)+m^2) + 8p_1^\nu \cdot p_1^\sigma p_2^\mu + 8p_1^\nu p_1^\mu p_2^\sigma$$

$$+4m^2(p_3^\nu g^{\mu\sigma} - p_3^\mu g^{\nu\sigma} + p_3^\sigma g^{\mu\nu}) - \tau'' \cdot p_{1\alpha} p_{3\beta} \cdot p_{2\delta}$$

$$\tau'' = Tr[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\gamma^\delta\gamma^\mu] \overset{\text{propiedades trazas}}{=} g^{\alpha\nu} Tr[\gamma^\beta\gamma^\sigma\gamma^\delta\gamma^\mu] - g^{\alpha\beta} Tr[\gamma^\nu\gamma^\sigma\gamma^\delta\gamma^\mu]$$

$$+g^{\alpha\sigma} Tr[\gamma^\nu\gamma^\beta\gamma^\delta\gamma^\mu] - g^{\alpha\delta} Tr[\gamma^\nu\gamma^\beta\gamma^\sigma\gamma^\mu] + g^{\alpha\mu} Tr[\gamma^\nu\gamma^\beta\gamma^\sigma\gamma^\delta]$$

$$= 4\cdot\{ g^{\alpha\nu}(g^{\beta\sigma}g^{\delta\mu} - g^{\beta\delta}g^{\sigma\mu} + g^{\beta\mu}g^{\sigma\delta}) - g^{\alpha\beta}(g^{\nu\sigma}g^{\mu\delta} - g^{\nu\delta}g^{\mu\sigma} + g^{\mu\nu}g^{\delta\sigma})$$

$$+ g^{\alpha\sigma}(g^{\nu\beta}g^{\delta\mu} - g^{\nu\delta}g^{\mu\beta} + g^{\mu\nu}g^{\beta\delta}) - g^{\alpha\delta}(g^{\nu\beta}g^{\mu\sigma} - g^{\nu\sigma}g^{\mu\beta} + g^{\mu\nu}g^{\beta\sigma})$$

$$+ g^{\alpha\mu}(g^{\nu\beta}g^{\sigma\delta} - g^{\nu\sigma}g^{\beta\delta} + g^{\nu\delta}g^{\beta\sigma})\}$$

$$\hookrightarrow p_{1\alpha}p_{3\beta}p_{2\delta}\,\tau' = 4\cdot\{p_1^\nu(p_3^\sigma p_2^\mu - g^{\mu\sigma}(p_3p_2) + p_3^\mu p_2^\sigma) - (p_1p_3)\cdot(g^{\nu\sigma}p_2^\mu - g^{\mu\sigma}p_2^\nu + g^{\mu\nu}p_2^\sigma)$$

$$+ p_1^\sigma(p_3^\nu p_2^\mu - p_3^\mu p_2^\nu + g^{\mu\nu}\cdot(p_2p_3)) - (p_1p_2)\cdot(p_3^\nu g^{\mu\sigma} - g^{\nu\sigma}p_3^\mu + g^{\mu\nu}p_3^\sigma)$$

$$+ p_1^\mu(p_3^\nu p_2^\sigma - (p_3\cdot p_2)g^{\nu\sigma} + p_3^\sigma p_2^\nu)\}$$

$$\downarrow$$

$$\tau' = 4\{p_1^\nu\cdot(-p_3^\sigma p_2^\mu + \overset{g^{\mu\sigma}}{(p_3p_2)} - p_3^\mu p_2^\sigma + 2p_1^\sigma p_2^\mu - 2(p_1p_2)g^{\mu\sigma} + 2p_1^\mu p_2^\sigma)$$

$$+ m^2(p_3^\nu g^{\mu\sigma} - p_3^\mu g^{\nu\sigma} + p_3^\sigma g^{\mu\nu} - 2p_1^\nu g^{\mu\sigma})$$

$$+ (p_1p_3)(g^{\nu\sigma}p_2^\mu - g^{\mu\sigma}p_2^\nu + g^{\mu\nu}p_2^\sigma)$$

$$+ p_1^\sigma(p_3^\mu p_2^\nu - p_2^\mu p_3^\nu - (p_2p_3)g^{\mu\nu}) + (p_1p_2)(p_3^\nu g^{\mu\sigma} - g^{\nu\sigma}p_3^\mu + g^{\mu\nu}p_3^\sigma)\}$$

$$+ p_1^\mu((p_2p_3)g^{\nu\sigma} - p_3^\nu p_2^\sigma - p_2^\nu p_3^\sigma)$$

$$\hookrightarrow \tilde{\tau} \equiv \tau'\cdot[g_{\mu\nu}(k+p_3)_\sigma + g_{\nu\sigma}(p_4-p_3)_\mu - g_{\mu\sigma}(p_4+k)_\nu] = (\tau_1+\tau_2+\tau_3+\tau_4+\tau_5 + \tau_6)\cdot 4$$

$$\tau_1 = -(p_1p_2)(p_3(k+p_3)) - (p_1p_3)(p_2(p_4-p_3)) + (p_2p_3)(p_1(k+p_4))$$

$$+ (p_2p_3)\cdot((p_1(k+p_3)) + (p_1(p_4-p_3)) - 4(p_1(p_4+k)))$$

$$- (p_1p_3)\cdot(p_2(k+p_3)) - (p_1p_2)(p_3(p_4-p_3)) + (p_2p_3)(p_1(k+p_4))$$

$$+2(p_1p_2)\cdot(p_1(k+p_3)) + 2p_1^2\cdot(p_2(p_4-p_3)) - 2(p_1p_2)\cdot((k+p_4)\cdot p_1)$$

$$+ 2(p_1p_2)\cdot(-(p_1(k+p_3)) - (p_1(p_4-p_3)) + 4(p_1(p_4+k)))$$

$$+ 2p_1^2(p_2(k+p_3)) + 2(p_1p_2)(p_1(p_4-p_3)) - 2(p_1p_2)\cdot(p_1(k+p_4))$$

$p_3^2 = p_4^2 = 0$

$k = p_3 + p_4$

$$= (p_1p_2)\cdot\{-(p_3p_4) - (p_3p_4) + 2(p_1p_4) + 4(p_1p_3) - 2p_1p_3 - 4(p_1p_4) - 2p_1p_4 - 4p_1p_3$$

$$- 2p_1p_4 + 2p_1p_3 + 8p_1p_3 + 16p_1p_4 + 2p_1p_4 - 2p_1p_3 - 2p_1p_3 - 4p_1p_4\}$$

$$+ (p_1p_3)\cdot\{-p_2p_4 + p_2p_3 - 2p_2p_3 - p_2p_4\}$$

$$+ (p_2p_3)\cdot\{2p_1p_4 + p_1p_3 + 2p_1p_3 + p_1p_4 + p_1p_4 - p_1p_3 - 8p_1p_4 - 4p_1p_3 + 2p_1p_4 + p_1p_3\}$$

$$+2m^2\cdot\{p_2\cdot(p_4 - p_3 + k + p_3)\}$$

$$= (p_1p_2)\cdot\{-2(p_3p_4) + 8(p_1p_4) + 4(p_1p_3)\} - 2(p_1p_3)(p_2p_4)$$

$$- (p_1p_3)(p_2p_3) + (p_2p_3)\cdot\{-2(p_1p_4) - (p_1p_3)\} + 2m^2\{2p_2p_4 + p_2p_3\}$$

$$= 2(p_1p_2)\cdot\{2(p_1p_3) + 4(p_1p_4) - (p_3p_4)\} - 2(p_1p_3)(p_2p_3) - 2(p_2p_3)(p_1p_4) + 2m^2\{2p_2p_4 + p_2p_3\}$$

$$\frac{Z_2}{m^2} = \cdot\, p_3\cdot(k+p_3) + p_3\cdot(p_4-p_3) - 4\,(p_3(p_4+k))$$

$$- p_3\,(k+p_3) - 4\,p_3\,(p_4-p_3) + (p_3\,(p_4+k))$$

$$+ 4\,(p_3\,(k+p_3)) + p_3\,(p_4-p_3) - (p_3\,(p_4+k))$$

$$-2\,p_1\,(k+p_3) - 2\,p_1\,(p_4-p_3) + 8\,(p_1\,(p_4+k))$$

$p_3^2=p_4^2=0$ ↖

$k=p_3+p_4$

$$= p_3p_4\cdot(1-4+1) - 8\,p_3p_4 + 4\,p_3p_4 - 4\,p_1p_3 - 2\,p_1p_4 - 2p_1p_4 + 2p_1p_3$$

$$+ 8p_1p_3 + 16\,p_1p_4$$

$$= -6\,p_3p_4 + 6\,p_1p_3 + 12\,p_1p_4 = \underline{6\,(p_1p_3 + 2p_1p_4 - p_3p_4)}\ \checkmark$$

$$\frac{Z_3}{(p_1p_3)} = (p_2(k+p_3)) + 4p_2(p_4-p_3) - p_2(p_4+k) - p_2(k+p_3) - p_2\,(p_4-p_3) + 4\,p_2\,(k+p_4)$$

$$+ 4\,p_2\,(k+p_3) + p_2\,(p_4-p_3) - p_2\,(p_4+k)$$

$$= p_2(4p_4 - 4p_3 - 2p_4 - p_3 + 4\,p_3 + 8p_4 + 8p_3 + 4\,p_4 - 2p_4 - p_3)$$

$$= p_2\,(12p_4 + 6\,p_3) = \underline{6\,(p_2p_3 + 2\,p_2p_4)}\ \checkmark$$

$$Z_4 = (p_1(k+p_3))\cdot(p_2p_3) + (p_1p_2)\cdot(p_3(p_4-p_3)) - (p_1p_3)\cdot(p_2(p_4+k))$$

$$- (p_2p_3)\cdot(p_1(k+p_3)) - (p_1p_3)(p_2(p_4-p_3)) + (p_1p_2)(p_3(p_4+k))$$

$$- 4(p_2p_3)(p_1(k+p_3)) - (p_2p_3)(p_1\,(p_4-p_3)) + (p_1\,(p_4+k))\cdot(p_2p_3)$$

$$= (p_1p_2)\cdot(p_3p_4 + 2p_3p_4) + (p_1p_3)(-2p_2p_4 - p_2p_3 - p_2p_4 + p_2p_3)$$

$$+ (p_2p_3)(-8p_1p_3 - 4p_1p_4 - p_1p_4 + p_1p_3 + 2p_1p_4 + p_1p_3)$$

$$= 3(p_1p_2)(p_3p_4) - 3\,(p_1p_3)(p_2p_4) + (p_2p_3)(-6\,p_1p_3 - 3\,p_1p_4)$$

$$= \underline{3\,(p_1p_2)(p_3p_4)} - \underline{3\,(p_1p_3)(p_2p_4)} - \underline{(p_2p_3)(6(p_1p_3)} + \underline{3\,(p_1p_4))}\ \checkmark$$

$$\frac{Z_5}{(p_1p_2)} = p_3\,(k+p_3) + p_3(p_4-p_3) - 4\,p_3\,(k+p_4) - p_3\,(k+p_3) - 4\,p_3(p_4-p_3) + p_3(p_4+k)$$

$$+ 4\,(p_3\,(k+p_3)) + p_3\,(p_4-p_3) - p_3\,(p_4+k)$$

$$= -2\,p_3p_4 - 8\,p_3p_4 + 4p_3p_4 = \underline{-6\,p_3p_4}$$

$$Z_6 = (p_2p_3)\cdot\{p_1\,(k+p_3) + 4\,p_1(p_4-p_3) - p_1\,(p_4+k)\}$$

$$- (p_1p_3)(p_2\,(k+p_3)) - (p_1\,(p_4-p_3))\cdot(p_2p_3) + (p_1p_2)\,(p_3(k+p_4))$$

$$- (p_1p_2)\,(p_3\,(k+p_3)) - (p_2p_3)(p_1\,(p_4-p_3)) + (p_1p_3)\,(p_2\,(k+p_4))$$

$$= (p_2p_3)\cdot\{2p_1p_3 + p_1p_4 + 4p_1p_4 - 4p_1p_3 - 2p_1p_4 - p_1p_3 - 2p_1p_4 + 2p_1p_3\}$$

$$+ (p_1p_3)(-2p_2p_3 - p_2p_4 + 2p_2p_4 + p_2p_3)$$

$$+ (p_1p_2)\cdot(2p_3p_4 - p_3p_4)$$

$$= (p_2p_3)\cdot(-1(p_1p_3) + (p_1p_4)) + (p_1p_3)\cdot(p_2p_4 - p_2p_3) + (p_1p_2)(p_3p_4)$$

$$= \underline{(p_1p_4)(p_2p_3)} - \underline{2(p_1p_3)(p_2p_3)} + \underline{(p_1p_3)(p_2p_4)} + \underline{(p_1p_2)(p_3p_4)}\ \checkmark$$

Juntando todo: $\frac{Z}{4} = (p_1p_2)\cdot\{4(p_1p_3) + 8(p_1p_4) - 2(p_3p_4) + 3\,(p_3p_4) - 6p_3p_4 + (p_3p_4)\}$

$$+ (p_1p_3)\cdot\{-2\,(p_2p_3)\ \underset{\hookrightarrow -2(p_2p_4)}{+\,6m^2} + 6(p_2p_3) + 12(p_2p_4) - 3\,(p_2p_4) - 6\,(p_2p_3) - 2(p_2p_3) + (p_2p_4)\}$$

$$+ (p_2p_3)\cdot\{-2\,(p_1p_4) + 2m^2\ \underset{\hookrightarrow -3p_1p_4}{+}\ p_1p_4\} + p_2p_4\cdot\{4m^2\}$$

$$+ (p_1p_4)\cdot\{12m^2\} + (p_3p_4)\cdot\{-6\,m^2\}$$

↳ $\frac{\tilde{\Sigma}}{4} = (p_1p_2)\cdot\{\cdot 4(p_1p_3) + 8(p_1p_4) - 4(p_3p_4)\}$

$+ (p_1p_3)\cdot\{-4(p_2p_3) + 8(p_2p_4) + 6m^2\}$

$+ (p_2p_3)\cdot\{-4(p_1p_4) + 2m^2\} + 4m^2(p_2p_4) + 12m^2(p_1p_4) - 6m^2(p_3p_4)$

Sustituyendo variables de Mandelstam: (p. 9b):

$$\frac{\tilde{\Sigma}}{4} = \left(\frac{s}{2} - m^2\right)\cdot\{2(m^2-t) + 4(m^2-u) - 2s\}$$

$$+ \frac{m^2-t}{2}\cdot\{-2(m^2-u) + 4(m^2-t) + 6m^2\}$$

$$+ \frac{m^2-u}{2}\cdot\{-2(m^2-u) + 2m^2\} + 2m^2(m^2-t) + 6m^2(m^2-u) - 3m^2s$$

$$= (s-2m^2)\cdot(3m^2 - t - 2u - s) + (m^2-t)\cdot(4m^2 + \underset{-2t}{u}) + (m^2-u)\cdot u$$

$$+ 2m^4 - 2m^2t + 6m^4 - 6m^2u - 3m^2s$$

$$= 3m^2s - st - 2us - s^2 - 6m^4 + 2m^2t + 4m^2u + 2m^2s + 4m^4 + m^2u - \underset{-2m^2t+2t^2}{4m^2t}$$

$$- tu + m^2u - u^2 + 2m^4 - 2m^2t + 6m^4 - 6m^2u - 3m^2s$$

$$= -s^2 - u^2 - st - 2us - tu + m^2(+2s - 6t + 6m^2)$$

$$= -(s+u)^2 - t(u-s) + m^2(2s - 6t + 6m^2)$$

$$= -(s+u)^2 + 2t^2 - t(u+s) + 2m^2(s - 3t + 3m^2)$$

$$= -(s+u)[\underbrace{s+u+t}_{2m^2}] + 2t^2 + 2m^2(s - 3t + 3m^2)$$

$$= 2m^2\cdot(s - 3t + 3m^2 - s - u) + 2t^2 = 2t^2 + 2m^2(3m^2 - u - 3t)$$

↳ $u = 2m^2 - s - t$

$$= 2t^2 + 2m^2\cdot(m^2 + s - 2t)$$

$$\tilde{\Sigma} = 2^3[t^2 + m^2\cdot(m^2 + s - 2t)]$$

$$D_{13} \equiv \overline{\sum} M_1^+ M_3 = -\frac{g_s^4\cdot 2^3}{2\cdot 3s(t-m^2)}\cdot[t^2 + m^2(m^2 + s - 2t)] \in \mathbb{R}$$

$$D_{13} + D_{31} \equiv \overline{\overline{\sum}} M_1^+ M_3 + \overline{\overline{\sum}} M_3^+ M_1 = \overset{=2D_{13}}{2\overline{\overline{\sum}} M_1^+ M_3} = -\frac{2^3 g_s^4}{3s(t-m^2)}\cdot(t^2 + m^2\cdot(m^2 + s - 2t))$$

Por analogía, cambiando $1 \leftrightarrow 2 \Rightarrow t \Leftrightarrow u$; s se mantiene.

$$D_{23} + D_{32} \equiv \overline{\overline{\sum}} M_2^+ M_3 + \overline{\overline{\sum}} M_3^+ M_2 = -\frac{2^3 g_s^4}{3s(u-m^2)}\cdot(u^2 + m^2(m^2 + s - 2u))$$

Rearreglo:

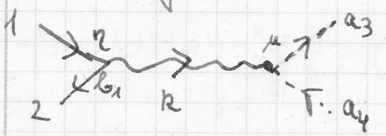
$$2D_{13} \equiv 2\sum M_1^+ M_3 = -\frac{2^3 g_s^4}{3s(t-m^2)}\cdot(t^2 \underbrace{+m^4 - 2m^2t}_{(t-m^2)^2} + m^2s) = -\frac{2^3 g_s^4}{3}\left(\frac{t-m^2}{s} + \frac{m^2}{t-m^2}\right)$$

$$2D_{23} = 2\sum M_2^+ M_3 = -\frac{2^3 g_s^4}{3}\left(\frac{u-m^2}{s} + \frac{m^2}{u-m^2}\right)$$

[illegible]

Como $\overline{\sum} \equiv \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^2 \sum_{r_1,r_2,r_3,r_4} \cdot \sum_{\alpha_1,\alpha_2} \sum_{a_3,a_4}$ y en todos los diagramas

hemos usado que $\sum_{r_3=0}^{3} \epsilon_3^{*\mu} \epsilon_3^{\nu} = -g^{\mu\nu}$, estamos sumando polarizaciones que no son físicas. Hemos de corregir con el diagrama de ghosts: $(q\bar{q} \to \phi\bar{\phi})$



Aplico reglas de Feynman

$$G = (\bar{v}_2 \gamma^\eta u_1) \cdot \frac{g_s}{2} \cdot (\lambda^{b_1})_{\alpha_2\alpha_1} \cdot \frac{-g_{\eta\mu}}{k^2+i\epsilon} \delta^{b_1 b_2} (-ig_s f_{b_2 a_4 a_3})\, \overset{\text{ghost saliente}}{p_3^{\mu}}$$

$$= \frac{ig_s^2}{2k^2} (\bar{v}_2 \gamma^\mu p_{3\mu} u_1)(\lambda^b)_{\alpha_2\alpha_1} f_{b a_4 a_3} = \frac{-ig_s^2}{2k^2} (\bar{v}_2 \not{p}_3 u_1)(\lambda^b)_{\alpha_2\alpha_1} \overset{\text{permuto y c. signo}}{f_{b a_3 a_4}}$$

$\Downarrow$

$$G^+G = \frac{g_s^4}{4k^4} \cdot (\bar{v}_2 \not{p}_3 u_1)(\bar{u}_1 \not{p}_3 v_2)(\lambda^{\tilde{b}})_{\alpha_1\alpha_2} (\lambda^b)_{\alpha_2\alpha_1} f_{\tilde{b} a_3 a_4} f_{b a_3 a_4}$$

Como los fantasmas (escalares) no tienen polarización, no habrá $\sum_{r_3,r_4}$
pero sí sobre $a_3, a_4 = 1, \dots 8$

$$\overline{\sum} G^+G = \frac{1}{2^2 3^2} \frac{g_s^4}{4k^4} \cdot Tr[(\not{p}_2 - m)\not{p}_3(\not{p}_1+m)\not{p}_3] \sum_{\substack{b,\tilde{b}\\a_3,a_4}} \underbrace{Tr_c(\lambda^{\tilde{b}}\lambda^b)}_{2\delta^{b\tilde{b}}} f_{\tilde{b} a_3 a_4} f_{b a_3 a_4}$$

$$= \frac{g_s^4}{2^3 3^2 k^4} \cdot \hat{T} \cdot \overbrace{\sum_{b,a_3,a_4} f_{b a_3 a_4} f_{b a_3 a_4}}^{= F\,(\text{pág. } 12) = 3\cdot 2^3} = \frac{g_s^4}{3k^4} \cdot \hat{T}$$

$$\hat{T} = Tr[\not{p}_2\not{p}_3\not{p}_1\not{p}_3] - m^2 Tr[\not{p}_3\not{p}_3] = 4\cdot\{2(p_2p_3)(p_1p_3) - (p_1p_2)\overset{0}{p_3^2} - m^2 \overset{0}{p_3^2}\}$$
$$= 8(p_1p_3)(p_2p_3) \underset{\text{mandelstam}}{=} 2\cdot(m^2-t)\cdot(m^2-u)$$

$$\hookrightarrow \overline{\sum} G^+G = \frac{2}{3} \cdot \frac{g_s^4}{s^2} \cdot (m^2-t)(m^2-u)$$

Como $\mathcal{L}_{FP}$ es antihermítico, la sección eficaz será proporcional a $-G^+G$ y será por tanto negativa. Además, en el proceso original, nuestros gluones son partículas idénticas en el estado final, mientras que en el proceso de ghosts, $\phi\bar{\phi}$ son distinguibles. Por ello, tenemos que añadir una contribución $\bar{\phi}\phi$ a la $\phi\bar{\phi}$ (sin interferencias) (como si simetrizases) para que el sistema combinado sí sea de partículas idénticas. O lo que es lo mismo, cambiar 3 por 4 en el diagrama anterior ($t \leftrightarrow u$, s se mantiene). Inmediatamente vemos que el resultado no cambia, y por tanto la contribución de ghosts al proceso será:

$$D_g \equiv -2\overline{\sum} G^+G = -\frac{4}{3}\frac{g_s^4}{s^2}(m^2-t)(m^2-u)$$

→ + añadir factor $\frac{1}{2}$ al proceso global al final por ser ya todo partículas idénticas. Equivalente

Juntando todo: $D_T = D_{11} + 2D_{12} + 2D_{13} + D_{22} + 2D_{23} + D_{33} + D_G$

$$= \frac{2^5 g_s^4}{3^3} \cdot \left\{ \frac{tu + m^2(s-2t-3m^2)}{(t-m^2)^2} + \frac{tu + m^2(s-2u-3m^2)}{(u-m^2)^2} \right\}$$

$$- \frac{2^3}{3^3} \cdot \frac{g_s^4 \cdot m^2 \cdot (s-4m^2)}{(t-m^2)(u-m^2)} - \frac{2^3 g_s^4}{3} \cdot \left(\frac{u-m^2}{s} + \frac{m^2}{u-m^2} + \frac{t-m^2}{s} + \frac{m^2}{t-m^2} \right)$$

$$- \frac{2}{3} \frac{g_s^4}{s^2} (s^2 + 7(t^2+u^2) + 4tu + 34m^2 s - 18m^4) - \frac{4}{3} \frac{g_s^4}{s^2} \cdot (m^2-t)(m^2-u)$$

$$= \frac{2^5}{3^3} g_s^4 \cdot \left\{ \frac{tu + \dots}{\dots} + \frac{tu + \dots}{\dots} \right\} - \frac{2^3}{3^3} g_s^4 m^2 \frac{(s-4m^2)}{(t-m^2)(u-m^2)}$$

giuntar

$$- \frac{2^3 g_s^4}{3} \cdot \left\{ \frac{\underbrace{u+t}_{=-s} - 2m^2}{s} \right\} - \frac{2^3 g_s^4 m^2}{3(t-m^2)(u-m^2)} \underbrace{(t - m^2 + u - m^2)}_{=-s}$$

$$- \frac{2}{3} \frac{g_s^4}{s^2} (s^2 + 7(t^2+u^2) + 4tu + 34m^2 s - 18m^4 + 2m^4 + 2tu - 2m^2 t - 2m^2 u)$$

$$= \frac{2^5}{3^3} g_s^4 \cdot \left\{ \frac{tu + \dots}{\dots} + \frac{tu + \dots}{\dots} \right\} + \frac{2^5}{3^3} \frac{g_s^4 m^4}{(t-m^2)(u-m^2)} + \frac{2^3}{3} g_s^4 m^2 s \underset{\overset{\shortparallel}{8/9}}{\left(1 - \frac{1}{9}\right)}$$

$$- \frac{2}{3} \frac{g_s^4}{s^2} \left\{ s^2 - 4s^2 + 7(t^2+u^2) + 6tu - 16m^4 + m^2(34s - 2t - 2u) \right\}$$

$$-3s^2 + 7t^2 + 7u^2 + 6tu = 4(t^2+u^2) + 3(t^2+u^2-s^2+2tu)$$

$$\hookrightarrow (t+u)^2 = t^2+u^2+2ut = (2m^2-s)^2 = 4m^4 - 4m^2 s + s^2$$

$$\downarrow$$

$$= 4(t^2+u^2) + 3\cdot(4m^4 - 4m^2 s) = 4(t^2+u^2) + 4\cdot 3m^2(m^2-s)$$

$$\{s - 4s^2 + 7(\dots)\dots\} = 4(t^2+u^2) + m^2(34s - 2t - 2u - 16m^2 + 12m^2 - 12s)$$

$$= 4(t^2+u^2) + m^2(22s - 2t - 2u - 4m^2) = 4(t^2+u^2) + m^2(22s - 2(2m^2-s) - 4m^2)$$

$$= 4(t^2+u^2) + m^2(24s - 8m^2) = 4\{t^2 + u^2 + 6m^2 s - 2m^4\}$$

$$\Downarrow$$

$$D_T = \frac{2^5 g_s^4}{3^3} \left\{ \frac{tu + m^2(s-2t-3m^2)}{(t-m^2)^2} + \frac{m^2(m^2+2s)}{(t-m^2)(u-m^2)} + \frac{tu + m^2(s-2u-3m^2)}{(u-m^2)^2} - \frac{9}{4s^2}(t^2+u^2+6m^2 s - 2m^4) \right\}$$

Y la sección eficaz, sustituyendo $g = \sqrt{4\pi\alpha_s}$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda} \cdot \frac{2^5}{3^3} 2^4 \pi^2 \alpha_s^2 \cdot \{\dots\} = \frac{2^5}{3^3} \pi \alpha_s^2 \cdot \{ \quad \}$$

$$\boxed{\frac{d\sigma}{dt} = \frac{32\pi\alpha_s^2}{27} \cdot \left\{ \frac{tu + m^2(s-2t-3m^2)}{(t-m^2)^2} + \frac{m^2(m^2+2s)}{(t-m^2)(u-m^2)} + \frac{tu + m^2(s-2u-3m^2)}{(u-m^2)^2} - \frac{9}{4s^2} \cdot (t^2+u^2+6m^2 s - 2m^4) \right\}}$$

En el límite $m \to 0$:

$$\boxed{\frac{d\sigma}{dt}(q\bar{q} \to gg) = \frac{32}{27}\pi \alpha_s^2 \left\{ \frac{u}{t} + \frac{t}{u} - \frac{9}{4} \cdot \frac{t^2+u^2}{s^2} \right\}}$$

→ Coincide con Peskin (p. 572) (17.75)

5b) Producción de pares $gg \to q\bar{q}$

Es el proceso inverso al 5). Cambiarías $\begin{smallmatrix} 1 \leftrightarrow 3 \\ 2 \leftrightarrow 4 \end{smallmatrix}$, no cambia (t, u, s) pero sí un factor global: cuando hacíamos $\sum$, promediamos $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{2}\right)^2$ por ser quarks partículas iniciales. Ahora serán gluones (8) y se promedia como $\left(\frac{1}{8}\right)^2 \cdot \left(\frac{1}{2}\right)^2$. (Este razonamiento constituye el principio de balance detallado + tma. de los estados finales)

No reescribo por ~~ahorrar~~ el caso con $m \neq 0$.

Por tanto, multiplicando (17.75) × $\frac{3^2}{8^2}$ y un factor $(-1) \cdot (-1)$ ... fermiones cruzados queda:

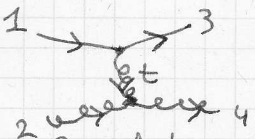
$$\boxed{\frac{d\sigma}{dt}(gg \to q\bar{q}) = \frac{\pi}{6} \alpha_s^2 \left\{ \frac{u}{t} + \frac{t}{u} - \frac{9}{4} \frac{t^2+u^2}{s^2} \right\}}$$

→ Peskin p. 572 (17.76)

Hay una diferencia extra, en el caso 5) hay un factor $\frac{1}{2}$ por espacio fásico que no hemos añadido, al ser gg partículas idénticas. Este factor también estaría en los procesos que acaben en quarks idénticos, pero no en 5b).

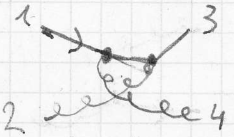
5c) Scattering quark-gluón // antiquark-gluón $qg \to qg$; $\bar{q}g \to \bar{q}g$

Idem quark o antiquark, pues sólo cambiaría signo m, y todas las m están al cuadrado (ver 5). Podemos relacionar los diagramas por simetría de cruce ($s \leftrightarrow t$) ya que rotamos 90°:
u se mantiene



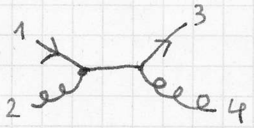
Canal t
(antes, en 5, era canal s, hemos rotado 90°)

+



Canal u
(=)

+



Canal s
(antes canal t)

Ver notas Vicente Giménez

Además, sólo habrá promedio $\left(\frac{1}{3} \cdot \frac{1}{8}\right) \cdot \left(\frac{1}{2}\right)^2$ → Factor relativo $\frac{3}{8}$

Y un signo menos global que se debe a un factor $(-1)^{FC}$ de simetría de cruce, donde FC es el nº de fermiones "cruzados" ≡ fermiones que pasen de ser incoming a outgoing o viceversa.
(en otros procesos todo eran fermiones y NC era siempre par)

En límite $m \to 0$:

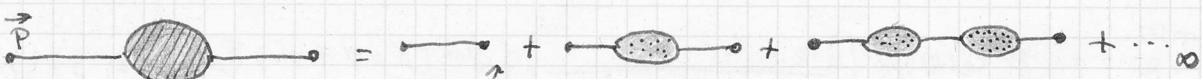
$$\boxed{\frac{d\sigma}{dt}\left(\begin{matrix} qg \to qg \\ \bar{q}\bar{q} \to \bar{q}g \end{matrix}\right) = \frac{4\pi \alpha_s^2}{9s^2} \cdot \left\{ -\frac{u}{s} - \frac{s}{u} + \frac{9}{4} \cdot \frac{s^2+u^2}{t^2} \right\}}$$

Peskin p. 572 (17.77)

El proceso $gg \to gg$ es muy largo, conviene usar paquete Feyncalc de Mathematica. Resultado en Peskin p. 572, ec. (17.78).

II. DIVERGENCIAS DE FUNCIONES DE GREEN

a) Autoenergía del quark



↳ bola, incluye todos los diagramas imaginables, y lo descomponemos en contribuciones irreducibles ⟶ No puedes dividir el diagrama (bola roja) en dos cortando una sola línea interna. La bola azul será suma de diagramas con nº de bolas irreducibles creciente hasta ∞.

Por tanto, tenemos que:

$$\underbrace{iS(\not{p})}_{\text{propagador}}\ \text{quark incluyendo autoenergía} \equiv iS^{(0)}(\not{p}) + iS^{(0)}(\not{p})\underbrace{[-i\Sigma(\not{p})]}_{\text{autoenergía del quark}}\cdot iS^{(0)}(\not{p}) + iS^{(0)}(\not{p})[-i\Sigma(\not{p})]\, iS^{(0)}(\not{p})[-i\Sigma(\not{p})]\, iS^{(0)}(\not{p}) + \dots$$

Aplicamos la sumación de Dyson, con lo que:

$$iS(\not{p}) = iS^{(0)}(\not{p})\cdot\sum_{n=0}^{\infty}[-i\Sigma(\not{p})\, iS^{(0)}(\not{p})]^n \longrightarrow \text{serie geométrica}$$

$$= i\,\frac{S^{0}(\not{p})}{1-\Sigma(\not{p})S^{(0)}(\not{p})} = i\cdot\frac{1}{\frac{1}{S^{0}(\not{p})}-\Sigma(\not{p})}$$

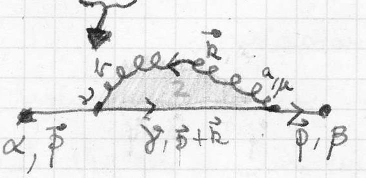
De las reglas de Feynman a tree-level, tenemos que $S^{0}(\not{p}) = \frac{1}{\not{p}-m+i\epsilon}$

↳ $S(\not{p}) = \frac{1}{\not{p}-m-\Sigma(\not{p})+i\epsilon}$; $\Sigma(\not{p})$ se podrá factorizar, por argumentos dimensionales y de dependencia (variables), como:

$$\Sigma(\not{p}) = \Sigma_1(p^2) + (\not{p}-m)\Sigma_2(p^2)$$

Nosotros calcularemos $\Sigma(\not{p})$ ...bola roja a 1 loop:

$-i\Sigma^{(2)}_{\beta\alpha}(\not{p})$ corresponde con:



[Ounito sabor]

↓ Reglas Feynman: (2 vértices V + 2 propagadores I → factor $f = (i)^{I+V}(-1)^{V} = +1$

$$-i\Sigma^{(2)}_{\beta\alpha}(\not{p}) = f\int\underbrace{\frac{d^4k}{(2\pi)^4}}_{\text{loop}} g_s\left(\frac{\lambda^a}{2}\right)_{\beta\gamma}\gamma^{\mu}\cdot\overbrace{\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\epsilon}}^{\text{prop. quark}}\cdot\overbrace{\delta_{ab}\,\frac{-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}}{k^2+i\epsilon}}^{\text{prop. gluón}}\cdot g_s\left(\frac{\lambda^b}{2}\right)_{\gamma\alpha}\gamma^{\nu}$$

$$= \frac{g_s^2}{(2\pi)^4}\cdot\frac{1}{4}\underbrace{(\lambda^a\lambda^a)_{\beta\alpha}}_{=4\cdot C_2(R)\,\delta_{\beta\alpha} = 4\cdot\frac{N^2-1}{2N}\delta_{\beta\alpha}}\cdot\int d^4k\cdot\frac{\{\gamma^{\mu}(\not{p}+\not{k}+m)\gamma^{\nu}\cdot(-g_{\mu\nu}) + (1-\xi)\cdot\frac{\not{k}(\not{p}+\not{k}+m)\not{k}}{k^2+i\epsilon}\}}{((k+p)^2-m^2+i\epsilon)\cdot(k^2+i\epsilon)}$$

$$= -\frac{g_s^2}{(2\pi)^4}\cdot C_2(R)\,\delta_{\beta\alpha}\int\frac{d^4k}{((k+p)^2-m^2+i\epsilon)(k^2+i\epsilon)}\cdot\left\{\gamma^{\mu}(\not{p}+\not{k}+m)\gamma_{\mu} - (1-\xi)\frac{(\not{k}\not{p}\not{k}+\not{k}\not{k}\not{k}+m\not{k}\not{k})}{k^2+i\epsilon}\right\}$$

Uso que $\not{a}\not{b} = 2ab - \not{b}\not{a} \longrightarrow \not{k}(2(pk) - \not{k}\not{p}) + \not{k}\not{k}\not{k} + m^2\not{k}\not{k}$

$$= 2(pk)\cdot\not{k} + \underset{\underset{k^2}{\parallel}}{\not{k}\not{k}}(\not{k}-\not{p}+m)$$

$k^2 \longrightarrow$ cancela con $\frac{1}{k^2+i\epsilon}$

↓

$$-i\Sigma^{(2)}_{\beta\alpha}(\not{p}) = -\frac{g_s^2}{(2\pi)^4}C_2(R)\,\delta_{\beta\alpha}\int\frac{d^4k}{((k+p)^2-m^2+i\epsilon)(k^2+i\epsilon)}\cdot\left\{\gamma^{\mu}(\not{p}+\not{k}+m)\gamma_{\mu} - (1-\xi)(\not{k}-\not{p}+m) - \frac{2(pk)\not{k}}{k^2+i\epsilon}(1-\xi)\right\}$$

Por otro lado, sustituyo $2kp$:

$$(k+p)^2 = k^2+p^2+2kp$$

$$\hookrightarrow 2kp = (k+p)^2 - k^2 - p^2 \overset{\text{Truco: } \pm m^2}{=} ((k+p)^2-m^2) - k^2 - (p^2-m^2)$$

$$-i\Sigma^{(2)}_{\beta\alpha} = -\frac{g_s^2}{(2\pi)^4} C_2(R)\,\delta_{\beta\alpha}\int \frac{d^4k \cdot 1 \cdot}{((k+p)^2-m^2+i\epsilon)\cdot(k^2+i\epsilon)} \cdot \Big\{ \gamma^\mu(\not{k}+\not{p}+m)\gamma_\mu + (1-\xi)(\not{p}-m)$$

$$-(1-\xi)\not{k} - \frac{(1-\xi)\not{k}\,[((k+p)^2-m^2)-k^2-(p^2-m^2)]}{k^2+i\epsilon}\Big\}$$

se cancelan

$$= \dots\int\frac{d^4k}{\dots}\Big\{\gamma^\mu(\not{k}+\not{p}+m)\gamma_\mu + (1-\xi)(\not{p}-m) + (1-\xi)\frac{(p^2-m^2)}{k^2+i\epsilon}\cdot\not{k}$$

$$-(1-\xi)\frac{((k+p)^2-m^2)}{k^2+i\epsilon}\cdot\not{k}\Big\}$$

Analicemos el último término:

$$\hookrightarrow \int\frac{d^4k}{(k^2+i\epsilon)\cancel{((k+p)^2-m^2+i\epsilon)}}\cdot(-)\cdot(1-\xi)\cdot\frac{\cancel{((k+p)^2-m^2)}}{k^2+i\epsilon}\cdot k^\mu\gamma_\mu$$

$$= -(1-\xi)\gamma^\mu\int_{-\infty}^{+\infty}\frac{d^4k\,k^\mu}{k^4} \longrightarrow$$ Al ser integral de función impar entre $\pm\infty$, se cancelará.

$\downarrow$

$$-i\Sigma^{(2)}_{\beta\alpha} = -\frac{g_s^2}{(2\pi)^4}\cdot C_2(R)\,\delta_{\beta\alpha}\int\frac{d^4k}{((k+p)^2-m^2+i\epsilon)(k^2+i\epsilon)}\cdot\Big\{\gamma^\mu(\not{k}+\not{p}+m)\gamma_\mu + (1-\xi)(\not{p}-m) + (1-\xi)\frac{(p^2-m^2)}{k^2+i\epsilon}\cdot\not{k}\Big\}$$

Hacemos unos cambios de notación:

- $\epsilon \to \eta$
- $k^\mu \leftrightarrow -k^\mu$ (cambio variable integración)
- $4 \to 4+2\epsilon$
- Si un término de la integral no depende de índices Lorentz, lo nombraremos de acuerdo a

$$I(A,B;p^2,m^2) \equiv I(A,B)$$

- Si el término tiene índices, sólo puede depender de p^μ, con lo que:

$$I^\mu(A,B;p^2,m^2) \equiv p^\mu\cdot I'(A,B)$$ único elemento tetravector libre, pues la k se integra.

donde las I se corresponden a las del Pascual y Tarrach (apéndice C):
(regularización D-dimensional)

$$I(A,B;q^2,m^2) = \frac{1}{\nu^{2\epsilon}}\cdot\int\frac{d^Dk}{(2\pi)^D}\cdot\frac{1}{[(k-q)^2-m^2+i\eta]^A\,(k^2+i\eta)^B}$$ (p. 249)

$$I^\mu(A,B;q^2,m^2) = \frac{1}{\nu^{2\epsilon}}\cdot\int\frac{d^Dk}{(2\pi)^D}\cdot\frac{k^\mu}{[(k-q)^2-m^2+i\eta]^A\,(k^2+i\eta)^B} = q^\mu\cdot I'(A,B)$$ (p. 250)

$\Rightarrow$

$$-i\Sigma^{(2)}_{\beta\alpha} = -ig_s^2\,C_2(R)\,\frac{\delta_{\beta\alpha}\nu^{2\epsilon}}{\nu^{2\epsilon}}\int\frac{d^Dk}{(2\pi)^D}\cdot\frac{1}{(\underset{(-k+p)^2}{\underset{\parallel}{(k-p)^2}}-m^2+i\eta)(k^2+i\eta)}\cdot\Big\{\gamma^\mu(\not{p}+m)\gamma_\mu$$

$$= - g_s^2 C_2(R)\, \delta_{\beta\alpha}\, \gamma^{2\epsilon} \{ \gamma^\mu (\not{p} + m) \gamma_\mu \cdot I(1,1) - \gamma^{\tilde{\mu}} \gamma_\mu p^\mu I'(1,1) \cdot \gamma_{\tilde{\mu}}$$

$$+ (1-\xi)(\not{p} - m) \cdot I(1,1) - (1-\xi)(p^2 - m^2) p^\mu I'(1,2) \gamma_\mu \}$$

Queremos separar factores $\propto p^2, m^2$ de los $\propto (\not{p} - m)$

Para ello: ↳ para escribir $\Sigma_{\beta\alpha}^{(2)}(\not{p}) = \Sigma_{\beta\alpha_1}^{(2)}(p^2) + (\not{p} - m) \Sigma_{\beta\alpha_2}^{(2)}(p^2)$

1: $\gamma^\mu (\not{p} + m) \gamma_\mu = (2p^\mu - \not{p}\gamma^\mu + m\gamma^\mu)\gamma_\mu =$ ↳ $\gamma^\mu \gamma_\mu = D = 4 + 2\epsilon$

$$= (2\not{p} - \not{p} D + mD) = -2(1+\epsilon) \cdot (\not{p} \underbrace{- m + m}_{0}) + m(4 + 2\epsilon)$$

$$= -2(1+\epsilon)(\not{p} - m) + 2m //$$

2: $\gamma^{\tilde{\mu}} \not{p} \gamma_{\tilde{\mu}} = (2p^\mu - \not{p}\gamma^\mu)\gamma_\mu = 2\not{p} - D\not{p} = -2(1+\epsilon)(\not{p} - m + m)$

$$= -2(1+\epsilon)(\not{p} - m) - 2m(1+\epsilon) //$$

4: $\not{p} = (\not{p} - m) + m$

⇓

$$\Sigma_{\beta\alpha_1}^{(2)}(p^2) = -i g_s^2 C_2(R) \gamma^{2\epsilon} \delta_{\alpha\beta} \cdot \{ 2m\, I(1,1) + 2m(1+\epsilon) I'(1,1) - m(1-\xi)(p^2 - m^2) I'(1,2) \}$$

$$= i g_s^2 C_2(R) \gamma^{2\epsilon} \delta_{\alpha\beta}\, m \{ 2 I(1,1) + 2(1+\epsilon) I'(1,1) - (p^2 - m^2)(1-\xi) I'(1,2) \}$$

$$\Sigma_{\beta\alpha_2}^{(2)}(p^2) = i g_s^2 C_2(R) \gamma^{2\epsilon} \delta_{\alpha\beta} \cdot \{ -2(1+\epsilon) \cdot I(1,1) + 2(1+\epsilon) \cdot I'(1,1) + (1-\xi) \cdot I(1,1)$$
$$- (1-\xi)(p^2 - m^2) I'(1,2) \}$$

$$= -i g_s^2 C_2(R) \gamma^{2\epsilon} \delta_{\alpha\beta} \cdot \{ -(1 + 2\epsilon + \xi) \cdot I(1,1) + 2(1+\epsilon) I'(1,1) - (1-\xi)(p^2 - m^2) I'(1,2) \}$$

I, I': apéndice C del Pascual y Tarrach

p. 249

(C.88): $I(1,1) = \frac{i}{(4\pi)^2} \cdot \{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(-\frac{q^2}{\gamma^2}\right) - \frac{m^2}{q^2} \ln\left(-\frac{m^2}{q^2}\right) - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{m^2}{q^2}\right) + 2 \}$

(C.90): $I'(1,1) = \frac{i}{(4\pi)^2} \cdot \frac{1}{2} \cdot \{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(-\frac{q^2}{\gamma^2}\right) - \frac{m^2}{q^2}\left(2 - \frac{m^2}{q^2}\right) \overset{\ln\left(-\frac{m^2}{q^2}\right)}{\wedge} - \left(1 - 2\frac{m^2}{q^2} + \frac{m^4}{q^4}\right) \cdot \ln\left(1 - \frac{m^2}{q^2}\right) - \frac{m^2}{q^2} + 2 \}$

$I'(1,2) = \frac{i}{(4\pi)^2} \cdot \frac{1}{q^2} \cdot \{ -\frac{m^2}{q^2} \ln\left(-\frac{m^2}{q^2}\right) + \frac{m^2}{q^2} \ln\left(1 - \frac{m^2}{q^2}\right) + 1 \}$

⇓ $q = p$

$$\Sigma_{\beta\alpha_1}^{(2)}(q^2) = g_s^2 C_2(R) \gamma^{2\epsilon} \delta_{\alpha\beta} \cdot \frac{m \cdot}{(4\pi)^2} \cdot \left\{ \left(-\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(-\frac{q^2}{\gamma^2}\right)\right) \cdot 3 + 4 - \frac{2m^2}{q^2} \ln\left(-\frac{m^2}{q^2}\right) \right.$$
$$- 2\left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{m^2}{q^2}\right) - \frac{m^2}{q^2}\left(2 - \frac{m^2}{q^2}\right) \ln\left(-\frac{m^2}{q^2}\right) - \left(1 - \frac{2m^2}{q^2} + \frac{m^4}{q^4}\right) \ln\left(1 - \frac{m^2}{q^2}\right) - \frac{m^2}{q^2} + 2$$
$$\left. - (1-\xi)\left(\cancel{1} - \frac{m^2}{q^2}\right) \cdot \left\{ -\frac{m^2}{q^2} \ln\left(-\frac{m^2}{q^2}\right) + \frac{m^2}{q^2} \ln\left(1 - \frac{m^2}{q^2}\right) + 1 \right) - \cdot \frac{\epsilon}{\epsilon} \right\}$$ ⇢ resto factores $\propto \epsilon \to 0$

[illegible] también la parte finita por completitud aunque sólo

$$= \dots \Big\{ -3\Big(\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\Big(-\frac{q^2}{\mu^2}\cdot\frac{m^2}{m^2}\Big)\Big) + 5 - (1-\xi)\overset{(1-m^2/q^2)}{\cancel{1}} - \frac{m^2}{q^2}$$

$$+ \ln\Big(-\frac{m^2}{q^2}\Big)\cdot\Big(-\frac{2m^2}{q^2} - \frac{2m^2}{q^2} + \frac{m^4}{q^4} + (1-\xi)\Big(\frac{m^2}{q^2} - \frac{m^4}{q^4}\Big)\Big)$$

$$+ \ln\Big(1-\frac{m^2}{q^2}\Big)\cdot\Big(-2 + \frac{2m^2}{q^2} - 1 + \frac{2m^2}{q^2} - \frac{m^4}{q^4} - (1-\xi)\Big(\frac{m^2}{q^2} - \frac{m^4}{q^4}\Big)\Big)\Big\}$$

$\frac{1}{\hat{\epsilon}} \equiv \frac{1}{\epsilon} - \ln 4\pi + \gamma_E$

$$= \dots \Big\{ -3\Big(\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\Big(+\frac{m^2}{\mu^2}\Big)\Big) - 3\ln\Big(-\frac{q^2}{m^2}\Big) + 5 - 1 + \frac{\cancel{m^2}}{q^2} - \cancel{\frac{m^2}{q^2}} + \xi\Big(1-\frac{m^2}{q^2}\Big)$$

$$+ \ln\Big(-\frac{m^2}{q^2}\Big)\cdot\Big(-\frac{3m^2}{q^2} - \xi\frac{m^2}{q^2}\Big(1-\frac{m^2}{q^2}\Big)\Big)$$

$$+ \ln\Big(1-\frac{m^2}{q^2}\Big)\cdot\Big(-3 + \frac{4m^2}{q^2} - \cancel{\frac{m^4}{q^4}} + \cancel{\frac{m^4}{q^4}} - \frac{m^2}{q^2} + \xi\frac{m^2}{q^2}\Big(1-\frac{3m^2}{q^2}\Big)\Big)\Big\}$$

$$= \dots \Big\{ -3\cdot\frac{1}{\hat{\epsilon}} \overset{+4 - 3\ln(+\frac{m^2}{\mu^2})}{\vee} - 3\ln\Big(-\frac{q^2}{m^2}\Big) + \xi\Big(1-\frac{m^2}{q^2}\Big) + \xi\frac{m^2}{q^2}\Big(1-\frac{m^2}{q^2}\Big)\Big[-\ln\Big(-\frac{m^2}{q^2}\Big) + \ln\Big(1-\frac{m^2}{q^2}\Big)\Big]$$

$$+ 3\frac{m^2}{q^2}\Big(-\ln\Big(-\frac{m^2}{q^2}\Big) + \ln\Big(1-\frac{m^2}{q^2}\Big)\Big) - 3\ln\Big(1-\frac{m^2}{q^2}\Big)\Big\}$$

$$= \dots \Big\{ -3\cdot\frac{1}{\hat{\epsilon}} \overset{-3\ln(\frac{m^2}{\mu^2})}{\vee} - 3\cdot\ln\Big(1-\frac{q^2}{m^2}\Big) + \xi\Big(1-\frac{m^2}{q^2}\Big) + 4 + \xi\frac{m^2}{q^2}\Big(1-\frac{m^2}{q^2}\Big)\ln\Big[1-\frac{q^2}{m^2}\Big)$$

$$+ 3\frac{m^2}{q^2}\ln\Big(1-\frac{q^2}{m^2}\Big)\Big\}$$

$$= \dots \Big\{ -3\cdot\frac{1}{\hat{\epsilon}} \overset{-3\ln(\frac{m^2}{\mu^2})}{\vee} + \ln\Big(1-\frac{q^2}{m^2}\Big)\cdot\Big[-3 + \frac{3m^2}{q^2} + \xi\frac{m^2}{q^2}\Big(1-\frac{m^2}{q^2}\Big)\Big] + 4 + \xi\Big(1-\frac{m^2}{q^2}\Big)\Big\}$$

$$= \dots \Big\{ -3\frac{1}{\hat{\epsilon}} + \overset{3\ln\frac{m^2}{\mu^2}}{\vee} \ln\Big(1-\frac{q^2}{m^2}\Big)\cdot\Big(-3 + \xi\frac{m^2}{q^2}\Big)\cdot\Big(1-\frac{m^2}{q^2}\Big) + 4 + \xi\Big(1-\frac{m^2}{q^2}\Big)\Big\}$$

→ coincide con PyT

$$\Sigma^{(2)}_{pol\,2}(q^2) = g_s^2 C_2(R)\cdot\mu^{2\epsilon}\cdot\delta_{\alpha\beta}\cdot\frac{1}{(4\pi)^2}\cdot\Big\{-(1+2\epsilon+\xi)\cdot\Big(-\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\Big(-\frac{q^2}{\mu^2}\Big) - \frac{m^2}{q^2}\ln\Big(-\frac{m^2}{q^2}\Big)$$

$$- \Big(1-\frac{m^2}{q^2}\Big)\ln\Big(1-\frac{m^2}{q^2}\Big) + 2\Big) + \cancel{2}(1+\epsilon)\cdot\frac{1}{\cancel{2}}\Big(-\frac{1}{\epsilon} + \ln 4\pi - \gamma_E - \ln\Big(-\frac{q^2}{\mu^2}\Big) - \frac{m^2}{q^2}\Big(2-\frac{m^2}{q^2}\Big)\ln\Big(-\frac{m^2}{q^2}\Big)$$

$$- \Big(1 - \frac{2m^2}{q^2} + \frac{m^4}{q^4}\Big)\cdot\ln\Big(1-\frac{m^2}{q^2}\Big) - \frac{m^2}{q^2} + 2\Big) - (1-\xi)\Big(1-\frac{m^2}{q^2}\Big)\cdot\Big(-\frac{m^2}{q^2}\ln\Big(-\frac{m^2}{q^2}\Big)$$

$$+ \frac{m^2}{q^2}\ln\Big(1-\frac{m^2}{q^2}\Big) + 1\Big)\Big\}$$

$$= \dots\cdot\Big\{ + \xi\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\Big(-\frac{q^2}{\mu^2}\Big) + \frac{m^2}{q^2}\ln\Big(-\frac{m^2}{q^2}\Big) + \Big(1-\frac{m^2}{q^2}\Big)\ln\Big(1-\frac{m^2}{q^2}\Big) - 2 + \Big(1-\frac{m^2}{q^2}\Big)\Big(-\frac{m^2}{q^2}\cdot$$

$$\cdot\ln\Big(-\frac{m^2}{q^2}\Big) + \frac{m^2}{q^2}\ln\Big(1-\frac{m^2}{q^2}\Big) + 1\Big)\Big] + \ln\Big(-\frac{m^2}{q^2}\Big)\cdot\Big(\underbrace{\frac{m^2}{q^2} - \frac{2m^2}{q^2} + \frac{m^4}{q^4} + \Big(1-\frac{m^2}{q^2}\Big)\cdot\frac{m^2}{q^2}}_{=0}\Big)$$

$$+ \ln\Big(1-\frac{m^2}{q^2}\Big)\cdot\Big(1 \underbrace{- \frac{m^2}{q^2} - 1 + \frac{2m^2}{q^2} - \frac{m^4}{q^4} - \Big(1-\frac{m^2}{q^2}\Big)\cdot\frac{m^2}{q^2}}_{=0}\Big) - 1 \underbrace{+ \frac{2\epsilon}{\epsilon} - \frac{\epsilon}{\epsilon}}\Big\}$$

$$=\dots\ \xi\Big[\frac{1}{\epsilon}-\ln 4\pi+\gamma_E+\ln\frac{m^2}{\nu^2}-\ln\left(-\frac{m^2}{q^2}\right)+1+\frac{m^2}{q^2}\ln\left(-\frac{m^2}{q^2}\right)+\ln\left(1-\frac{m^2}{q^2}\right)\cdot\left(+1+\frac{m^2}{q^2}+\left(1-\frac{m^2}{q^2}\right)\frac{m^2}{q^2}\right)$$

$$+\left(1-\frac{m^2}{q^2}\right)\cdot\left(-\frac{m^2}{q^2}\right)\cdot\ln\left(-\frac{m^2}{q^2}\right)-\frac{m^2}{q^2}\Big]$$

$$=\dots\ \xi\Big[\underset{+\ln\frac{m^2}{\nu^2}}{\frac{1}{\hat\epsilon}}+\ln\left(-\frac{m^2}{q^2}\right)\cdot\left(-1+\underbrace{\frac{m^2}{q^2}-\frac{m^2}{q^2}}_{=0}+\frac{m^4}{q^4}\right)-1-\frac{m^2}{q^2}+\ln\left(1-\frac{m^2}{q^2}\right)\cdot\left(+1-\frac{m^4}{q^4}\right)$$

$$=\dots\ \xi\Big[\frac{1}{\hat\epsilon}+\left(1-\frac{m^4}{q^4}\right)\ln\left(1-\frac{q^2}{m^2}\right)-1-\frac{m^2}{q^2}+\ln\frac{m^2}{\nu^2}\Big]$$

→ coincide con P y T

Hasta ahora hemos regularizado. Ahora renormalizamos:
Renombro m anteriores por m_0. Por Dyson hemos visto que:

$$S(\not q)=\frac{1}{\not q-m_0-\Sigma(\not q)}\equiv Z_{2F}S_R(\not q,\nu)$$

→ Queremos escribir de esta forma
↳ divergencias en Z
↳ S_R renormalizado

$$\left(S_R^{-1}(\not q,\nu)=Z_{2F}\not q-Z_{2F}m_0-Z_{2F}\Sigma(\not q)=Z_{2F}\not q-Z_{2F}m_0-Z_{2F}\Sigma_1(q^2)-(\not q-m_0)Z_{2F}\Sigma_2(q^2)\right)$$

Para ello queremos separar Σ en sus partes finita Σ_R y divergente.

$$\Sigma^{(2)}(\not q)=\Sigma_1^{(2)}(q^2)+\Sigma_2^{(2)}(q^2)(\not q-m_0)=\underbrace{\Sigma_R^{(2)}(\not q)}_{\text{parte finita}}+\underbrace{C_4m_0-C_{2F}\not q}_{\text{parte divergente}}$$

ctes arbitrarias, tendrá esa estructura por dimensiones.

…$\delta\alpha\beta$

↓ Omito índices de color y sabor, pues es diagonal en ellos.

$$\Sigma_R^{(2)}(\not q)=\Big(\underbrace{\Sigma_1^{(2)}(q^2)-(C_4-C_{2F})m_0}_{\Sigma_{1R}^{(2)}}\Big)+(\not q-m_0)\Big(\underbrace{\Sigma_2^{(2)}(q^2)+C_{2F}}_{\Sigma_{2R}^{(2)}}\Big)$$

Como queremos cancelar las divergencias, (esquema $\overline{MS}$)

$$m_0(C_4-C_{2F})=\mathrm{DIV}\left[\Sigma_1^{(2)}(q^2)\right]=-\frac{3}{\hat\epsilon}m_0g_s^2\cdot C_2(R)\left(\frac{1}{4\pi}\right)^2\nu^{2\epsilon}=-\frac{3}{\hat\epsilon}\frac{\alpha_{s0}\cdot\nu^{2\epsilon}}{4\pi}C_2(R)\cdot m_0$$

$$C_{2F}=-\mathrm{DIV}\left[\Sigma_2^{(2)}(q^2)\right]=-\frac{\xi_0}{\hat\epsilon}\cdot g_s^2C_2(R)\cdot\left(\frac{1}{4\pi}\right)^2\cdot\nu^{2\epsilon}=-\frac{\xi_0}{\hat\epsilon}\cdot\frac{\alpha_{s0}\cdot\nu^{2\epsilon}}{4\pi}C_2(R)$$

$$C_4=-\frac{1}{\hat\epsilon}\frac{\alpha_{0}^2\nu^{2\epsilon}}{4\pi}C_2(R)\cdot(3+\xi_0)$$

$$a_0=\frac{\alpha_{s0}\cdot\nu^{2\epsilon}}{4\pi}\ ,\quad C_2(R)\equiv C_F$$

⇓

$$\boxed{\begin{aligned}Z_{2F}&=1-C_{2F}=1+\frac{\xi_0\alpha_{s0}\nu^{2\epsilon}}{4\pi}\cdot C_2(R)\cdot\frac{1}{\hat\epsilon}=1+\xi_0a_0\,C_2(R)\cdot\frac{1}{\hat\epsilon}\\ Z_4&=1-C_4=1+(3+\xi_0)\cdot\frac{\alpha_{s0}\nu^{2\epsilon}}{4\pi}\cdot\frac{1}{\hat\epsilon}\cdot C_2(R)=1+a_0(3+\xi_0)C_2(R)\cdot\frac{1}{\hat\epsilon}\end{aligned}}$$

Y queda:

$$\Sigma_R^{(2)}(\not q)=\frac{\alpha_{s0}}{4\pi}\cdot\nu^{2\epsilon}C_2(R)m_0\cdot\underbrace{\left\{-3\ln\frac{m_0^2}{\nu^2}+4+\xi_0\left(1-\frac{m_0^2}{q^2}\right)-\left(3-\xi\frac{m^2}{q^2}\right)\left(1-\frac{q^2}{m^2}\right)\ln\left(1-\frac{q^2}{m^2}\right)\right\}}_{\Sigma_{R1}^{(2)}(q^2,\nu^2)}$$

$$+\frac{\alpha_{s0}}{4\pi}\nu^{2\epsilon}C_2(R)\cdot(\not q-m)\cdot\left\{\ln\frac{m^2}{\nu^2}-1-\frac{m^2}{q^2}+\left(1-\frac{m^4}{q^4}\right)\ln\left(1-\frac{q^2}{m^2}\right)\right\}\rightarrow\Sigma_{R2}^{(2)}(q^2,\nu^2)$$

Todavía no hemos acabado. Falta demostrar que, habiendo elegido C_2, C_4 de esa manera (recogiendo partes divergentes de Σ: $\mathrm{DIV}(\Sigma)=q, m_0$ [illegible])

Para ello, explotaremos al máximo la expansión en términos de α y despreciaremos términos de órdenes superiores (α^2). Este proceso nos llevará a definir una masa renormalizada.

Para homogeneizar la notación con las transparencias de Pich, llamaré:

$C_{2F} \equiv -\Delta\Sigma_2$; $m_0(C_4 - C_{2F}) \equiv \Delta\Sigma_1$ } Es fácil ver que:

$$\hookrightarrow \Sigma^{(2)}(\not{p}) = \Sigma_{1R}^{(2)} + \Delta\Sigma_1 + (\Sigma_{2R} + \Delta\Sigma_{2R})(\not{p} - m_0)$$

$\Downarrow$ omito superíndice (2) del orden por simplicidad

$$S^{-1}(\not{p}) = \not{p} - m_0 - \Sigma(\not{p}) = (\not{p} - m_0)\cdot(1 - \Sigma_2(p^2)) - \Sigma_1(p^2)$$

$$= (\not{p} - m_0)(1 - \Sigma_{2R} - \Delta\Sigma_2) - \Sigma_{1R} - \Delta\Sigma_1$$

Hemos visto que Σ_{iR}, $\Delta\Sigma_i$ son proporcionales a α, con lo que despreciaremos los productos cruzados & términos con α^2 (expansión en α a orden más bajo).

Buscamos sacar Z_{2F}^{-1} de factor común: $Z_{2F} = 1 - C_{2F} = 1 + \Delta\Sigma_2$

$(Z_4 = -\Delta\Sigma_1/m_0 + \Delta\Sigma_2 + 1)$

$$S^{-1}(\not{p}) \equiv \frac{1}{1+\Delta\Sigma_2}\cdot\left\{(\not{p} - m_0)(1 - \Sigma_{2R} - \Delta\Sigma_2)(1 + \Delta\Sigma_2) - \Sigma_{1R}(1 + \Delta\Sigma_2) - \Delta\Sigma_1(1 + \Delta\Sigma_2)\right\}$$

$$= Z_{2F}^{-1}\cdot\left\{\left(\not{p} - m_0 - \frac{\Delta\Sigma_1}{1 - \Sigma_{2R} - \Delta\Sigma_2}\right)\cdot(1 - \Sigma_{2R} - \Delta\Sigma_2)(1 + \Delta\Sigma_2) - \Sigma_{1R}(1 + \Delta\Sigma_2)\right\}$$

$\downarrow$ Me quedo a orden α (quito productos cruzados):

$$\simeq Z_{2F}^{-1}\cdot\left\{\left(\not{p} - m_0 - \frac{\Delta\Sigma_1}{1 - \Sigma_{2R} - \Delta\Sigma_2}\right)\cdot\left(1 + \Delta\Sigma_2 - \Sigma_{2R} - \cancelto{\alpha^2}{\Sigma_{2R}\Delta\Sigma_2} - \Delta\Sigma_2 + \cancelto{\alpha^2}{\Delta\Sigma_2^2}\right) - \Sigma_{1R} - \cancelto{\alpha^2}{\Sigma_{1R}\Delta\Sigma_2}\right\}$$

$\Delta\Sigma_2 - \Delta\Sigma_2 = 0$ (cancelación exacta)

En el límite $\epsilon \to 0$, $\Delta\Sigma_2 \gg \Sigma_{2R}$

$\Downarrow$ Queda que

$$S^{-1}(\not{p}) \simeq Z_{2F}^{-1}\cdot\left\{\left(\not{p} - m_0 - \frac{\Delta\Sigma_1}{1 - \Delta\Sigma_2}\right)\cdot(1 - \Sigma_{2R}) - \Sigma_{1R}\right\}$$

Para "esconder" la divergencia, renormalizo la masa:

$$\boxed{m \equiv m_0 + \frac{\Delta\Sigma_1}{1 - \Delta\Sigma_2}} = m_0\left(1 + \frac{\Delta\Sigma_1/m_0}{1 - \Delta\Sigma_2}\right) = m_0\left(1 + \frac{\Delta\Sigma_1/m_0}{1 - \Delta\Sigma_2 + \underbrace{\frac{\Delta\Sigma_1}{m_0} - \frac{\Delta\Sigma_1}{m_0}}_{0}}\right)$$

salvo correcciones α^2, desarrollo numerador y denominador para sacar $(1 + \frac{\Delta\Sigma_1}{m_0})$ por separado

$$\simeq m_0\left(1 + \frac{\Delta\Sigma_1/m_0\,(1 + \frac{\Delta\Sigma_1}{m_0})}{(1 + \Delta\Sigma_2 - \frac{\Delta\Sigma_1}{m_0})(1 + \frac{\Delta\Sigma_1}{m_0})}\right) = m_0\left(1 + \frac{\Delta\Sigma_1/m_0}{1 + \Delta\Sigma_2 - \frac{\Delta\Sigma_1}{m_0}}\right)$$

$$= m_0\cdot\left(\frac{1 + \Delta\Sigma_2}{1 + \Delta\Sigma_2 - \frac{\Delta\Sigma_1}{m_0}}\right) = m_0\cdot\frac{Z_{2F}}{Z_4} \Rightarrow \boxed{m = m_0 Z_{2F} Z_4^{-1}}$$

IV.- QCD MATCHING

n: nº de sabores (omito F de n_F) ⟶ por simplicidad

N: nº de colores (omito C de N_C)

α : cte de acoplamiento fuerte $\equiv g_s^2/4\pi$ (omito s de α_s)

$n-1$ quarks ligeros ($m_q \simeq 0$)

a) $\underset{\mu > M}{\mathcal{L}_n} \longleftrightarrow \mathcal{L}_{n-1}$ (integrate out en la acción el quark pesado de masa M) para $\mu < M$

Por otro lado: $\mu \dfrac{d\alpha}{d\mu} = \beta\alpha \equiv \alpha \cdot \left(\sum_{k=1}^{\infty} \beta_k \left(\dfrac{\alpha}{\pi}\right)^k \right)$ donde α, β dependen de n.

Además, las constantes de acoplamiento del lagrangiano con n y $n-1$ sabores:

$$\alpha_n(\mu^2) = \alpha_{n-1} \cdot \left\{ 1 + \sum_{k=1}^{\infty} C_k(L) \left(\frac{\alpha_{n-1}}{\pi} \right)^k \right\} \quad \text{con } L \equiv \ln\frac{\mu}{M}$$

A primer orden, $k=1$:

Renombro, mudas

$$\mu' \frac{d\alpha}{d\mu'} = \frac{\alpha^2}{\pi}\beta_1 \rightarrow \frac{d\alpha}{\alpha^2} = \frac{\beta_1}{\pi}\frac{d\mu'}{\mu'} \rightarrow \int_{\tilde{\mu}_0}^{\mu}$$ ... elegimos cte $\tilde{\mu}_0$ como referencia

$$\hookrightarrow - \frac{1}{\alpha}\Bigg|_{\alpha(\tilde{\mu}_0^2)}^{\alpha(\mu^2)} = \frac{\beta_1}{\pi} \ln\left(\frac{\mu}{\tilde{\mu}_0}\right)$$

$$\frac{1}{\alpha(\tilde{\mu}_0^2)} - \frac{1}{\alpha(\mu^2)} = \frac{\beta_1}{\pi}\ln\left(\frac{\mu}{\tilde{\mu}_0}\right) \rightarrow \alpha(\mu^2) = \frac{1}{\underset{\overset{\shortparallel}{\alpha^0}}{\frac{1}{\alpha(\tilde{\mu}_0^2)}} - \frac{\beta_1}{\pi}\ln\left(\frac{\mu}{\tilde{\mu}_0}\right)} = \frac{\alpha^0}{1 - \frac{\alpha^0\beta_1}{\pi}\ln\left(\frac{\mu}{\tilde{\mu}_0}\right)}$$

α^0 (por simplicidad)

$$\ln\left(\frac{\mu}{\tilde{\mu}_0}\right) = \frac{1}{2}\ln\left(\frac{\mu^2}{\tilde{\mu}_0^2}\right)$$

Aparte:

$$\alpha_n = \alpha_{n-1}\left\{ 1 + C_1(L)\frac{\alpha_{n-1}}{\pi} \right\}$$

↓ Sustituyo α en función β teniendo en cuenta que $\alpha, \alpha^0, \beta_1$ dependen de n:

$\hookrightarrow \beta_{1,x} \equiv (2x - 11N)/6$

$$\frac{\alpha^0{}_n}{1 - \alpha^0{}_n \frac{\beta_{1n}}{\pi}\ln\frac{\mu}{\tilde{\mu}_0}} = \frac{\alpha^0{}_{n-1}}{1 - \alpha^0{}_{n-1}\frac{\beta_{1,n-1}}{\pi}\ln\frac{\mu}{\tilde{\mu}_0}} \cdot \left\{ 1 + C_1(L) \cdot \frac{\alpha^0{}_{n-1}}{1 - \alpha^0{}_{n-1}\frac{\beta_{1,n-1}}{\pi}\ln\frac{\mu}{\tilde{\mu}_0}} \right\}$$

Si $\mu = M \rightarrow L = 0$. Cond. inicial: $C_1(L=0) = 0$

⇓

$$\frac{\alpha^0{}_n}{1 - \frac{\alpha^0{}_n}{\pi}\beta_{1,n}\ln\frac{M}{\tilde{\mu}_0}} = \frac{\alpha^0{}_{n-1}}{1 - \frac{\alpha^0{}_{n-1}\beta_{1,n-1}}{\pi} \cdot \ln\frac{M}{\tilde{\mu}_0}}$$

Si fijamos $\tilde{\mu}_0 \equiv M$ nos queda una relación simple:

$$\alpha^0{}_n(M^2) = \alpha^0{}_{n-1}(M^2) \equiv \alpha^0_M$$ → matching α entre $\mathcal{L}^{n_F}, \mathcal{L}^{n_F-1}$ a $\mu = M$

Volviendo a la expresión completa, podemos despejar $C_1(L)$:

$$C_1(L) = \quad \pi - \alpha^0_M \quad \beta_{1,n-1}\ln\frac{\mu}{M} \qquad 6\pi - \quad L\,\alpha^0_M \cdot (2n - 11N - 2)$$

$$\boxed{C_1(L) = * \; \frac{2\alpha^0_M \ln\frac{\mu}{M}}{6\pi - \alpha^0_M \ln\frac{\mu}{M}(2n - 11N)}}$$

$*$ → Se puede obtener de forma más general derivando:

$$\mu \frac{d\alpha_n}{d\mu} = \beta \alpha_n = \ldots$$

b) A dos loops: (sólo la β)

$$\mu' \frac{d\alpha}{d\mu'} = \frac{\alpha^2 \beta_1}{\pi} + \frac{\alpha^3}{\pi^2}\beta_2 \qquad (\alpha, \beta \text{ dependen de } n)$$

↓

$$\frac{d\mu'}{\mu'} = \frac{d\alpha}{\alpha^2 \cdot \left(\frac{\beta_1}{\pi}\right) + \alpha^3\left(\frac{\beta_2}{\pi^2}\right)} = \frac{\frac{d\alpha}{\alpha^2}}{\left(\frac{\beta_1}{\pi}\right) + \alpha\left(\frac{\beta_2}{\pi^2}\right)}$$

Integral tipo:

$$\frac{dx}{ax^3 + bx^2} = \frac{dx}{x^2(ax+b)} \longrightarrow \text{descomposición en fracciones simples:}$$

$$\frac{1}{(x-0)^2(a)\cdot(x + \frac{b}{a})} = \frac{1}{a}\left\{\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x + \frac{b}{a}}\right\}$$

$$\hookrightarrow A_1 x\left(x + \frac{b}{a}\right) + A_2\left(x + \frac{b}{a}\right) + A_3 \cdot x^2 = 1$$

x^2: $A_1 + A_3 = 0$

x: $A_1 \frac{b}{a} + A_2 = 0$

x^0: $A_2 \frac{b}{a} = 1 \rightarrow A_2 = a/b$

$A_1 = -\frac{a}{b} A_2 = -\frac{a^2}{b^2}$

$A_3 = a^2/b^2$

$$\hookrightarrow \int_{x_{min}}^{x_{max}} \frac{dx}{ax^3 + bx^2} = \int_{x_{min}}^{x_{max}} \frac{dx}{a} \cdot \left(\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x + \frac{b}{a}}\right) = \frac{1}{a}\left[A_1 \ln x - \frac{A_2}{x} + A_3 \ln\left(x + \frac{b}{a}\right)\right]_{x_{min}}^{x_{max}}$$

$$= \frac{1}{a}\left[-\frac{a^2}{b^2}\ln x - \frac{a}{b} \cdot \frac{1}{x} + \frac{a^2}{b^2}\ln\left(x + \frac{b}{a}\right)\right]_{x_{min}}^{x_{max}} = \left[\frac{a}{b^2}\ln\left(\frac{x + \frac{b}{a}}{x}\right) - \frac{1}{b} \cdot \frac{1}{x}\right]_{x_{min}}^{x_{max}}$$

$$= \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right) - \frac{a}{b^2}\cancel{\ln a} - \frac{1}{bx}\Bigg|_{x_{min}}^{x_{max}} = \frac{1}{b} \cdot \left[\frac{a}{b}\ln\frac{ax+b}{x} - \frac{1}{x}\right]_{x_{min}}^{x_{max}}$$

se va, es cte

o bien $\frac{a}{b}\ln\left(1 + \frac{b}{ax}\right)$

$x \equiv \alpha$
$a \equiv \beta_2/\pi^2$
$b \equiv \beta_1/\pi$

↓

$$\ln\frac{\mu}{\mu_0} = \frac{\pi}{\beta_1} \cdot \left[\frac{\beta_2}{\pi\beta_1}\ln\left(\frac{\frac{\beta_2}{\pi^2}\alpha + \frac{\beta_1}{\pi}}{\alpha}\right) - \frac{1}{\alpha}\right]_{\alpha(\mu_0^2)}^{\alpha(\mu^2)}$$

$$= \frac{\pi}{\beta_1}\left[-\frac{1}{\alpha(\mu^2)} + \frac{1}{\alpha(\mu_0^2)} + \ln\underbrace{\left(\frac{\frac{\beta_2}{\pi^2}\alpha(\mu^2) + \frac{\beta_1}{\pi}}{\frac{\beta_2}{\pi^2}\alpha(\mu_0^2) + \frac{\beta_1}{\pi}} \cdot \frac{\alpha(\mu_0^2)}{\alpha(\mu^2)}\right)^{\frac{\beta_2}{\pi\beta_1}}}\right]$$

→ Función implícita trascendente
↓
2 alternativas:
Calcular con ordenador
o aproximar

$\alpha \approx \alpha_0(1 + \alpha_0\beta_1 + \ldots)$ → serie geométrica perturbativa

⑳

resto de términos se van
\+ invierto y saco signo −.

$$\ln\frac{\mu}{\mu_0} \approx \frac{\pi}{\beta_1}\left[-\frac{1}{\alpha(\mu^2)}+\frac{1}{\alpha(\mu_0^2)}-\frac{\beta_2}{\pi\beta_1}\cdot\ln\left(\frac{\alpha(\mu^2)}{\alpha(\mu_0^2)}\right)\right]$$

$$= \frac{\pi}{\beta_1}\left(\frac{1}{\alpha^0}-\frac{1}{\alpha}\right)-\frac{\beta_2}{\beta_1^2}\ln\left(\frac{\alpha}{\alpha^0}\right)$$

→ Para poder despejar $\alpha(\mu)$, y obtener una expresión analítica estimaremos α en el interior del logaritmo como el resultado a 1 loop $(\beta_2 = 0)$:

ver 2 pág antes

$$\ln\frac{\mu}{\mu_0} \approx \frac{\pi}{\beta_1}\left(\frac{1}{\alpha^0}-\frac{1}{\alpha}\right)-\frac{\beta_2}{\beta_1^2}\ln\left(\frac{1}{1-\frac{\beta_1\alpha^0}{\pi}\ln\frac{\mu}{\mu_0}}\right)$$

invierto

$$= \frac{\pi}{\beta_1}\left(\frac{1}{\alpha^0}-\frac{1}{\alpha}\right)+\frac{\beta_2}{\beta_1^2}\ln\left(1-\frac{\beta_1\alpha^0}{\pi}\ln\frac{\mu}{\mu_0}\right) = \text{Taylor:}$$

↳ $\ln(1+x) \approx \ln(1) + \left.\frac{1}{1+x}\right|_{x=0}\cdot x + \dots$

$\approx x + \mathcal{O}(x^2)$ ⟶ despreciamos

con $x \equiv -\frac{\beta_1\alpha^0}{\pi}\ln\frac{\mu}{\mu_0}$

$$\approx \frac{\pi}{\beta_1}\left(\frac{1}{\alpha^0}-\frac{1}{\alpha}\right)-\frac{\beta_2\alpha^0}{\beta_1\pi}\ln\frac{\mu}{\mu_0}$$

↳ $$\frac{\pi}{\beta_1\alpha} = \frac{\pi}{\beta_1\alpha^0}-\ln\frac{\mu}{\mu_0}-\frac{\beta_2\alpha^0}{\beta_1\pi}\ln\frac{\mu}{\mu_0}$$

↓

$$\frac{1}{\alpha} = \frac{1}{\alpha^0}-\frac{\beta_1}{\pi}\ln\left(\frac{\mu}{\mu_0}\right)-\frac{\beta_2\alpha^0}{\pi^2}\ln\left(\frac{\mu}{\mu_0}\right) \Longleftarrow \ln\frac{\mu}{\mu_0} = \frac{1}{2}\ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

↳ $$\alpha \overset{\text{a 2 loops}}{\approx} \frac{\alpha^0}{1-\frac{\beta_1}{2}\left(\frac{\alpha^0}{\pi}\right)\ln\left(\frac{\mu^2}{\mu_0^2}\right)-\frac{\beta_2}{2}\left(\frac{\alpha^0}{\pi}\right)^2\ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

donde $\alpha^0 \equiv \alpha(\mu_0^2)$
$\alpha \equiv \alpha(\mu^2)$
y α, α^0 depende de n
a través de β ya que:

Para n, $n-1$ sabores: cambian β_i

$$\begin{cases} \beta_{1,n} = \dfrac{2n - 11N}{6} \\ \beta_{2,n} = -\dfrac{51}{4}+\dfrac{19}{12}n \end{cases}$$

$N=3$

Por ejemplo, para:

$\mu < M$: usamos α_{n-1} (a 2 loops) con $\mu^0 \neq M$ en general

$\mu = M$: aparece un nuevo sabor
↳ condición de matching ⟹ Transición SIN discontinuidad.
↳ pero la derivada (pendiente) sí será discontinua.

$\mu > M$: usamos $\alpha_n = \alpha_{n-1}\left\{1 + c_1(L)\frac{\alpha_{n-1}}{\pi}\right\}$

con $\alpha^0_M \equiv \alpha_{n-1}(M^2)$

c) Dibujo $\alpha(\mu^2)$ para el caso en que $M \equiv m_{bottom} \approxeq 4,4$ GeV $\left(\begin{matrix}\text{cambio de}\\ n-1=4 \text{ sabores}\\ \text{a } n=5\end{matrix}\right)$ entre $\mu = 3$ GeV hasta 100 GeV. Elijo $\mu_0 \equiv \underset{91,1876}{\overset{}{M_Z}}$, $\alpha(\mu_0) = 0,1184$ como referencia (lo usual).

$\beta_{1,5} = -\frac{23}{6}$ $\quad\beta_{2,5} = -\frac{29}{6}$

$\beta_{1,4} = -\frac{25}{6}$ $\quad\beta_{2,4} = -\frac{77}{12}$

$\begin{pmatrix}\mu \equiv Q\\ \mu_0 \equiv Q_0\end{pmatrix}$ – Empezaré con α_{n-1} entre 3 GeV y M, y luego usaré condición de matching. entre M y 100 GeV.

↳ Ver gráfica adjunta