Problemas de Teoría Cuántica de Campos

Grado de Física 2010 - 2011

1. Comprobar que el tensor totalmente antisimétrico $\epsilon^{\mu\nu\rho\sigma}$ satisface las identidades:

   a) $\epsilon^{\alpha\beta\mu\nu}\,\epsilon_{\alpha\beta\rho\sigma} \;=\; -2\,\left(g^{\mu}_{\rho}g^{\nu}_{\sigma} - g^{\mu}_{\sigma}g^{\nu}_{\rho}\right)$

   b) $\epsilon^{\alpha\beta\gamma\mu}\,\epsilon_{\alpha\beta\gamma\nu} \;=\; -6\,g^{\mu}_{\nu}$

   c) $\epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{\alpha\beta\gamma\delta} \;=\; -24$

2. En el sistema en reposo $k^\mu = (m,\vec{0})$ definimos el cuadrivector polarización $s^\mu = (0,\vec{n})$, con $\vec{n}^{\,2} = 1$. Demostrar que en el sistema en movimiento, $k^\mu = (E,\vec{k})$, el cuadrivector s^μ viene dado por

$$s^\mu \;=\; \left(\frac{\vec{k}\,\vec{n}}{m}\,,\,\vec{n} + \frac{(\vec{k}\,\vec{n})\,\vec{k}}{m\,(E+m)}\right)\,.$$

3. Considerar el proceso de colisión $a+b \to 1+2$, en función de las variables de Mandelstam $s=(p_a+p_b)^2$, $t=(p_a-p_1)^2$ y $u=(p_a-p_2)^2$.

   a) Demostrar que $s+t+u \;=\; m_a^2+m_b^2+m_1^2+m_2^2\,.$

   b) Demostrar que en el sistema Centro de Masas el ángulo de dispersión entre los trimomentos $\vec{p}_a$ y $\vec{p}_1$ satisface la relación

$$\cos\theta \;=\; \frac{s(t-u)+(m_a^2-m_b^2)(m_1^2-m_2^2)}{\lambda^{1/2}(s,m_a^2,m_b^2)\;\lambda^{1/2}(s,m_1^2,m_2^2)}\,.$$

   c) Demostrar que la condición $\cos^2\theta \le 1$ restringe el rango físico de variación de la variable t: $t_+ \le t \le t_-$, donde

$$t_\pm \;=\; \frac{1}{4s}\,\left\{(m_a^2-m_b^2-m_1^2+m_2^2)^2 - \left[\lambda^{1/2}(s,m_a^2,m_b^2) \pm \lambda^{1/2}(s,m_1^2,m_2^2)\right]^2\right\}.$$

   d) Expresar las energías, trimomentos y ángulos de dispersión en el Sistema Laboratorio, $p_b^\mu = (m_b,\vec{0}\,)$, en función de s, t y u.

   e) Relacionar los sistemas Laboratorio y Centro de Masas mediante un boost a lo largo de la dirección de movimiento de la partícula b. Obtener las energías y trimomentos en Centro de Masas a partir de los resultados en el Laboratorio.

   f) Demostrar que para partículas muy energéticas ($\sqrt{s} \gg m_i$)

$$\tan\theta_1^L \;\approx\; \frac{2m_b}{\sqrt{s}}\;\tan\frac{\theta}{2}\,,$$

   donde θ_1^L es el ángulo de dispersión de la partícula 1 en el sistema Laboratorio. Comprobar que esta relación es exacta en el caso $m_a = m_b = m_1 = m_2$.

4. Considerar la desintegración a dos partículas $A \to 1 + 2$. Calcular las energías y trimomentos de las partículas finales en el sistema en reposo de la partícula A.

5. Considerar la desintegración $A \to 1 + 2 + 3$, en función de las variables invariantes $t_i \equiv (P - p_i)^2 = (p_j + p_k)^2 \equiv s_{jk}$, donde P^μ es el cuadrimomento de la partícula A y los índices $i \neq j \neq k$ etiquetan los cuadrimomentos de las 3 partículas finales.

   a) Demostrar que $t_1 + t_2 + t_3 \; = \; M_A^2 + m_1^2 + m_2^2 + m_3^2$.

   b) Calcular las energías y trimomentos de las partículas finales en el sistema en reposo de la partícula A, $P^\mu = (M_A, \vec{0}\,)$.

   c) Tomando como variables independientes s_{13} y s_{23}, demostrar que el rango cinemático permitido para estas variables viene dado por:

   $$s_{23}^{+} \;\leq\; s_{23} \;\leq\; s_{23}^{-}\,, \qquad\qquad (m_1 + m_3)^2 \;\leq\; s_{13} \;\leq\; (M_A - m_2)^2\,,$$

   donde

   $$s_{23}^{\pm} \;=\; \frac{1}{4 s_{13}} \; \left\{ (M_A^2 - m_1^2 - m_2^2 + m_3^2)^2 - \left[\lambda^{1/2}(M_A^2, s_{13}, m_2^2) \pm \lambda^{1/2}(s_{13}, m_1^2, m_3^2) \right]^2 \right\} .$$

   d) Representar gráficamente la región cinemática permitida en el plano (s_{13}, s_{23}) (diagrama de Dalitz). Particularizar el resultado en los casos

   - $m_1 = m_2 = m_3 = 0$
   - $m_1 = m\,, \quad m_2 = m_3 = 0$
   - $m_1 = m_3 = m\,, \quad m_2 = 0$

6. Considerar una partícula libre de masa m, que en $t = 0$ se encuentra en la posición $\vec{x}_0$. Estudiar la amplitud de probabilidad cuántica de encontrar la partícula en la cuadriposición $(t, \vec{x})$ en los casos

   a) no-relativista: $E = \vec{p}^{\,2}/(2m)$,

   b) relativista: $E = +\sqrt{m^2 + \vec{p}^{\,2}}$.

   Discutir el resultado obtenido para puntos que no están conectados causalmente (separación de tipo espacial $|\vec{x} - \vec{x}_0| > t$).

7. Estudiar la colisión de una partícula sin espín, descrita por el campo de Klein-Gordon, con un potencial escalón $V(z) \; = \; V_0 \, \theta(z)$.

8. Reescribir la ecuación de Klein-Gordon como $\left[\, \theta \equiv \frac{1}{2}\,(\phi + \frac{i}{m}\dot{\phi}) \;\;,\;\; \chi \equiv \frac{1}{2}\,(\phi - \frac{i}{m}\dot{\phi}) \,\right]$

   $$i\,\frac{\partial \Phi}{\partial t} = H_0\,\Phi \quad ; \quad \Phi = \begin{pmatrix} \theta \\ \chi \end{pmatrix} \quad ; \quad H_0 = -\frac{\vec{\nabla}^2}{2m} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + m \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

   Utilizando el cambio de variable $\phi \equiv \mathrm{e}^{-imt}\tilde{\phi}$, estudiar el límite no relativista, $\vec{p} \ll m$, y comprobar que se obtiene la ecuación de Schrödinger. Calcular las primeras correcciones relativistas.

9. Demostrar las identidades:

    a) $\gamma_5\,\sigma^{\mu\nu} = \frac{i}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma}$

    b) $\left[\sigma^{\alpha\beta}, \gamma^{\mu}\right] = -2i\,\left(g^{\mu\alpha}\gamma^{\beta} - g^{\mu\beta}\gamma^{\alpha}\right)$

10. Demostrar las siguientes propiedades del álgebra de matrices de Dirac ($\not{a} \equiv a_\mu\gamma^\mu$):

    a) $\gamma^\mu\gamma_\mu = 4\,I_4$

    b) $\gamma^\mu\gamma^\nu\gamma_\mu = -2\,\gamma^\nu$

    c) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\,g^{\nu\rho}$

    d) $\sigma^{\mu\nu}\sigma_{\mu\nu} = 12\,I_4$

    e) $\gamma^\mu\gamma_5\gamma_\mu = -4\,\gamma_5$

    f) $\gamma^\mu\gamma^\nu\gamma_5\gamma_\mu = 2\,\gamma^\nu\gamma_5$

    g) $\sigma^{\mu\nu}\gamma_5\sigma_{\mu\nu} = 12\,\gamma_5$

    h) $\gamma^\mu\sigma^{\rho\sigma}\gamma_\mu = 0$

    i) $\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\mu = 2\,\left[g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu\right]$

    j) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\,\gamma^\sigma\gamma^\rho\gamma^\nu$

    k) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\mu = 2\,\left(\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\tau + \gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma\right)$

    l) $\gamma^\mu\not{a}\gamma_\mu = -2\,\not{a}$

    m) $\gamma^\mu\not{a}\not{b}\gamma_\mu = 4\,a\cdot b = 4\,a^\mu b_\mu$

    n) $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\,\not{c}\not{b}\not{a}$

    ñ) $\gamma^\mu\not{a}\not{b}\not{c}\not{d}\gamma_\mu = 2\,\left(\not{c}\not{b}\not{a}\not{d} + \not{d}\not{a}\not{b}\not{c}\right) = 2\,\left(\not{b}\not{c}\not{d}\not{a} + \not{a}\not{d}\not{c}\not{b}\right)$

    o) $\not{a}\not{b} = 2\,a\cdot b - \not{b}\not{a}$

    p) $\not{a}\not{a} = a^2$

11. Demostrar las siguientes identidades:

    a) $\mathrm{Tr}\left(\gamma^\mu\right) = 0$

    b) $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4\,g^{\mu\nu}$

    c) $\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2k+1}}\right) = 0$

    d) $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\,\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$

    e) $\mathrm{Tr}\left(\not{a}\not{b}\right) = 4\,a\cdot b$

f) $\mathrm{Tr}\,(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = 4\,\left[(a_1\cdot a_2)\,(a_3\cdot a_4) - (a_1\cdot a_3)\,(a_2\cdot a_4) + (a_1\cdot a_4)\,(a_2\cdot a_3)\right]$

g) $$\mathrm{Tr}\,(\not{a}_1\not{a}_2\cdots\not{a}_{2k}) \;=\; a_1{\cdot}a_2\mathrm{Tr}\,(\not{a}_3\not{a}_4\cdots\not{a}_{2k}) - a_1{\cdot}a_3\mathrm{Tr}\,(\not{a}_2\not{a}_4\cdots\not{a}_{2k}) + a_1{\cdot}a_4\mathrm{Tr}\,(\not{a}_2\not{a}_3\cdots\not{a}_{2k}) \\ -\;\cdots\;+\;a_1\cdot a_{2k}\,\mathrm{Tr}\,(\not{a}_2\not{a}_3\cdots\not{a}_{2k-1})$$

h) $\mathrm{Tr}\,(\gamma_5) = 0$

i) $\mathrm{Tr}\,(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2k+1}}) = 0$

j) $\mathrm{Tr}\,(\gamma_5\not{a}\not{b}) = 0$

k) $\mathrm{Tr}\,(\gamma_5\not{a}\not{b}\not{c}\not{d}) = -4\,i\,\epsilon^{\alpha\beta\gamma\delta}\,a_\alpha b_\beta c_\gamma d_\delta$

l) $\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2k-1}}\gamma^{\mu_{2k}}) = \mathrm{Tr}\,(\gamma^{\mu_{2k}}\gamma^{\mu_{2k-1}}\ldots\gamma^{\mu_2}\gamma^{\mu_1})$

12. El conjunto de 16 matrices $\Gamma^n \equiv (I_4\,,\,i\gamma_5\,,\,\gamma^\mu\,,\,\gamma_5\gamma^\mu\,,\,\sigma^{\mu\nu})$ forman una base del espacio vectorial de las matrices de Dirac 4×4. Si definimos, $\Gamma_n \equiv (\Gamma^n)^{-1} = (I_4\,,\,-i\gamma_5\,,\,\gamma_\mu\,,\,\gamma_\mu\gamma_5\,,\,\sigma_{\mu\nu})$, comprobar que se satisfacen las siguientes propiedades:

    a) $\gamma^0\,(\Gamma^n)^\dagger\,\gamma^0 = \Gamma^n \qquad , \qquad \mathrm{Tr}\,(\Gamma^n\Gamma_m) = 4\,\delta^n_m \qquad (1\le n,m\le 16)\,.$

    b) La expansión de una matriz 4×4 arbitraria X en esta base viene dada por:

    $$X = \frac{1}{4}\sum_n \mathrm{Tr}\,(\Gamma_n X)\;\Gamma^n\,.$$

    c) Comprobar que el conjunto Γ^n forma una álgebra multiplicativa que no tiene más elementos.

    d) Estudiar las propiedades de transformación bajo el grupo de Lorentz de los bilineales $\bar\psi\,\Gamma^n\psi$.

13. Encontrar la matriz de transformación de un campo de Dirac, $S(\Lambda)$, correspondiente a un boost con velocidad v en la dirección $\hat z$.

14. Obtener la solución general de la ecuación de Dirac en un sistema de referencia arbitrario, aplicando una transformación de Lorentz sobre las soluciones en el sistema en reposo.

15. Dadas dos soluciones arbitrarias de la ecuación de Dirac, $\psi_1(x)$ y $\psi_2(x)$, demostrar que se cumple la siguiente identidad (descomposición de Gordon):

    $$\bar\psi_2\,\gamma_\mu\,\psi_1 \;=\; \frac{i}{2m}\left(\bar\psi_2\,\overset{\leftrightarrow}{\partial}_\mu\,\psi_1\right) \;+\; \frac{1}{2m}\,\partial^\nu\left(\bar\psi_2\,\sigma_{\mu\nu}\,\psi_1\right)\,.$$

16. Demostrar que

    $$\bar u_s(\vec{k_1})\,\gamma_\mu\,u_r(\vec{k_2}) \;=\; \frac{1}{2m}\,\bar u_s(\vec{k_1})\,\left[(k_1+k_2)_\mu \;+\; i\;(k_1-k_2)^\nu\;\sigma_{\mu\nu}\right]\,u_r(\vec{k_2})\,.$$

17. Estudiar la colisión de un electrón, con momento k y polarización $+\frac{1}{2}$ en la dirección $\hat{z}$, con un potencial escalón $V(z) = V_0\,\theta(z)$.

18. Encontrar un Lagrangiano que genere las ecuaciones de
    - *a*) Schrödinger
    - *b*) Klein-Gordon
    - *c*) Dirac

19. Demostrar que el operador

$$\mathcal{Q} = i \sum_i Q_i \int d^3x \, \left\{ \pi_i^\dagger(x)\, \phi_i^\dagger(x) - \pi_i(x)\, \phi_i(x) \right\}$$

es el generador ($U = \exp\{i\theta\mathcal{Q}\}$) de la transformación

$$\phi_i'(x) = U\,\phi_i(x)\,U^\dagger = \mathrm{e}^{-i\theta Q_i}\,\phi_i(x) \qquad , \qquad \phi_i'^\dagger(x) = U\,\phi_i^\dagger(x)\,U^\dagger = \mathrm{e}^{i\theta Q_i}\,\phi_i^\dagger(x)\,.$$

20. Demostrar que el operador momento angular

$$\mathcal{M}^{ij} = \int d^3x \, \left\{ x^i\,T^{0j} - x^j\,T^{0i} + \pi_m\,\Sigma^{ij}\,\phi_m \right\}$$

satisface

$$\left[\mathcal{M}^{ij}, \phi_m(x)\right] = -i\,\left(x^i\partial^j - x^j\partial^i + \Sigma^{ij}\right)\,\phi_m(x)\,.$$

21. Considerar un campo de Klein–Gordon real.
    - *a*) Obtener las expresiones de los operadores cuadrimomento P^μ y momento angular $\mathcal{M}^{\mu\nu}$ en función de operadores creación y destrucción.
    - *b*) Comprobar que P^μ y $\mathcal{M}^{\mu\nu}$ satisfacen las reglas de conmutación de los generadores del grupo de Poincaré:

$$[P^\mu, P^\nu] = 0 \qquad , \qquad [\mathcal{M}^{\mu\nu}, P^\alpha] = i\,(g^{\mu\alpha}\,P^\nu - g^{\nu\alpha}\,P^\mu)\;,$$
$$\left[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\alpha\beta}\right] = i\,\left(g^{\mu\alpha}\,\mathcal{M}^{\nu\beta} - g^{\mu\beta}\,\mathcal{M}^{\nu\alpha} - g^{\nu\alpha}\,\mathcal{M}^{\mu\beta} + g^{\nu\beta}\,\mathcal{M}^{\mu\alpha}\right)\;.$$

22. Suponer que cuantizamos el campo de Klein-Gordon real con reglas de anticonmutación (en lugar de las reglas canónicas de conmutación):

$$\left\{a(\vec{k}), a^\dagger(\vec{k}')\right\} = \Delta_{\vec{k}\vec{k}'} \qquad , \qquad \left\{a(\vec{k}), a(\vec{k}')\right\} = \left\{a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right\} = 0\,.$$

Demostrar que dados dos puntos espacio-temporales x^μ e y^μ separados espacialmente, $(x-y)^2 < 0$,

$$[\phi(x), \phi(y)] \neq 0 \qquad , \qquad \{\phi(x), \phi(y)\} \neq 0\,,$$

y por lo tanto no se satisface el requisito de microcausalidad.

23. Considerar un campo de Dirac libre. Demostrar que en términos de operadores creación-destrucción, el Hamiltoniano y el operador carga adoptan la forma

$$\mathcal{H} \,=\, \sum_{k,r} k^0 \left(a_r^\dagger(\vec{k})\, a_r(\vec{k}) \,-\, b_r(\vec{k})\, b_r^\dagger(\vec{k}) \right) \,,$$

$$\mathcal{Q} \,=\, Q \sum_{k,r} \left(a_r^\dagger(\vec{k})\, a_r(\vec{k}) \,+\, b_r(\vec{k})\, b_r^\dagger(\vec{k}) \right) \,.$$

Interpretar físicamente el resultado, suponiendo que cuantizamos el campo de Dirac con a) conmutadores o b) anticonmutadores.

24. Comprobar que el operador densidad de corriente de la ecuación de Dirac, $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, satisface la relación

$$[j^\mu(x), j^\nu(y)] \,=\, 0 \qquad , \qquad \text{si} \quad (x-y)^2 < 0 \,,$$

como requiere el principio de microcausalidad para dos observables situados en puntos del espacio-tiempo que no están conectados causalmente.

25. Definimos las componentes levógiras y dextrógiras del campo de Dirac ψ como

$$\psi_L \,\equiv\, \frac{1}{2}\,(1-\gamma_5)\,\psi \qquad , \qquad \psi_R \,\equiv\, \frac{1}{2}\,(1+\gamma_5)\,\psi \,.$$

Comprobar que

a) $\gamma_5\,\psi_L \,=\, -\psi_L$ y $\gamma_5\,\psi_R \,=\, \psi_R$.

b) ψ_L y ψ_R se transforman de forma independiente bajo transformaciones de Lorentz $\mathcal{L}_+^\uparrow$.

c) El Lagrangiano de Dirac puede escribirse como

$$\mathcal{L} \,=\, i\,\overline{\psi_L}\not{\partial}\psi_L \,+\, i\,\overline{\psi_R}\not{\partial}\psi_R \,-\, m\,\left(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\right) \,.$$

d) En el límite de masa nula las ecuaciones de movimiento de ψ_L y ψ_R se desacoplan; ψ_L describe entonces fermiones de helicidad negativa y ψ_R fermiones de helicidad positiva.

e) Reescribir tanto el Lagrangiano como las ecuaciones de movimiento en términos de spinores de dos componentes, utilizando la representación de Weyl.

26. Considerar el Lagrangiano

$$\mathcal{L}_M \,=\, i\,\overline{\psi_L}\not{\partial}\psi_L \,-\, \frac{1}{2}\,m\,\left(\overline{\psi_L^c}\psi_L \,+\, \overline{\psi_L}\psi_L^c\right)$$

donde $\psi_L^c \equiv C\,\overline{\psi_L}^T$, con C la matriz de conjugación de carga.

a) Comprobar que $\gamma_5\,\psi_L^c = \psi_L^c$. El campo ψ_L^c describe por tanto un spinor dextrógiro.

b) Comprobar que $\mathcal{L}_M$ es invariante bajo transformaciones de Lorentz $\mathcal{L}_+^\uparrow$.

c) Comprobar que el término de masas en $\mathcal{L}_M$ solo existe si el campo ψ_L satisface reglas de anticonmutación.

d) Escribir la ecuación de movimiento del campo ψ_L.

e) Comprobar que el término cinético es invariante bajo transformaciones de fase $\psi'_L = \mathrm{e}^{i\theta}\,\psi_L$, mientras que el término de masas no lo es. Escribir la corriente de Noether asociada y calcular su divergencia.

f) Definimos el campo $\psi_M \equiv \psi_L + \psi_L^c$, que satisface $\psi_M = \psi_M^c$. Comprobar que (salvo derivadas totales)

$$\mathcal{L}_M = \frac{1}{2}\, i\,\overline{\psi_M}\partial\!\!\!/\psi_M - \frac{1}{2}\, m\,\overline{\psi_M}\psi_M\,.$$

g) Encontrar las ecuaciones de movimiento de ψ_M. Demostrar que describe fermiones que son iguales a sus propios antifermiones, con las dos helicidades.

h) Reescribir todas las ecuaciones anteriores en términos de spinores de Weyl.

27. Considerar el Lagrangiano

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\Phi\,\partial_\mu\Phi - \frac{1}{2}\,M^2\,\Phi^2 + \frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\,m^2\,\varphi^2 - \frac{g}{2}\,\Phi\,\varphi^2\,,$$

que describe la interacción entre dos campos escalares reales. Calcular las amplitudes de los procesos:

a) $\Phi(\vec{p}_1) \to \varphi(\vec{p}_2)\,\varphi(\vec{p}_3)$.

b) $\Phi(\vec{p}_1)\Phi(\vec{p}_2) \to \varphi(\vec{p}_3)\,\varphi(\vec{p}_4)$.

c) $\Phi(\vec{p}_1)\varphi(\vec{p}_2) \to \Phi(\vec{p}_3)\,\varphi(\vec{p}_4)$.

d) $\varphi(\vec{p}_1)\varphi(\vec{p}_2) \to \varphi(\vec{p}_3)\,\varphi(\vec{p}_4)$.

28. Estudiar el proceso elástico d) del problema anterior en el régimen cinemático $E_i, m \ll M$. Construir un Lagrangiano efectivo $\mathcal{L}(\varphi)$, que solo incluya el campo ligero φ, y que genere la misma amplitud de colisión a bajas enegías (al orden más bajo en la constante de acoplamiento g).

29. Considerar el Lagrangiano

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\Phi\,\partial_\mu\Phi - \frac{1}{2}\,M^2\,\Phi^2 + \partial^\mu\varphi^\dagger\,\partial_\mu\varphi - m^2\,\varphi^\dagger\varphi - g\,\Phi\,\varphi^\dagger\varphi\,,$$

que describe la interacción entre un campo escalar real (neutro) Φ y un campo complejo (cargado) φ. Calcular las amplitudes de los procesos:

a) $\Phi(\vec{p}_1) \to \varphi(\vec{p}_2)\,\bar{\varphi}(\vec{p}_3)$.

b) $\Phi(\vec{p}_1)\Phi(\vec{p}_2) \to \varphi(\vec{p}_3)\,\bar{\varphi}(\vec{p}_4)$.

c) $\Phi(\vec{p}_1)\varphi(\vec{p}_2) \to \Phi(\vec{p}_3)\,\varphi(\vec{p}_4)$.

d) $\varphi(\vec{p}_1)\bar{\varphi}(\vec{p}_2) \to \varphi(\vec{p}_3)\,\bar{\varphi}(\vec{p}_4)$.

30. Considerar la siguiente interacción entre un campo escalar real y un campo de Dirac:

$$\mathcal{L} \,=\, \frac{1}{2}\,\partial^\mu\Phi\,\partial_\mu\Phi - \frac{1}{2}\,M^2\,\Phi^2 + \bar{\psi}\,\left(i\,\gamma^\mu\partial_\mu - m\right)\,\psi - i\,g\,\left(\bar{\psi}\,\gamma_5\,\psi\right)\,\Phi\,.$$

Calcular las amplitudes de los procesos:

a) $\Phi(\vec{p}_1) \rightarrow \psi(\vec{p}_2)\,\bar{\psi}(\vec{p}_3)$.

b) $\Phi(\vec{p}_1)\Phi(\vec{p}_2) \rightarrow \psi(\vec{p}_3)\,\bar{\psi}(\vec{p}_4)$.

c) $\Phi(\vec{p}_1)\psi(\vec{p}_2) \rightarrow \Phi(\vec{p}_3)\,\psi(\vec{p}_4)$.

d) $\psi(\vec{p}_1)\bar{\psi}(\vec{p}_2) \rightarrow \psi(\vec{p}_3)\,\bar{\psi}(\vec{p}_4)$.

31. Calcular las anchuras de desintegración y secciones eficaces correspondientes a los procesos del problema anterior:

a) $\Gamma(\Phi \rightarrow \psi\,\bar{\psi})$.

b) $\sigma(\Phi\Phi \rightarrow \psi\bar{\psi})$.

c) $\sigma(\Phi\psi \rightarrow \Phi\psi)$.

d) $\sigma(\psi\bar{\psi} \rightarrow \psi\bar{\psi})$.

32. Estudiar el proceso elástico d) del problema anterior en el régimen cinemático $E_i, m \ll M$. Construir un Lagrangiano efectivo $\mathcal{L}(\psi)$, que solo incluya el campo de Dirac ligero ψ, y que genere la misma amplitud de colisión a bajas enegías (al orden más bajo en la constante de acoplamiento g).

33. La desintegración del muón está descrita por el Lagrangiano de Fermi

$$\mathcal{L}_F \,=\, \frac{G_F}{\sqrt{2}}\,\left[\bar{e}\,\gamma^\alpha\,(1-\gamma_5)\,\nu_e\right]\,\left[\bar{\nu}_\mu\,\gamma_\alpha\,(1-\gamma_5)\,\mu\right]\,.$$

a) Estimar $\Gamma(\mu^- \rightarrow e^-\bar{\nu}_e\nu_\mu)$ mediante argumentos dimensionales.

b) Calcular la vida media del muón.

34. En el modelo estandar electrodébil, los acoplamientos leptónicos del bosón cargado $W^\pm$ estan descritos por el Lagrangiano de interacción

$$\mathcal{L}_W \,=\, \frac{g}{2\sqrt{2}}\,\left\{W_\alpha^\dagger\,\sum_{l=e,\mu,\tau}\,\left[\bar{\nu}_l\,\gamma^\alpha\,(1-\gamma_5)\,l\right]\;+\;\mathrm{h.c}\right\}\,.$$

a) Calcular las anchuras de desintegración leptónicas $\Gamma(W^- \rightarrow l^-\bar{\nu}_l),\;\; l = e, \mu, \tau$.

b) Comparar los resultados con las medidas experimentales y obtener un test cuantitativo de la universalidad de la interacción: la constante de acoplamiento g es la misma para los 3 leptones.

c) Relacionar la constante electrodébil g con la constante de Fermi G_F.

PROBLEMAS TQC

①

a) $\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2!\,(g^{\mu}{}_{\rho}\,g^{\nu}{}_{\sigma} - g^{\mu}{}_{\sigma}\,g^{\nu}{}_{\sigma})$

$\epsilon^{0123} = +1$

● Manera 1:

- El resultado debe depender de los índices no sumados → $= A^{\mu\nu}{}_{\rho\sigma}$
- $A^{\mu\nu}{}_{\rho\sigma}$ debe ser totalmente antisimétrico al intercambiar $\mu \leftrightarrow \nu$, pero
- A debe tener que ser con el tensor métrico g (2 índices Lorentz)
- Para tener 4 índices, debe ser producto de 2 gs.
- No pueden aparecer $g^{\mu\nu}$ ó $g_{\rho\sigma}$ porque no son antisimétricos.

↳ Deben ser $g^{\mu}{}_{\rho}\cdot g^{\nu}{}_{\sigma}$

- Para preservar la antisimetría, hay que añadir (restar) otro producto de gs. → única manera de combinar gs para tener 4 índices y que sea antisimétrico.

Por tanto, proponemos una dependencia del tipo:

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = a\, g^{\mu}{}_{\rho}\, g^{\nu}{}_{\sigma} + b\, g^{\mu}{}_{\sigma}\, g^{\nu}{}_{\rho}$$

↓ permuto $\mu\nu$, cambio signo:

$$\epsilon^{\alpha\beta\nu\mu}\epsilon_{\alpha\beta\rho\sigma} = -\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -(a\, g^{\mu}{}_{\rho}\, g^{\nu}{}_{\sigma} + b\, g^{\mu}{}_{\sigma}\, g^{\nu}{}_{\rho}) = a\, g^{\nu}{}_{\rho}\, g^{\mu}{}_{\sigma} + b\, g^{\nu}{}_{\sigma}\, g^{\mu}{}_{\rho}$$

↳ De donde se deduce que $a = -b$

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = a(g^{\mu}{}_{\rho}\, g^{\nu}{}_{\sigma} - g^{\mu}{}_{\sigma}\, g^{\nu}{}_{\rho})$$ → combinación antisimétrica adecuada

→ Debemos calcular la constante a, para lo que empleamos un caso particular (sin pérdida de generalidad).

$$\left.\begin{matrix}\mu = 2 = \rho \\ \nu = 3 = \sigma\end{matrix}\right\} \quad \epsilon^{\alpha\beta 23}\epsilon_{\alpha\beta 23} = \epsilon^{0123}\epsilon_{0123} + \epsilon^{1023}\epsilon_{1023}$$

$$= 1\,(-1) + (-1)\cdot 1 = -2 = a\,(\underset{1}{\underset{\shortparallel}{g^{2}{}_{2}}}\,\underset{1}{\underset{\shortparallel}{g^{3}{}_{3}}} - \cancel{g^{2}{}_{3}}\,g^{3}{}_{2}) = a$$

↳ $a = -2$

Otra forma de obtener a es teniendo en cuenta las permutaciones que puedes realizar en el espacio de Minkowski con los índices sumados (α, β) → $2!$, y el signo negativo por el determinante de la métrica.

- Manera 2:

- Asimilar el producto de $\epsilon \cdot \epsilon$ a un determinante de la métrica 4x4
- Reducirlo a un det 2 × 2 al contar permutaciones (2!) y introducir el signo negativo debido a la métrica

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2! \cdot \begin{vmatrix} g^{\mu}{}_{\rho} & g^{\mu}{}_{\sigma} \\ g^{\nu}{}_{\rho} & g^{\nu}{}_{\sigma} \end{vmatrix} = -2!\left(g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} - g^{\mu}{}_{\sigma}g^{\nu}{}_{\rho}\right) \checkmark$$

Ambas maneras son poco rigurosas o formales (demostración de palabra). La última manera es demostrar todos los casos particulares de μ, ν, ρ, σ, lo cual no es muy elegante pero sí plenamente riguroso. Lo más formal sería hacer el determinante 4 × 4 completo (sin reducirlo a ojo como en manera 2) → explicado en página siguiente

b) $\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -6g^{\mu}{}_{\nu}$

- Manera 1:

Uso a) con $\mu \Rightarrow \gamma$; $\nu \Rightarrow \mu$; $\rho \Rightarrow \gamma$; $\sigma \Rightarrow \nu$ → $g^{\gamma}{}_{\gamma} = \sum_{\gamma=0}^{3} \overset{1}{\overset{\|}{g^{\gamma}{}_{\gamma}}} = 1+1+1+1=4$

$$\Rightarrow = -2\left(\overset{4}{g^{\gamma}{}_{\gamma}}g^{\mu}{}_{\nu} - g^{\gamma}{}_{\nu}g^{\mu}{}_{\gamma}\right) = -2\left(4g^{\mu}{}_{\nu} - g^{\mu}{}_{\nu}\right) = -6g^{\mu}{}_{\nu} \checkmark$$

- Manera 2:

(Repetir procedimiento de a))

- Hay 2 índices libres → es suficiente con una g
- μ y ν (libres) son independientes, no están en el mismo ϵ, ya no deben ser antisimétricos
- Si $\alpha \neq \beta \neq \gamma$ (contribución no nula), μ y ν están fijados y son el mismo porque sólo hay índices del 0 al 3 y no se pueden repetir. Si $\mu \neq \nu$ → contribución nula.

$\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\mu} = -1$ siempre que no haya índices repetidos y ningún índice esté sumado!! $\alpha, \beta, \gamma, \mu$ no como índices, no sumados.

↓

si no: 4!

↳ Por tanto, proponemos $\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} = -a \cdot g^{\mu}{}_{\nu}$

Falta calcular a:

→ Con caso particular:

$$\epsilon^{0123}\epsilon_{0123} + \epsilon^{1023}\epsilon_{1023} + \epsilon^{0213}\epsilon_{0213} + \epsilon^{1203}\epsilon_{1203} + \epsilon^{2013}\epsilon_{2013} + \epsilon^{2103}\epsilon_{2103}$$

$= -6 = -a\,g^{3}{}_{3} = -a \longrightarrow a = 6 = 3!$

- Contando permutaciones de los índices sumados

$a = \text{N}^{\text{o}}$ permut $\{\alpha, \beta, \gamma\} = 3!$

- Manera 3: Con un determinante 1 × 1, añadiendo signo y contando permutaciones → $-3!\,|g^{\mu}{}_{\nu}|$ ✓

c) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24$

- Manera 1:

Uso b) con $\mu = \nu = \delta$ ó añades una $g^{\nu}{}_{\mu}$

↳ $= -6 \cdot g^{\delta}{}_{\delta} = -6 \cdot 4 = -24$

↳ $= -6 \cdot g^{\mu}{}_{\nu}\, g^{\nu}{}_{\mu} = -6\, g^{\mu}{}_{\mu} = -6 \cdot 4$
$= -24$

- Manera 2:

↳ No hay índices libres → $= \text{cte} = a \cdot (-1)$ ∴ métrica por bajar 4 índices (↳ $\det(g) = -1$)

↳ Calcular cte con caso particular → cuentas permutaciones

↳ a = Nº permutaciones de los 4 índices sumados $= 4!$

→ $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -4!$ // ✓

⇒ El método del determinante es el más robusto, y es más útil cuando hay pocos índices repetidos (evitas pensar qué combinaciones de gs debe haber).

⇒ El método de contar permutaciones es muy útil cuando hay muchos índices repetidos y la dificultad está en calcular la constante, pues no hay apenas gs.

En general, el determinante está definido como:

$$\epsilon_{i_1 \dots i_n}\epsilon_{j_1 \dots j_n} = \det \begin{bmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \delta_{i_1 j_3} & \dots & \delta_{i_1 j_n} \\ \vdots & & & & \\ \delta_{i_n j_1} & \dots & & & \delta_{i_n j_n} \end{bmatrix}$$

Para 4 índices, en el espacio de Minkowski, con convenio $\epsilon^{0123} = +1$ pero $\epsilon_{0123} = -1$

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha'\beta'\gamma'\delta'} = \underset{g^{00}}{1} \cdot \underset{g^{ii}}{(-1)^3} \cdot \det \begin{bmatrix} \delta^{\alpha}_{\alpha'} & \delta^{\alpha}{}_{\beta'} & \dots \\ & & \end{bmatrix}$$

↳ $\delta^{\alpha}{}_{\alpha'} = g^{\alpha}{}_{\alpha'}$

$$= - \begin{vmatrix} g^{\alpha}{}_{\alpha'} & g^{\alpha}{}_{\beta'} & g^{\alpha}{}_{\gamma'} & g^{\alpha}{}_{\delta'} \\ g^{\beta}{}_{\alpha'} & g^{\beta}{}_{\beta'} & g^{\beta}{}_{\gamma'} & g^{\beta}{}_{\delta'} \\ g^{\gamma}{}_{\alpha'} & g^{\gamma}{}_{\beta'} & g^{\gamma}{}_{\gamma'} & g^{\gamma}{}_{\delta'} \\ g^{\delta}{}_{\alpha'} & g^{\delta}{}_{\beta'} & g^{\delta}{}_{\gamma'} & g^{\delta}{}_{\delta'} \end{vmatrix}$$

que permite resolver los casos a, b y c mecánicamente.

→ Se puede encontrar esta relación en CORE 2.1 - Compendium of Relations (disponible en internet)

② $k^\mu = (m, 0)$ $\quad \longrightarrow \quad k'^\mu = (E, \vec{k}')$

$s^\mu = (0, \hat{n})$ $\quad \hookrightarrow s'^\mu = \left(\frac{\vec{k}'\cdot\vec{n}}{m}, \frac{(\vec{k}'\cdot\vec{n})\vec{k}'}{m(E+m)} \right)$

boost $\Lambda^\mu{}_\nu$ con $\gamma = \frac{E}{m}$, $\vec{k}' = \gamma m \vec{\beta} = E \cdot \vec{\beta}$

$s'^\mu = \Lambda^\mu{}_\nu s^\nu$

$\vec{k}'$ orientación arbitraria

- Manera general (boost arbitrario sin rotación de ejes de coordenadas)

$$s'^\mu = \Lambda^\mu{}_\nu s^\nu = \begin{bmatrix} \gamma & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & 1+(\gamma-1)\beta_x^2/\beta^2 & (\gamma-1)\beta_x\beta_y/\beta^2 & (\gamma-1)\beta_x\beta_z/\beta^2 \\ +\gamma\beta_y & (\gamma-1)\beta_y\beta_x/\beta^2 & 1+(\gamma-1)\beta_y^2/\beta^2 & (\gamma-1)\beta_y\beta_z/\beta^2 \\ +\gamma\beta_z & (\gamma-1)\beta_z\beta_x/\beta^2 & (\gamma-1)\beta_z\beta_y/\beta^2 & 1+(\gamma-1)\beta_z^2/\beta^2 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ n_x \\ n_y \\ n_z \end{pmatrix}$$

$$= \begin{bmatrix} +\gamma(n_x\beta_x + n_y\beta_y + n_z\beta_z) \\ n_x + (\gamma-1)/\beta^2 \cdot \beta_x(\beta_x n_x + \beta_y n_y + \beta_z n_z) \\ n_y + (\gamma-1)/\beta^2 \cdot \beta_y(\quad " \quad) \\ n_z + (\gamma-1)/\beta^2 \cdot \beta_z(\quad " \quad) \end{bmatrix} = \begin{pmatrix} +\gamma\vec{\beta}\cdot\hat{n} \\ n_x + \frac{(\gamma-1)}{\beta^2}\beta_x \cdot \vec{\beta}\cdot\hat{n} \\ n_y + \quad " \quad \beta_y \quad " \\ n_z + \quad " \quad \beta_z \quad " \end{pmatrix}$$

$$= \left(+\gamma\vec{\beta}\cdot\hat{n}, \vec{n} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\hat{n})\cdot\vec{\beta} \right)$$

$$= \left(\frac{\vec{K}'}{m}\cdot\hat{n}, \vec{n} + \left(\frac{\gamma-1}{\beta^2}\right)\frac{\vec{K}'}{\gamma m}\left(\frac{\vec{K}'}{\gamma m}\cdot\hat{n}\right) \right) = \left(\frac{\vec{k}'\cdot\hat{n}}{m}, \vec{n} + \frac{\gamma-1}{\gamma^2\beta^2}\cdot\frac{1}{m^2}(\vec{k}'\cdot\hat{n})\cdot\vec{k}' \right)$$

$$\hookrightarrow \frac{(\gamma-1)}{\gamma^2\beta^2} = \frac{\gamma-1}{\gamma^2}\cdot\frac{\gamma^2}{\gamma^2-1} = \frac{\gamma-1}{(\gamma-1)(\gamma+1)} = \frac{1}{\gamma+1} = \frac{1}{\frac{E}{m}+1} = \frac{m}{E+m}$$

$$\hookrightarrow s'^\mu = \left(\frac{\vec{k}'\cdot\hat{n}}{m}, \vec{n} + \frac{(\vec{k}'\cdot\hat{n})\cdot\vec{k}'}{m(E+m)} \right) \quad // \quad ✓$$

- Manera particular $\quad \rightarrow$ para luego generalizar por simetría

$\vec{k}' = (0, 0, \vec{k}') = k'\hat{z}$, $\vec{\beta} = \beta\hat{z}$

$K'^\mu = (\gamma m, 0, 0, \gamma\beta m) \quad \rightarrow \quad \frac{K'_z}{\beta} = E = \gamma m$; ó $(k'_z)^2 = \underbrace{\gamma^2 m^2}_{E^2} - m^2$

$= m^2\gamma^2\left(1 - \frac{1}{\gamma^2}\right)$

$= E^2(1 - (1-\beta^2)) = E^2\beta^2$

$$s'^\mu = \Lambda^\mu{}_\nu s^\nu = \begin{pmatrix} \gamma & & & \gamma\beta \\ & 1 & & \\ & & 1 & \\ \gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \hat{n} \end{pmatrix} = \begin{pmatrix} \gamma\beta n_z \\ n_x \\ n_y \\ \gamma n_z \end{pmatrix} = \begin{pmatrix} \frac{K'_z}{m} n_z \\ n_x \\ n_y \\ \frac{E}{m} n_z \end{pmatrix} = \begin{pmatrix} K'_z n_z/m \\ n_x \\ n_y \\ n_z + n_z\left(\frac{E}{m}-1\right) \end{pmatrix}$$

$$\frac{E}{m} - 1 = \gamma - 1 = (\gamma-1)\frac{(\gamma+1)}{(\gamma+1)} = \frac{\gamma^2-1}{\gamma+1} = \frac{\beta^2\gamma^2}{(\gamma+1)} = \frac{K_z'^2/m^2}{\left(\frac{E}{m}+1\right)} = \frac{K_z'^2}{m(E+m)}$$

$$s'^{\mu} = \begin{pmatrix} \dfrac{k'_z n_z}{m} \\ n_x \\ n_y \\ n_z + \dfrac{n_z k'_z k'_z}{m(E+m)} \end{pmatrix}$$

Por simetría, si hubiésemos elegido $\vec{k}' = (0, k'_y, 0)$ habríamos obtenido
(n_z no es dirección privilegiada)
↳ término extra se debía a k'_z

$$s'^{\mu} = \begin{pmatrix} \dfrac{k'_y n_y}{m} \\ n_x \\ n_y + \dfrac{n_y k'_y k'_y}{m(E+m)} \\ n_z \end{pmatrix}$$

Para un $\vec{k}'$ general:

$$s'^{\mu} = \begin{pmatrix} \dfrac{\vec{k}' \cdot \vec{n}}{m} \\ \hat{n} + \dfrac{(\vec{k} \cdot \vec{n})}{m(E+m)} \cdot \vec{k} \end{pmatrix}$$ ✓

③

$a + b \longrightarrow 1 + 2$

$s = (p_a + p_b)^2$, $t = (p_a - p_1)^2$, $u = (p_a - p_2)^2$ → Variables de Mandelstam (invariantes)

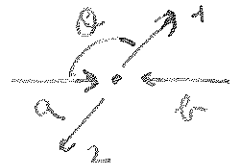
a) $s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2$

Sabemos que: $p_i^2 = m_i^2$
$p_a + p_b = p_1 + p_2$

↳ $s + t + u = p_a^2 + p_b^2 + 2p_a p_b + p_a^2 + p_1^2 - 2p_a p_1 + p_a^2 + p_2^2 - 2p_2 p_a$

$= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 - 2p_a \underbrace{(p_1 + p_2 - p_b)}_{p_a} = m_a^2 + m_b^2 + m_1^2 + m_2^2$ ✓

b) $\cos\theta = \ldots$; $\theta \equiv$ ángulo entre $\vec{p_a}$ y $\vec{p_1}$



$t = m_a^2 + m_1^2 - 2p_a p_1 = m_a^2 + m_1^2 - 2E_a E_1 - 2\overbrace{\vec{p_a}\,\vec{p_1}}^{|\vec{p_a}||\vec{p_1}|\cdot\cos\theta}$

$\cos\theta = \dfrac{t - m_a^2 - m_1^2 + 2E_a E_1}{2|\vec{p_a}|\cdot|\vec{p_1}|}$ ∀ stma. de referencia

En sistema centro de masas:

$$s=(p_a+p_b)^2 \longrightarrow p_a=(E_a,\vec{p}) \ ; \ p_b=(E_b,-\vec{p}) \ \Big\| \ p_1=(E_1,\vec{p}\,'), p_2=(E_2,\vec{p}\,')$$

$$\sqrt{s}=p_a+p_b=E_a+E_b=E_1+E_2=p_1+p_2$$

$$s=(p_a+p_b)^2= m_a^2+m_b^2+2p_ap_b=m_a^2+m_b^2+2E_aE_b-2\vec{p}(-\vec{p})$$

$$=m_a^2+m_b^2+2E_aE_b+2(E_a^2-m_a^2)=-m_a^2+m_b^2+2E_a(\underbrace{E_b+E_a}_{\sqrt{s}})$$

$$E_a^{CM}=\frac{s+m_a^2-m_b^2}{2\sqrt{s}}$$

$$\vec{p}^{\,2}=E_a^2-m_a^2=\frac{1}{4s}\left[s^2+(m_a^2-m_b^2)^2+2s(m_a^2-m_b^2)-4s\,m_a^2\right]$$

$$|\vec{p}|=\frac{1}{2\sqrt{s}}\left[s^2+m_a^4+m_b^2-2m_a^2m_b^2-2s\,m_a^2-2s\,m_b^2\right]$$

$$=\frac{1}{2\sqrt{s}}\lambda^{1/2}(s,m_a^2,m_b^2) \ ; \quad \lambda(x,y,z)\equiv x^2+y^2+z^2-2xy-2yz-2xz$$

↳ Función de Källen

Idem para $|\vec{p}\,'|$

→ Intercambiar $1,2 \rightarrow a,b$

$$s=(p_a+p_b)^2=(p_1+p_2)^2=-m_1^2+m_2^2+2E_1\sqrt{s}$$

$$E_1^{CM}=\frac{s+m_1^2-m_2^2}{2\sqrt{s}} \longrightarrow |\vec{p}\,'|=\frac{1}{2\sqrt{s}}\lambda^{1/2}(s,m_1^2,m_2^2)$$

$$\cos\theta_{CM}=\frac{t-m_a^2-m_1^2+2E_a^{CM}E_1^{CM}}{2\cdot\frac{1}{4s}\lambda^{1/2}(s,m_a^2,m_b^2)\lambda^{1/2}(s,m_1^2,m_2^2)}$$

$$\hookrightarrow 2s(t-m_a^2-m_1^2+2E_a^{CM}E_1^{CM})=2s\left(t-m_a^2-m_1^2+2\left(\frac{s+m_a^2-m_b^2}{4s}\right)\left(\frac{s+m_1^2-m_2^2}{1}\right.\right.$$

$$=2s(t-m_a^2-m_1^2)+s^2+sm_1^2-sm_2^2+sm_a^2+m_a^2m_1^2-m_2^2m_a^2-sm_b^2-m_1^2m_b^2 \ \swarrow +m_b^2m_2^2$$

$$=2st-sm_a^2-sm_1^2+s^2-sm_2^2+m_a^2m_1^2-m_2m_a^2-sm_b^2-m_1^2m_b^2+m_b^2m_2^2$$

$$=2st-s(\underbrace{m_a^2+m_1^2+m_2^2+m_b^2}_{s+t+u})+s^2+m_a^2m_1^2+m_b^2m_2^2-m_a^2m_2^2-m_b^2m_1^2$$

$$=2st-s^2-st-su+s^2+m_a^2(m_1^2-m_2^2)+m_b^2(m_2^2-m_1^2)$$

$$=st-su+(m_1^2-m_2^2)(m_a^2-m_b^2)$$

$$\cdot\ \cos\theta_{CM}=\frac{s(t-u)+(m_a^2-m_b^2)(m_1^2-m_2^2)}{\lambda^{1/2}(s,m_a^2,m_b^2)\ \lambda^{1/2}(s,m_1^2,m_2^2)}$$

c) $\cos^2\theta \leq 1 \rightarrow \cos\theta_{lim} = \pm 1$

Aparte, sustituyo $u(t, s, m_i) \rightarrow s+t+u = \ldots \rightarrow$ ver a)

$$\pm 1 = \frac{s\left(t_{\substack{1\\2}} - (m_a^2 + m_b^2 + m_1^2 + m_2^2 - s - t_{\substack{1\\2}})\right) + (m_a^2 - m_b^2)(m_1^2 - m_2^2)}{\underbrace{\lambda^{1/2}(s, m_a^2, m_b^2)}_{\lambda^{1/2}_{sab}} \cdot \lambda^{1/2}(s, m_1^2, m_2^2) = \lambda^{1/2}_{s12}} \rightarrow$$

2 soluciones $t_{1,2}$, no tienen que ver con las partículas 1-2

signo ± 1 del coseno $\equiv$ arriba / $\equiv$ abajo
signo $\substack{+1\\-1}$

$$\Rightarrow \pm \lambda^{1/2}_{sab}\lambda^{1/2}_{s12} = 2st_{\substack{1\\2}} + s^2 - sm_a^2 - sm_b^2 - sm_1^2 - sm_2^2 + (m_a^2 - m_b^2)(m_1^2 - m_2^2)$$

$\downarrow \cdot 2$

$$\pm 2\lambda^{1/2}_{sab}\lambda^{1/2}_{s12} = 4st_{\substack{1\\2}} + \overbrace{s^2 - 2sm_a^2 - 2sm_b^2 + s^2}^{2s^2} - 2sm_1^2 - 2sm_2^2 + 2(m_a^2 - m_b^2)(m_1^2 - m_2^2)$$

$$= 4st_{\substack{1\\2}} + s^2 - 2sm_a^2 - 2sm_b^2 \overbrace{+ (m_a^4 + m_b^4 - 2m_a^2m_b^2) - (m_a^4 + m_b^4 - 2m_a^2m_b^2)}^{0}$$
$$+ s^2 - 2sm_1^2 - 2sm_2^2 \underbrace{+ (m_1^4 + m_2^4 - 2m_1^2m_2^2) - (m_1^4 + m_2^4 - 2m_1^2m_2^2)}_{0}$$
$$+ 2(m_a^2 - m_b^2)(m_1^2 - m_2^2)$$

$$= 4st_{\substack{1\\2}} + \lambda_{sab} + \lambda_{s12} - (m_a^2 - m_b^2)^2 - (m_1^2 - m_2^2)^2 + 2(m_a^2 - m_b^2)(m_1^2 - m_2^2)$$
$$= 4st_{\substack{1\\2}} + \lambda_{sab} + \lambda_{s12} - \left[(m_a^2 - m_b^2) - 2(m_a^2 - m_b^2)(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2\right]$$
$$= 4st_{\substack{1\\2}} + \lambda_{sab} + \lambda_{s12} - \left[(m_a^2 - m_b^2) - (m_1^2 - m_2^2)\right]^2$$

$$\hookrightarrow 4st_{\substack{1\\2}} = \left[m_a^2 - m_b^2 - m_1^2 + m_2^2\right]^2 - \left(\lambda_{sab} + \lambda_{s12} \mp 2\lambda^{1/2}_{sab}\lambda^{1/2}_{s12}\right)$$
$$= \qquad '' \qquad - \left(\lambda^{1/2}_{sab} \mp \lambda^{1/2}_{s12}\right)^2$$

$$t_{\substack{1\\2}} = \frac{1}{4s}\left\{(m_a^2 - m_b^2 - m_1^2 - m_2^2)^2 - \left[\lambda^{1/2}(s, m_a^2, m_b^2) \mp \lambda^{1/2}(s, m_1^2, m_2^2)\right]^2\right\}$$ ✓

$\cos\theta_1 = +1 \rightarrow \theta_1 = 0 \rightarrow t_1$
$\cos\theta_2 = -1 \rightarrow \theta_2 = \pi \rightarrow t_2$

El segundo término con las λs resta $\rightarrow$ cuanto menor sea, mayor es t
$\hookrightarrow (\lambda^{1/2}_{sab} - \lambda^{1/2}_{s12})^2 < (\lambda^{1/2}_{sab} + \lambda^{1/2}_{s12})^2$

Por tanto t_1 (asociado a $\theta_1 = 0$, $\cos\theta_1 = 1$, pero signo $-$ entre las λs) es mayor que t_2 $\rightarrow t_1 > t_2 \rightarrow$ Cambio 1 y 2 por el signo entre las λs : $t_1 \equiv t_-$; $t_2 \equiv t_+$

Por tanto: $t_+ < t < t_-$ (rango de variación de t), donde

$$t_{\substack{1\\2}} = t_{\mp} = \frac{1}{4s}\left\{\ldots - \left[\lambda_{sab} \mp \lambda_{s12}\right]^2\right\}$$

$$\Downarrow$$

$$t_{\pm} = \frac{1}{4s}\left\{(m_a^2 - m_b^2 - m_1^2 + m_2^2)^2 - \left[\lambda^{1/2}(s, m_a^2, m_b^2) - \lambda^{1/2}(s, m_1^2, m_2^2)\right]^2\right\}$$ ✓

d) • E:

$\boxed{\text{En S. CM}}$: (por comparar) → repetido, ya hecho en b)

$(E_a, \vec{p}) + (E_b, -\vec{p}) = p_a + p_b$

$s = (p_a + p_b)^2 \overset{i}{\Rightarrow} m_a^2 + m_b^2 + 2E_aE_b - 2\vec{p}(-\vec{p}) = m_a^2 + m_b^2 - 2E_aE_b + 2\vec{p}^2$ ⋮

$= (E_a + E_b)^2 \rightarrow \sqrt{s} = E_a + E_b \rightarrow E_b = \sqrt{s} - E_a$

$\hookrightarrow E_b^2 = s + E_a^2 - 2\sqrt{s}\,E_a$

$\downarrow$

$m_b^2 + \vec{p}^2 = s + m_a^2 + \vec{p}^2 - 2\sqrt{s}\,E_a$

$\hookrightarrow E_a^{CM} = \dfrac{s + m_a^2 - m_b^2}{2\sqrt{s}} \longleftrightarrow E_b^{CM} = \dfrac{s + m_b^2 - m_a^2}{2\sqrt{s}}$

por simetría a ⟺ b o por $E_b = \sqrt{s} - E_a$

$\boxed{\text{En S. Lab}}$:

$(E_a, \vec{p}) + (m_b, \vec{0}) = p_a + p_b = \sqrt{s}$

$\downarrow$

$m_a^2 + m_b^2 + 2E_am_b = s \rightarrow E_a^L = \dfrac{s - m_a^2 - m_b^2}{2m_b}$ //

$\rightarrow E_b^L = m_b$

• $|\vec{p}|$:

$\boxed{\text{En S. c.M}}$:

$\vec{p}^2 = E_a^2 - m_a^2 = \dfrac{1}{4s}\left(s^2 + (m_a^2 - m_b^2)^2 + 2s(m_a^2 - m_b^2) - 4sm_a^2\right)$

$= \dfrac{1}{4s}\left(s^2 + m_a^4 + m_b^4 - 2m_a^2m_b^2 - 2sm_a^2 - 2sm_b^2\right)$

$\hookrightarrow |\vec{p}^{CM}| = \dfrac{1}{2\sqrt{s}} \cdot \lambda^{1/2}(s, m_a^2, m_b^2)$ //

→ Dirección es arbitraria

→ $a \xrightarrow[\vec{p}^{CM}]{} \cdot \xleftarrow[-\vec{p}^{CM}]{} b \quad \longmapsto x$

$\boxed{\text{En S. Lab}}$:

$\vec{p}_b = \vec{0}$

$\vec{p}_a = \vec{p} \Rightarrow \vec{p}^2 = E_a^2 - m_a^2 = \dfrac{s^2 + (m_a^2 + m_b^2)^2 - 2s(m_a^2 + m_b^2) - 4m_b^2m_a^2}{4m_b^2}$

$= \dfrac{s^2 + m_a^4 + m_b^4 + 2m_a^2m_b^2 - 2sm_a^2 - 2sm_b^2 - 4m_a^2m_b^2}{4m_b^2} = \dfrac{\lambda(s, m_a^2, m_b^2)}{4m_b^2}$

$|\vec{p}^L| = \dfrac{\lambda^{1/2}(s, m_a^2, m_b^2)}{2m_b}$ //

Dirección arbitraria

$a \xrightarrow{\vec{p}^L} \cdot_{b} \qquad \longmapsto x$

Para las partículas 1 y 2:

En $\boxed{\text{S. C. M}}$ → (ver b)

$E_1^{CM} = \dfrac{s + m_1^2 - m_2^2}{2\sqrt{s}}$ //

$\hookrightarrow$ cambiar

$\begin{matrix} a \\ b \end{matrix} \rightarrow \begin{matrix} 1 \\ 2 \end{matrix}$

(simetría CM)

$E_2^{CM} = \dfrac{s + m_2^2 - m_1^2}{2\sqrt{s}}$ //

$\Bigg\}\ |\vec{p}^{\,\prime CM}| = \dfrac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_1^2, m_2^2)$ //

$\begin{matrix} a \Leftrightarrow 1 \\ b \Leftrightarrow 2 \end{matrix}$

En $\boxed{\text{S. Lab}}$

$$u = (p_a - p_2)^2 \overset{p_a+p_b = p_1+p_2}{=} (p_1 - p_b)^2 = m_1^2 + m_b^2 - 2E_1^L m_b + 0$$

$$\to E_1^L = \frac{m_1^2 + m_b^2 - u}{2m_b}$$

$$E_2^L = E_a^L + m_b - E_1^L = \frac{s - m_a^2 - m_b^2 + 2m_b^2 - m_1^2 - m_b^2 + u}{2m_b}$$

$$= \frac{s - m_a^2 - m_1^2 + u + t - t}{2m_b} = \frac{m_b^2 + m_2^2 - t}{2m_b}$$

O bien:

$$t = (p_a - p_1)^2 = (p_2 - p_b)^2 = m_2^2 + m_b^2 - 2E_2^L m_b$$

$$|\vec{p}_1^{\,L}|^2 = (E_1^L)^2 - m_1^2 = \frac{m_1^4 + m_b^4 + u^4 + 2m_1^2 m_b^2 - 2m_1^2 u - 2m_b^2 u - 4m_1^2 m_b^2}{4m_b^2}$$

$$|\vec{p}_1^{\,L}| = \frac{\lambda^{1/2}(u, m_b^2, m_1^2)}{2m_b}$$

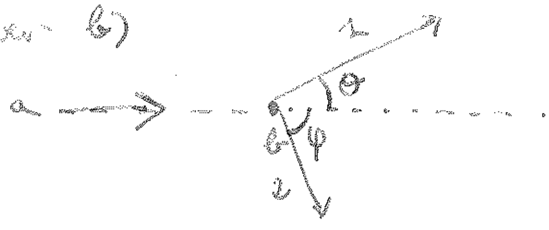
$$|\vec{p}_2^{\,L}|^2 = E_2^{L2} - m_2^2 = \frac{m_b^4 + m_2^4 + t^4 + 2m_b^2 m_2^2 - 2m_b^2 t - 2m_2^2 t - 4m_2^2 m_b^2}{4m_b^2}$$

$$\to |\vec{p}_2^{\,L}| = \frac{\lambda^{1/2}(t, m_b^2, m_2^2)}{2m_b}$$

Ángulos entre partículas:

En $\boxed{\text{SCM}}$ → ver b)

En $\boxed{\text{SLAB}}$:



$$t = (p_a - p_1)^2 = m_a^2 + m_1^2 - 2E_a^L E_1^L + 2|\vec{p}_a^{\,L}|\cdot|\vec{p}_1^{\,L}|\cdot\cos\theta^L$$

$$u = (p_a - p_2)^2 = m_a^2 + m_2^2 - 2E_a^L E_2^L + 2|\vec{p}_a^{\,L}|\cdot|\vec{p}_2^{\,L}|\cdot\cos\varphi^L$$

$$\to \cos\theta^L = \frac{t + 2E_a^L E_1^L - m_a^2 - m_1^2}{2|\vec{p}_a^{\,L}||\vec{p}_1^{\,L}|}$$ (idem para φ, no lo hago)

$$= \left[t + \frac{2}{4m_b^2}(s - m_a^2 - m_b^2)\cdot(m_1^2 + m_b^2 - u) - m_1^2\right] / (2|\vec{p}_a^{\,L}||\vec{p}_1^{\,L}|) \quad \cdot 2m_b$$

$$= \left[2m_b^2(t - m_1^2) + s m_1^2 + s m_b^2 - su - m_a^2 m_1^2 - m_a^2 m_b^2 + m_a^2 u - m_b^2 m_1^2 - m_b^4 + m_b^2 u\right] / 2p_a^L p_1^L$$

$$= \left[2m_b^2\cdot t + s(m_b^2 + m_1^2) + u(m_a^2 + m_b^2) - su - 3m_b^2 m_1^2 - m_a^2 m_1^2 - m_a^2 m_b^2 - m_b^4\right] / 4m_b p_a p_1$$

$$= \left[m_b^2(2t + \overbrace{s + u}^{-t+\Sigma m_i^2} - 3m_1^2 - m_a^2 - m_b^2) + s(m_1^2 - u) + u m_a^2 - m_a^2 m_1^2\right] / 4m_b p_a p_1$$

$$= \left[m_b^2(t + m_2^2 - 2m_1^2) + s m_1^2 + u m_a^2 - su - m_a^2 m_1^2\right] / 4m_b p_a^L p_1^L$$

→ puedes poner $t(s, u)$ pero no se simplifica mucho más.

e) Partimos del S. Lab

$S^L \; \bullet \xrightarrow{a} \quad \bullet^{b}$ $\overset{\uparrow Y}{\llcorner} \!\to x$ $\quad S' \equiv S_{CM}$ se mueve paralelo a A con velocidad β.

$S' \longrightarrow \beta$

$$\Longrightarrow p'^{\mu} = \Lambda^{\mu}{}_{\nu}\, p^{\nu}$$

$$\begin{pmatrix} E^{CM} \\ p_x^{CM} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E^L \\ p_x^L \end{pmatrix} \qquad p_Y^L = p_Y^{CM} \; ; \; p_x = \vec{p}\cdot\hat{x}$$

Sacamos β con el caso sencillo de p_b y p_a:

$$\begin{pmatrix} E_b^{CM} \\ p_b^{CM} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} m_b \\ 0 \end{pmatrix} \to \begin{matrix} E_b^{CM} = \gamma m_b \\ p_b^{CM} = -\gamma\beta m_b \end{matrix}$$

... mismo eje x, antiparalelos $\to p_b^{CM} \equiv \vec{p}_b^{\,CM}\cdot\hat{x}$; $p_a^{CM} = \vec{p}_a^{\,CM}\cdot\hat{x}$

$$p_b^{CM} = -p_a^{CM} = -(-\gamma\beta E_a^L + \gamma\, p_a^L) = -\gamma\beta m_b \to \beta\gamma(m_b + E_a^L) = \gamma\, p_a^L$$

$$\hookrightarrow \beta = \frac{p_a^L}{m_b + E_a^L}$$

$$\gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1-\left(\frac{p_a^L}{E_a^L + m_b}\right)^2} = \frac{(E_a^L + m_b)^2}{(E_a^L)^2 + m_b^2 + 2E_a^L m_b - (p_a^L)^2}$$

$$= \frac{(E_a^L + m_b)^2}{m_a^2 + m_b^2 + 2E_a^L m_b} \xrightarrow{\;s = (p_a + p_b) = m_a^2 + m_b^2 + 2E_a^L m_b\;} = \frac{(E_a^L + m_b)^2}{s}$$

$$\hookrightarrow \gamma = \frac{E_a^L + m_b}{\sqrt{s}}$$

$$\hookrightarrow \gamma\beta = \frac{p_a^L}{E_a^L + m_b}\cdot\frac{E_a^L + m_b}{\sqrt{s}} = \frac{p_a^L}{\sqrt{s}}$$

$$E_b^{CM} = \gamma m_b = \frac{E_a^L + m_b}{\sqrt{s}}\, m_b = \frac{m_b}{\sqrt{s}}\cdot\frac{s - m_a^2 - m_b^2 + 2m_b^2}{2m_b} = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}$$

$$p_b^{CM} = -\gamma\beta m_b = -m_b\frac{p_a^L}{\sqrt{s}} = -\frac{m_b}{\sqrt{s}}\cdot\frac{\lambda^{1/2}(s, m_a^2, m_b^2)}{2m_b} = -\frac{\lambda^{1/2}(s, m_a^2, m_b^2)}{2\sqrt{s}}$$

$$E_a^{CM} = \gamma E_a^L - \gamma\beta\, p_a^L = \frac{E_a^L + m_b}{\sqrt{s}}\, E_a^L - \frac{(p_a^L)^2}{\sqrt{s}} = \frac{m_a^2 + m_b E_a^L}{\sqrt{s}}$$

$$= \frac{2m_a^2 + s - m_a^2 - m_b^2}{2\sqrt{s}} = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}$$

$$p_a^{CM} = -p_b^{CM}$$

~~$E_1^{CM} = \frac{E_a^L + m_b}{\sqrt{s}} E_1^L - \frac{(|\vec{p}_1|^L \overset{\cos\theta}{\cdot} p_a^L)}{\sqrt{s}} \overset{\leftarrow p_1\sin\theta}{=} \frac{(s - m_a^2 + m_b^2)\cdot(m_1^2 + m_b^2 - u)}{2m_b\sqrt{s}\cdot 2m_b} - \frac{\lambda^{1/2}(u, m_b^2, m_1^2)}{2m_b}\cdot\frac{\lambda^{1/2}(t, m_b^2, m_2^2)}{2m_b}$~~

~~→ No lo hago, es más largo, descomponer en $\cos\theta^L$, $\sin\theta^L$ ...~~

↓ sig. página

$$E_1^{CM} = \gamma E_1^L - \gamma\beta(p_1^L)_x = \gamma E_1^L - \gamma\beta|\vec{p}_1^L|\cdot\cos\theta^L$$

$$= \frac{E_a^L + m_b}{\sqrt{s}} E_1^L - \frac{|\vec{p}_a^L|}{\sqrt{s}}|\vec{p}_1|\cdot\cos\theta$$

$$= \frac{1}{2\sqrt{s}}\left(2E_a^L E_1^L + 2m_b E_1^L - t - 2E_a^L E_1^L + m_a^2 + m_1^2\right)$$

$$= \frac{1}{2\sqrt{s}}\left(2m_b E_1^L - t + m_a^2 + m_1^2\right) = \frac{1}{2\sqrt{s}}\left(m_1^2 + m_b^2 - u - t + m_a^2 + m_1^2\right)$$

$$= \frac{1}{2\sqrt{s}}\left(m_1^2 \underbrace{+ m_2^2 - m_2^2}_{0} + m_a^2 + m_b^2 - u - t + m_1^2\right) \overset{s+t+u=\sum_i m_i^2}{=} \frac{(s + m_1^2 - m_2^2)}{2\sqrt{s}}$$

$E_2^{CM} = \gamma E_2^L - \gamma\beta(p_2^L)_x \rightarrow$ + complicado, mejor:

$$\Rightarrow E_a^{CM} + E_b^{CM} - E_1^{CM} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}$$

Idem con $(p_{1,2}^{CM})_{x,y}$ por componentes → no lo hago por ser largo.

f) ... ~~Largo~~ ...

$$p_{1x}^{CM} = |\vec{p}_1^{CM}|\cdot\cos\theta_1^{CM} = -\gamma\beta E_1^L + \gamma|\vec{p}_1^L|\cdot\cos\theta_1^L$$

$$p_{1y}^{CM} = |\vec{p}_1^{CM}|\cdot\sin\theta_1^{CM} = |\vec{p}_1^L|\cdot\sin\theta_1^L$$

$$\hookrightarrow \tan\theta_1^L = \frac{|\vec{p}_1^{CM}|\cdot\sin\theta_1^{CM}}{\frac{|\vec{p}_1^{CM}|\cos\theta_1^{CM}}{\gamma} + \beta E_1^L} = \frac{|\vec{p}_1^{CM}|\cdot\sin\theta_1^{CM}}{|\vec{p}_1^{CM}|\cdot\sqrt{s}\cdot\cos\theta_1^{CM} + E_1^L\cdot|\vec{p}_a^L|}\cdot(m_b + E_a^L)$$

$= \ldots$

Mejor:

$$(p^\mu)^{CM} = \overbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\Lambda^\mu{}_\nu} (p^\nu)^L \quad |\cdot\Lambda^{-1} \rightarrow \Lambda^{-1} = \Lambda(-\beta)$$

$$p^L = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot p^{CM} \quad ; \quad p \equiv \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$p_{1x}^L = |\vec{p}_1|^L\cdot\cos\theta_1^L = \gamma\beta E_1^{CM} + \gamma p_{1x}^{CM} = \gamma\beta E_1^{CM} + \gamma|\vec{p}_1|^{CM}\cos\theta^{CM}$$

$$p_{1x}^L = |\vec{p}_1|^L \sin\theta_1^L = p_{1y}^{CM} = |\vec{p}_1|^{CM}\cdot\sin\theta_{CM}$$

$$\hookrightarrow \frac{p_{1x}^L}{p_{1x}^L} = \tan\theta_1^L = \frac{|\vec{p}_1|^{CM}\sin\theta_{CM}}{\gamma\beta E_1^{CM} + \gamma|\vec{p}_1^{CM}|\cos\theta^{CM}}$$

$$= \frac{\sin\theta_{CM}}{\gamma\beta\frac{E_1^{CM}}{|\vec{p}_1^{\,CM}|} + \gamma\cos\theta^{CM}} = \frac{\sin\theta_{CM}}{\frac{|\vec{p}_b|^{CM}}{m_b}\cdot\frac{E_1^{CM}}{|\vec{p}_1^{\,CM}|} + \frac{E_b^{CM}}{m_b}\cos\theta_{CM}}$$

$$= \frac{\sin\theta_{CM}}{\frac{\lambda^{1/2}(s, m_a^2, m_b^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)}\cdot\frac{s+m_1^2-m_2^2}{m_b\cdot 2\sqrt{s}} + \frac{s-m_a^2+m_b^2}{m_b\, 2\sqrt{s}}\cos\theta_{CM}}$$

$\downarrow$ si $\sqrt{s} \gg m_i$ $\quad \swarrow \quad \lambda^{1/2}(s, m_i^2, m_j^2) \simeq s$

$$\approx \frac{\sin\theta_{CM}}{\frac{s}{s}\cdot\frac{s}{m_b 2\sqrt{s}} + \frac{s}{2m_b\sqrt{s}}\cos\theta_{CM}} = \frac{2m_b}{\sqrt{s}}\,\frac{\sin\theta^{CM}}{1+\cos\theta^{CM}} =$$

$$\tan\theta_1^L = \frac{2m_b}{\sqrt{s}}\,\frac{2\sin\theta_{CM}/2\cdot\cos\theta_{CM}/2}{\cos^2\theta_{CM}/2} = \frac{2m_b}{\sqrt{s}}\tan\frac{\theta_{CM}}{2}$$

La relación es exacta si $m_i = m_j = m$

$\Rightarrow \lambda^{1/2}(s, m_a^2, m_b^2) = \lambda^{1/2}(s, m_1^2, m_2^2)$ $\hookleftarrow$

$\downarrow$

$$\tan\theta_1^L = \frac{\sin\theta_{CM}}{\frac{1}{1}\cdot\frac{s+m^2-m^2}{m\cdot 2\sqrt{s}} + \frac{s+0}{m\cdot 2\sqrt{s}}\cos\theta_{CM}} = \frac{2m}{\sqrt{s}}\tan\frac{\theta^{CM}}{2}$$

④

Se pueden utilizar las fórmulas del ejercicio 3 en SCM con $m_b = 0$, $E_a = m_a$, $\vec{p} = 0$.

Si lo volvemos a derivar para este caso particular:

$$(m_a, 0) = (E_1, \vec{p}\,') + (E_2, -\vec{p}\,')$$

$()^2 \downarrow$

$$m_a^2 = m_1^2 + m_2^2 + 2E_1E_2 + 2(\vec{p}\,')^2$$

$\rightarrow E_2 = m_a - E_1$

$$\hookrightarrow m_a^2 = m_1^2 + m_2^2 + 2E_1(m_a - E_1) + 2\vec{p}\,'^2$$

$$m_a^2 - m_1^2 - m_2^2 = 2E_1 m_a - 2E_1^2 + 2\vec{p}\,'^2 = 2E_1 m_a - 2m_1^2$$

$$\hookrightarrow E_1 = \frac{m_a^2 + m_1^2 - m_2^2}{2m_a} /\!/$$

$$\hookrightarrow E_2 = m_a - E_1 = \frac{m_a^2 + m_2^2 - m_1^2}{2m_a} /\!/$$

$-4m_a^2m_1^2$

$$\hookrightarrow |\vec{p}\,'|^2 = E_1^2 - m_1^2 = \frac{m_a^4 + m_1^4 + m_2^4 + 2m_a^2m_1^2 - 2m_a^2m_2^2 - 2m_1^2m_2^2}{4m_a^2}$$

$$= \frac{\lambda(m_a^2, m_1^2, m_2^2)}{4m_a^2}$$

$$|\vec{p}\,'| = \frac{\lambda^{1/2}(m_a^2, m_1^2, m_2^2)}{2m_a}$$

→ El ángulo θ_{CM} es arbitrario al no haber dirección privilegiada al estar A en reposo.

→ Coincide con ejc. 3d) para SCM con $m_b = 0 \Rightarrow s = m_a^2$

⑤ $A \rightarrow 1 + 2 + 3$

$t_i = (P - p_i)^2$; $s_{jk} = (p_j + p_k)^2$

a) $$\sum_{i=1}^{3} t_i = \sum_{i=1}^{3} (P - p_i)^2 = \sum_{i=1}^{3}\left(M_A^2 + m_i^2 - 2M_A E_i + 2\vec{P}\cdot\vec{p_i}\right)$$

$$= \sum_{i=1}^{3}(M_A^2 + m_i^2) - 2E_A \underbrace{\sum_{i=1}^{3} E_i}_{=E_A} + 2\vec{P}\cdot\underbrace{\sum_{i=1}^{3}\vec{p_i}}_{=\vec{P}}$$ → x conservac. de $E, \vec{p}$

$$= \sum_{i=1}^{3} m_i^2 + 3M_A^2 - 2\underbrace{(E_A^2 - \vec{P}^2)}_{M_A^2} = m_1^2 + m_2^2 + m_3^2 + M_A^2 /\!/ \checkmark$$

b)

$$(M_A, \vec{0}) = \sum_{i=1}^{3} (E_i, \vec{p}_i)$$

$$t_1 = (P-p_1)^2 = m_A^2 + m_1^2 - 2E_1 m_a \equiv s_{23}$$

$$\hookrightarrow E_1 = \frac{m_A^2 + m_1^2 - s_{23}}{2m_A}$$

$$t_2 = (P-p_2)^2 = m_A^2 + m_2^2 - 2E_2 m_a \equiv s_{13}$$

$$\hookrightarrow E_2 = \frac{m_A^2 + m_2^2 - s_{13}}{2 m_a}$$

$$\hookrightarrow E_3 = M_A - \underset{\underset{s_{13}}{\parallel}}{\underset{t_2}{E_1}} - \underset{\underset{s_{23}}{\parallel}}{\underset{t_1}{E_2}} = (2m_a^2 - 2m_a^2 - m_1^2 - m_2^2 + s_{13} + s_{23})/2m_a$$

$$= \frac{s_{13} + s_{23} - m_1^2 - m_2^2}{2m_a} \overset{\pm t_3 = \pm s_{12}}{=} \frac{m_A^2 + m_3^2 - s_{12}}{2m_a}$$

Obsv: $t_3 = (P-p_3)^2 = s_{12} = m_A^2 + m_3^2 - 2E_3 m_A$

$$|\vec{p}_1|^2 = E_1^2 - m_1^2 = \frac{(m_A^4 + m_1^4 + s_{23}^2 + 2m_A^2 m_1^2 - 2m_A^2 s_{23} - 2m_1^2 s_{23} - 4m_1^2}{4 m_a^2}$$

$$\hookrightarrow |\vec{p}_1| = \frac{\lambda^{1/2}(m_A^2, m_1^2, \overset{t_1}{\underset{}{\overset{\parallel}{s_{23}}}})}{2m_A}$$

$$\hookrightarrow |\vec{p}_2| \overset{\text{idem } 1 \leftrightarrow 2}{=} \frac{\lambda^{1/2}(m_A^2, m_2^2, \overset{t_2}{\overset{\parallel}{s_{13}}})}{2m_A}$$

$$\hookrightarrow |\vec{p}_3|^2 = [s_{13}^2 + s_{23}^2 + m_1^4 + m_2^4 + 2s_{13}s_{23} - 2s_{13}m_1^2 - 2s_{13}m_2^2 - 2s_{23}m_1^2$$
$$- 2s_{23}m_2^2 + 2m_1^2 m_2^2 - 4m_a^2 m_3^2]/4m_a^2$$

o bien

$$|\vec{p}_3| = \frac{\lambda^{1/2}(M_a^2, m_3^2, \overset{t_3}{\overset{\parallel}{s_{12}}})}{2m_A}, \text{ con } t_3 = M_A^2 + \sum_{i=1}^{3} m_i^2 - (t_1 + t_2)$$

c) s_{13}, s_{23} independientes (2 grados libertad)

$$s_{13} = (P-p_2)^2 = m_A^2 + m_2^2 - 2E_2 m_A$$

$s_{13}(E_2) \to$ El máximo de s_{13} se da cuando E_2 es mínimo, es decir, cuando 2 está en reposo: $E_2 = m_2$

$$s_{13} \leq (s_{13})_{MAX} = m_A^2 + m_2^2 - 2m_2 m_A = (m_A - m_2)^2$$

Por otro lado:

$s_{13} = (p_1+p_3)^2 = m_1^2 + m_3^2 + 2E_1E_3 - 2\vec{p_1}\vec{p_3}$

Es intuitivo ver (demuestro) que s_{13} es mínimo cuando $\vec{p_1} = \vec{p_3}$

↳ no necesariamente $= 0$.

$E_1E_3 \underset{\wedge}{-} \vec{p_1}\vec{p_3}$ es mínimo cuando $\vec{p_1}$ es paralelo a $\vec{p_3}$

↳ $E_1E_3 - p_1p_3 = \gamma_1\gamma_3 m_1m_3 - \gamma_1\gamma_3 m_1m_3\beta_1\beta_3$

$= \gamma_1\gamma_3 m_1m_3(1-\beta_1\beta_3) = \frac{\gamma_3 m_1m_3}{\sqrt{1-\beta_1^2}}(1-\beta_1\beta_3)$

↳ Busco el mínimo variando γ_1 con γ_3 constante:

$$\frac{\partial}{\partial\beta_1}(E_1E_3 - p_1p_3) = \gamma_3 m_1m_3\left(\frac{(1-\beta_1\beta_3)}{(1-\beta_1^2)^{3/2}}\cdot\left(-\frac{1}{2}\right)\cdot(-2\beta_1) - \frac{\beta_3}{\sqrt{1-\beta_1^2}}\right) = 0$$

↳ $\gamma_3\left[\beta_1(1-\beta_1\beta_3) - \beta_3(1-\beta_1^2)\right] = 0 = (\beta_1-\beta_3)\gamma_3$

Idem con $\frac{\partial}{\partial\beta_3} \rightarrow (\beta_3-\beta_1)\gamma_1 = 0$ ↙ unimos condiciones

↳ $(\beta_1-\beta_3)^2\gamma_3^2 - (\beta_3-\beta_1)^2\gamma_1^2 = (\beta_1-\beta_3)^2(\gamma_3^2-\gamma_1^2) = 0$

↳ $(\beta_1-\beta_3)^2\left(\frac{1}{1-\beta_3^2} - \frac{1}{1-\beta_1^2}\right) = \frac{(\beta_1-\beta_3)^2(1-\beta_1^2-(1-\beta_3^2))}{(1-\beta_3^2)(1-\beta_1^2)} = 0$

↳ $(\beta_1-\beta_3)^2(\beta_3^2-\beta_1^2) = -(\beta_1-\beta_3)^2(\beta_3+\beta_1) = 0$

↳ Condición $\beta \equiv \beta_1 = \beta_3$, $\gamma \equiv \gamma_1 = \gamma_3$, además de que sean paralelos: $\vec{\beta_1} = \vec{\beta_3}$

$\Rightarrow (E_1E_3 - \vec{p_1}\vec{p_3})_{mín} = \gamma^2 m_1m_3 - \gamma^2\beta^2 m_1m_3 = m_1m_3\overbrace{\gamma^2(1-\beta^2)}^{=1}$

$= m_1m_3$

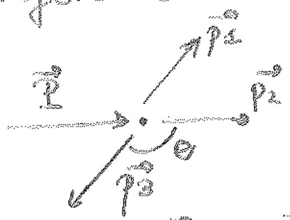
$(s_{13})_{mín} = m_1^2 + m_3^2 + 2m_1m_3 = (m_1+m_3)^2$

$\Rightarrow (m_1+m_3)^2 \leq s_{13} \leq (m_4-m_2)^2$ //

Idem para s_{23} cambiando $1 \leftrightarrow 2$

$(m_2+m_3)^2 \leq s_{23} \leq (m_4-m_1)^2$

Para s_{23}, como es un invariante, lo calcularemos en otro sistema de referencia más conveniente, en el que $\vec{p_1}+\vec{p_3}=\vec{P}-\vec{p_2}=0$



→ Aquí podemos poner una cota a $\vec{p_2}\cdot\vec{p_3}$ ($\theta<\pi$).

Calculamos $\vec{p_2}$ y $\vec{p_3}$ y E_2 y E_3 en este nuevo sistema:

$$s_{13}=(P-p_2)^2 = m_A^2+m_2^2-2E_AE_2+2(\vec{p_2})^2 = m_A^2-m_2^2+\overbrace{2m_2^2+2\vec{p_2}^{\,2}}^{2E_2^2}-2E_AE_2$$

$$= m_A^2-m_2^2-2E_2(E_A-E_2)$$

Como $(P-p_2)^2=(E_A-E_2,0)^2=(E_A-E_2)^2 \rightarrow \sqrt{s_{13}}=E_A-E_2$

$$s_{13}=m_A^2-m_2^2-2E_2\sqrt{s_{13}}$$

↳ $E_2=\dfrac{m_A^2-m_2^2-s_{13}}{2\sqrt{s_{13}}}$

↳ $|\vec{p_2}|^2 = E_2^2-m_2^2 = \dfrac{m_A^4+2m_2^2s_{13}+m_2^4+s_{13}^2-2m_A^2m_2^2-2m_A^2s_{13}-4m_2^2s_{13}}{4s_{13}}$

↳ $|\vec{p_2}| = \dfrac{\lambda^{1/2}(m_A^2,s_{13},m_2^2)}{2\sqrt{s_{13}}}$ //

Idem para $\vec{p_3}$:

$$s_{13}=(p_1+p_3)^2 = m_1^2+m_3^2+2E_1E_3-2\vec{p_3}(-\vec{p_3}) = m_1^2+m_3^2+2E_1E_3+2(\vec{p_3})^2$$

$$= m_1^2-m_3^2+2m_3^2+2(\vec{p_3})^2+2E_1E_3 = m_1^2-m_3^2+2E_3(E_1+E_3) \overset{\sqrt{s_{13}}=E_1+E_3}{=}$$

$$= m_1^2-m_3^2+2E_3\sqrt{s_{13}}$$

↳ $E_3=\dfrac{m_3^2+s_{13}-m_1^2}{2\sqrt{s_{13}}}$

↳ $|\vec{p_3}|^2 = E_3^2-m_3^2 = (m_3^4+s_{13}^2+m_1^4+2m_3^2s_{13}-2m_1^2m_3^2-2s_{13}m_1^2-4s_{13}m_3^2)/4s_{13}$

$|\vec{p_3}| = \lambda^{1/2}(s_{13},m_1^2,m_3^2)/2\sqrt{s_{13}}$ //

→ $s_{23}=(p_2+p_3)^2 = m_2^2+m_3^2+2E_2E_3-2\vec{p_2}\cdot\vec{p_3} = m_2^2+m_3^2+2E_2E_3-2|\vec{p_2}||\vec{p_3}|\cos\theta$

⇒ $s_{23}(m_i,s_{13},\theta)$ ya que $E_2, E_3, \vec{p_2}, \vec{p_3}$ sólo dependen de masas y de s_{13}.
Como hemos elegido s_{13} independiente, el rango de s_{23} sólo dependerá de θ para s_{13} fijo.

$s_{23}\Big(\underset{\overset{\shortparallel}{s_{23}^-}}{\text{MAX}}\Big) = s_{23}(\theta=\pi)$; $s_{23}(\text{MIN}) = s_{23}(\theta=0) \equiv s_{23}^+$ → se ve a primera vista → signo del coseno

$$s_{23}^{\pm} = m_2^2+m_3^2+2E_2E_3 \mp 2|\vec{p_2}|\cdot|\vec{p_3}| = m_2^2+m_3^2+2$$

$$+\frac{1}{4s_{13}}\cdot\left\{(m_A^2-m_2^2-s_{13})(m_3^2+s_{13}-m_1^2) \mp 2\lambda^{1/2}(m_A^2,s_{13},m_2^2)\lambda^{1/2}(s_{13},m_1^2,m_3^2)\right\}$$

$$=\frac{1}{4s_{13}}\Big\{4s_{13}m_2^2+4s_{13}m_3^2+2m_A^2m_3^2+2m_A^2s_{13}-2m_A^2m_1^2-2m_2^2m_3^2-2m_2^2s_{13}$$
$$+2m_2^2m_1^2-2s_{13}m_3^2-2s_{13}^2+2s_{13}m_1^2 \mp 2\lambda^{1/2}_{A\,13\,2}\lambda^{1/2}_{13\,1\,3}\Big\}$$

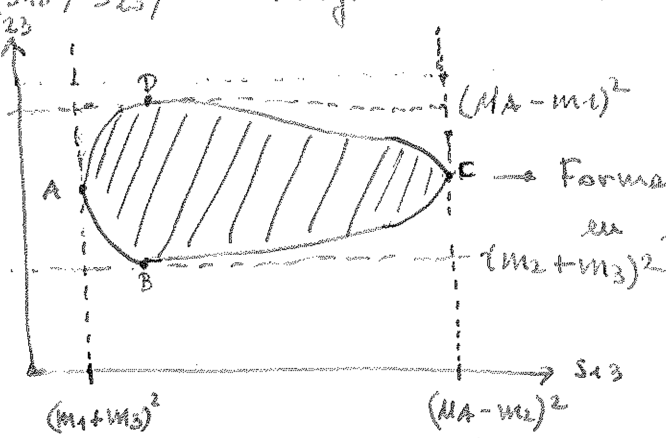
$$=\frac{1}{4s_{13}}\Big\{-(s_{13}^2+m_A^4+m_2^4-2s_{13}m_2^2-2s_{13}m_A^2-2m_A^2m_2^2)$$
$$-(s_{13}^2+m_1^4+m_3^4-2s_{13}m_3^2-2s_{13}m_1^2-2m_1^2m_3^2) \mp 2\lambda^{1/2}\lambda^{1/2}$$
$$+m_A^4+m_2^4+m_1^4+m_3^4-2m_A^2m_2^2-2m_1^2m_3^2$$
$$+2m_A^2m_3^2-2m_A^2m_1^2-2m_2^2m_3^2+2m_1^2m_2^2\Big\}$$

$$= \frac{1}{4s_{13}}\Big\{ -\lambda(m_A^2, s_{13}, m_2^2) - \lambda(s_{13}, m_1^2, m_3^2) \mp 2\lambda^{1/2}_{A12}\lambda^{1/2}_{1313}$$
$$+ m_A^4 + m_1^4 + m_2^4 + m_3^4 + 2m_A^2m_3^2 + 2m_1^2m_2^2 - 2m_A^2m_2^2 - 2m_1^2m_3^2$$
$$- 2m_2^2m_3^2 - 2m_A^2m_1^2\Big\}$$

$$= \frac{1}{4s_{13}}\Big\{(m_A^2 - m_1^2 - m_2^2 + m_3^2)^2 - \left[\lambda^{1/2}(m_A^2, s_{13}, m_2^2) \pm \lambda^{1/2}(s_{13}, m_1^2, m_3^2)\right]^2\Big\}$$

↳ Como s_{23} es un invariante y s_{13} también, esta expresión es válida ∀ S. de referencia aunque hayamos usado uno particular para derivarlo.

d) (s_{13}, s_{23}) → Diagrama de Dalitz



→ Forma arbitraria, pero solución única en los extremos (degeneración)

↓ ya que

$\lambda^{1/2}(s_{13\,min}, m_2^2, m_3^2) = 0$

$\lambda^{1/2}(s_{13\,max}, m_A^2, m_2^2) = 0$

$(m_1+m_3)^2$ ↓ $\vec{p}_1 = \vec{p}_3$ en SCM(A)

$(M_A - m_2)^2$ ↓ $\vec{p}_2 = 0$ en S.C.M(A)

↳ Veámoslo:

$$\lambda((M_A - m_2)^2, m_A^2, m_2^2) = (M_A - m_2)^4 + m_A^4 + m_2^4 - 2(m_A - m_2)^2 m_A^2 - 2(m_A - m_2)m_2^2$$
$$- 2m_A^2m_2^2 = ((M_A - m_2)^2 - m_A^2 - m_2^2)^2 - 4m_A^2m_2^2 =$$
$$= (\cancel{M_A^2} + \cancel{m_2^2} - \cancel{m_A^2} - \cancel{m_2^2} - 2m_Am_2)^2 - 4m_A^2m_2^2 = 0$$

Idem con

$$\lambda((m_1+m_3)^2, m_1^2, m_3^2) = ((m_1+m_3)^2 - m_1^2 - m_3^2)^2 - 4m_1^2m_3^2 = 0$$

↳ En esos casos $s_{23}^+ = s_{23}^-$ (degenera en un punto, valor fijo) → lógico porque s_{13} es extremal.
↳ A, C

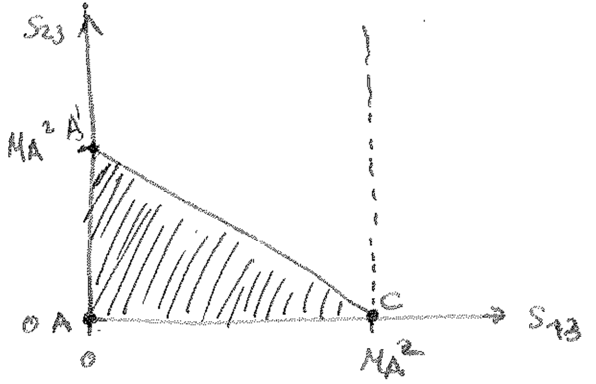
Aparte, hay unos s_{13} para los que $s_{23}^\pm$ es MÍNIMO / MÁXIMO, pero no lo calculo en general (largo... $\frac{\partial s_{23}^\pm}{\partial s_{13}} = 0$) ↳ B, D

También se podría ver si $s_{23}(A) \gtrless s_{23}(C)$, pero lo dejamos mejor para los casos particulares.

Es fácil si usas:

Por último, $(m_2+m_3)^2 \leq s_{23} \leq (M_A - m_1)^2$ → Ya tienes extremos

• $m_1 = m_2 = m_3 = 0$



$$S_{23}^{\pm} = \frac{1}{4S_{13}}\left\{M_A^4 - [(M_A^2 - S_{13}) \pm S_{13}]^2\right\}$$

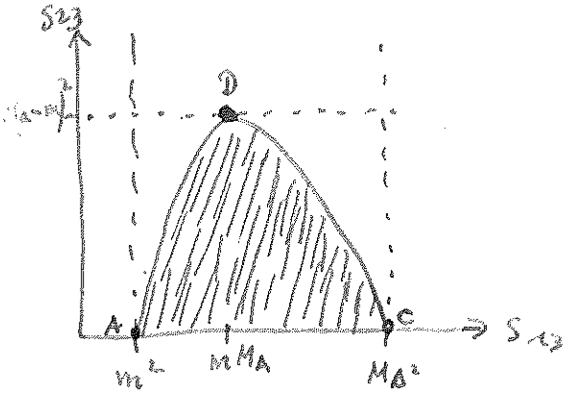
$$= \frac{1}{4S_{13}}\left\{M_A^4 - \left[M_A^2 - \begin{cases}0\\ S_{13}\end{cases}\right]^2\right\}$$

$$= \begin{cases} 0 \\ \frac{1}{4S_{13}}\left\{\cancel{M_A^4} - \cancel{M_A^4} - 4S_{13}^2 + 4S_{13}M_A^2\right\}\end{cases}$$

$$S_{23}^{\pm} = \begin{cases} 0 \\ M_A^2 - S_{13}\end{cases}$$

$S_{23}(S_{13} \to 0) \longrightarrow$ se rompe degeneración (divergencia) → en la página anterior estaba calculado para el caso general pero con $m_i \neq 0$. En ese caso sí es degenerado.

• $m_1 = m$, $m_2 = m_3 = 0$



$$S_{23}^{\pm} = \frac{1}{4S_{13}}\left\{(M_A^2 - m^2)^2 - [(M_A^2 - S_{13}) \pm (S_{13} - m^2)]^2\right\}$$

$$= \frac{1}{4S_{13}}\left\{(M_A^2 - m^2)^2 - \begin{cases}(M_A^2 - m^2)^2 \\ (M_A^2 + m^2 - 2S_{13})^2\end{cases}\right\}$$

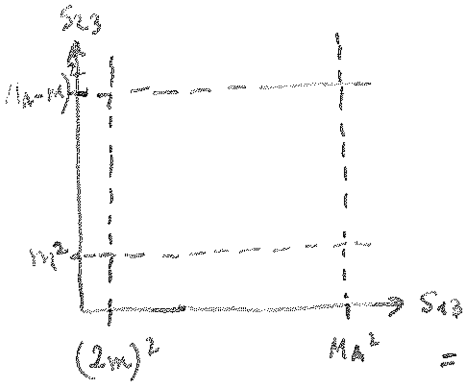
$$= \begin{cases} 0 \\ \frac{1}{4S_{13}}\left(\cancel{M_A^4} + \cancel{m^4} - 2m^2M_A^4 - \cancel{M_A^4} - \cancel{m^4} - 4S_{13}^2 - 2m^2M_A^2 + 4M_A^2S_{13} + 4m^2S_{13}\right)\end{cases}$$

$$= \begin{cases} 0 \quad \to \text{fijo} \\ -S_{13} + m^2 + M_A^2 - \frac{m^2M_A^2}{S_{13}} = (M_A^2 - S_{13}) + m^2\left(1 - \frac{M_A^2}{S_{13}}\right)\end{cases}$$

→ mínimo: $\frac{\partial S_{23}^-}{\partial S_{13}} = -1 + \frac{m^2M_A^2}{S_{13ext}^2} = 0 \to S_{13ext} = mM_A$

$$S_{23}(S_{13ext}) = M_A^2 - mM_A + m^2 - mM_A = (M_A - m)^2$$

• $m_1 = m_3 = m$, $m_2 = 0$



$$S_{23}^{\pm} = \frac{1}{4S_{13}}\left\{(M_A^2 - 2m^2)^2 - \left[(M_A^2 - S_{13}) \pm [(S_{13} - 2m^2)^2 - 4m^4]^{1/2}\right]^2\right\}$$

$$= \frac{1}{4S_{13}}\left\{(M_A^2 - 2m^2)^2 - \left[M_A^2 - S_{13} \pm (S_{13}^2 + \cancel{4m^4} - 4S_{13}m^2 - \cancel{4m^4})^{1/2}\right]^2\right\}$$

$$= \frac{1}{4S_{13}}\left\{(M_A^2 - 2m^2)^2 - \left[M_A^2 - S_{13} \pm \sqrt{S_{13}}\,(S_{13} - 4m^2)^{1/2}\right]^2\right\}$$

$$= \frac{1}{4S_{13}}\left\{\cancel{M_A^4} + 4m^4 - 4m^2M_A^2 - \cancel{M_A^4} - S_{13}^2 - S_{13}(S_{13} - 4m^2) + 2M_A^2S_{13} \mp 2M_A^2\sqrt{S_{13}}\,(S_{13} - 4m^2)^{1/2} \pm 2S_{13}^{3/2}(S_{13} - 4m^2)^{1/2}\right\}$$

$$S_{23}(4m^2) = \frac{1}{16m^2}\left\{4m^4 - 4m^2M_A^2 - 16m^4 + 8M_A^2m^2\right\} = \frac{1}{16m^2}(4m^2M_A^2 - 12m^4)$$

... No sigo → representar por ordenador

⑥

$\vec{X}(t=0)=\vec{X}_0$

a) $\hat{H}=\vec{p}^2/2m = E$ (operadores) $\rightarrow \hat{U}=e^{-iHt}$

$$\mathcal{P}(t,\vec{x}) = |\langle \vec{x}\, |\hat{U}|\vec{x}^0\rangle|^2$$

$$\left[\begin{matrix} x^0 = (0,\vec{x}_0) \\ x^\mu = (t,\vec{x}) \end{matrix}\right]$$

$$\langle \vec{x}\,|U|\vec{x}^0\rangle = \langle \vec{x}|U|\overbrace{\int \frac{d^3\vec{p}}{(2\pi)^3}|p\rangle\langle p|}^{I}\vec{x}_0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\langle \vec{x}|e^{-iHt}|p\rangle\cdot \overbrace{\langle p|\vec{x}_0\rangle}^{e^{i\vec{p}\vec{x}_0}}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\,\langle \vec{x}|\,e^{-i\vec{p}^2/2m}|p\rangle\, e^{i\vec{p}\vec{x}_0} = \int \frac{d^3\vec{p}}{(2\pi)^3}\langle \vec{x}|p\rangle e^{-iE(p)t}\langle p|\vec{x}_0\rangle$$

↳ propio de $|p\rangle$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\vec{p}(\vec{x}-\vec{x}_0)}\cdot e^{-iE(p)t} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-\frac{it}{2m}[\vec{p}^2+\frac{2m}{t}\vec{p}(\vec{x}-\vec{x}_0)]} =$$

↳ $\pm \frac{m^2}{t^2}(\vec{x}-\vec{x}_0)^2$

$$= e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}\int \frac{d^3\vec{p}}{(2\pi)^3}\cdot e^{-\frac{it}{2m}[\vec{p}+\frac{m}{t}(\vec{x}-\vec{x}_0)]^2}$$

↳ Integral de tipo

$$\int_{-\infty}^{+\infty} e^{-(x+b)^2 d^2}\,dx \overset{x+b=y}{=} \int_{-\infty}^{+\infty} e^{-y^2d^2}\,dy \overset{z=yd}{=} \frac{1}{d}\int_{-\infty}^{+\infty} e^{-z^2}\,dz \overset{\text{Gaussiana}}{=} \frac{1}{d}\sqrt{\pi}$$

→ En 3 dimensiones

$$\langle \vec{x}|U|\vec{x}^0\rangle = e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}\frac{1}{(2\pi)^3}\left(\frac{\sqrt{\pi}}{\sqrt{it/2m}}\right)^3 = e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}\left(\sqrt{\frac{2\pi m}{it}\cdot\frac{1}{(2\pi)^2}}\right)^3$$

$$= e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}\left(\frac{m}{2\pi t\, i}\right)^{3/2} \equiv U(t)$$

$$|U(t)|^2 = \left(\frac{m}{2\pi t}\right)^3 = \text{cte} \rightarrow \neq 0$$

→ Viola causalidad para intervalos espaciales

b) Caso relativista:

$$E = +\sqrt{m^2+\vec{p}^2}$$

Coordenadas esféricas

$$U(t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot e^{-i\sqrt{\vec{p}^2+m^2}\,t}\, e^{-i\vec{p}(\vec{x}-\vec{x}_0)} = \frac{1}{2\pi}\int d|\vec{p}|\cdot\vec{p}^2\, e^{-i\sqrt{\vec{p}^2+m^2}\,t}\, e^{-i|\vec{p}||\vec{x}-\vec{x}_0|\cdot d\Omega\cdot\cos\theta}$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty d|\vec{p}|\,|\vec{p}|^2 e^{-i\sqrt{\vec{p}^2+m^2}\,t}\underbrace{\int d\Omega\, e^{i|\vec{p}|\cdot|\vec{x}-\vec{x}_0|\cos\theta}}_{I_\Omega}\quad ; \quad d\Omega = \sin\theta\, d\theta\, d\varphi$$

θ: ángulo entre $\vec{p}$ y $\vec{x}-\vec{x}_0$

$$I_\Omega = \int_0^{2\pi} d\varphi\int_0^{\pi} d\theta\,\sin\theta\, e^{-i|\vec{p}||\vec{x}-\vec{x}_0|\cos\theta} = 2\pi\cdot(-1)\cdot\left.\frac{e^{-i|\vec{p}||\vec{x}-\vec{x}_0|\cos\theta}}{-i|\vec{p}||\vec{x}-\vec{x}_0|}\right|_0^{\pi}$$

$$= \frac{2\pi}{i|\vec{p}||\vec{x}-\vec{x}_0|}\left(e^{i|\vec{p}||\vec{x}-\vec{x}_0|} - e^{-i|\vec{p}||\vec{x}-\vec{x}_0|}\right) = \frac{4\pi}{|\vec{p}||\vec{x}-\vec{x}_0|}\sin(|\vec{p}|\cdot|\vec{x}-\vec{x}_0|)$$

$$u(t) = \frac{1}{(2\pi)^3}\frac{4\pi}{|\vec{x}-\vec{x}_0|}\int d|\vec{p}|\cdot|\vec{p}|\cdot\sin(|\vec{p}|\cdot|\vec{x}-\vec{x}_0|)\cdot e^{-i\sqrt{|\vec{p}|^2+m^2}\,t}$$

$$= \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int d|\vec{p}|\cdot|\vec{p}|\cdot\sin(|\vec{p}|\cdot|\vec{x}-\vec{x}_0|)\,e^{-i\sqrt{|\vec{p}|^2+m^2}\,t}$$

analizamos comportamiento asintótico

$$\sim \exp\left(-m\sqrt{|\vec{x}-\vec{x}_0|^2-t^2}\right) \qquad \text{si } |\vec{x}-\vec{x}_0|^2 \gg t^2$$

↳ space-like ✓

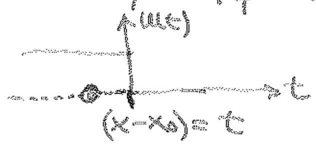
↓

Aproximación matemática, Peskin p. 14 ; método de fase estacionaria

↳ Como se ve, es distinta de cero para intervalos espaciales, viola causalid.

⇒ ES NECESARIA una T^a Cuántica de Campos

↳ queremos escalón cuando pase tipo-luz



⑦

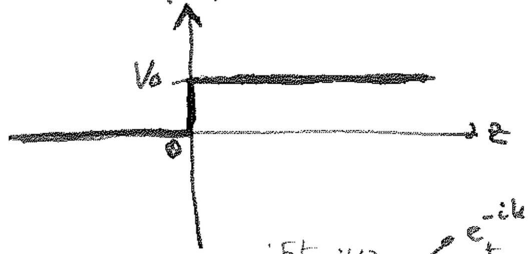
campo de Klein-Gordon: ϕ → M.Q. relativista

Soluciones c.l. ondas planas:

$\phi_{z<0} = a e^{+ipx} + b e^{-ipx}$

$\phi_{z>0} = c\, e^{ip'x}$

$V(z) = V_0\cdot\Theta(z)$



e_{+}^{-ikz} no xq va hacia la dcha. / izda.

$\phi_{inc} = a\cdot e^{-iEt} e^{ikz}$

$\phi_{ref} = b\cdot e^{-iEt} e^{-ikz}$

$\phi_{trans} = c\cdot e^{-iEt} e^{ik'z}$ ··· hacia dcha

$\to (\square+m^2)\phi = 0 \to -E^2+K^2+m^2=0 \to E^2 = K^2+m^2 \to E = +\sqrt{K^2+m^2}$ → $E>0$: incoming beam $E>0$

→ Como $V\neq 0 \to ((\underset{E}{\overset{\shortparallel}{i\partial_t}}-V_0)^2+\vec{\nabla}^2)\phi + m^2\phi = 0$

↳ ya que $E - V_0 = \sqrt{p^2+m^2}$

↳ $(-\partial^2_t + V_0^2 - 2iV_0\partial_t + \nabla^2 - m^2)\phi_t = 0$

↳ $(E-V_0)^2 - K'^2 - m^2 = 0$

$k' = \underset{\text{hacia dcha}}{\pm}\sqrt{(E-V_0)^2-m^2}$

→ Continuid ϕ, $\frac{\partial\phi}{\partial z}$ en $z=0$

$\phi_{inc}(z=0) + \phi_{ref}(z=0) = \phi_{trans}(z=0)$

$a+b = c$; $K(a-b) = K'c$

Se distinguen 3 casos:

- Potencial débil $V_0 < E - m \equiv E_{cin} \to K' \in \mathbb{R}$ → onda plana transmitida
- " intermedio $E-m < V_0 < E+m$ ó $|E-V_0| < m \to K'$ imaginario puro ↳ onda evanescente
- " fuerte $V_0 > E+m \to |E-V_0| > m \to K' \in \mathbb{R}$

↳ $E - V_0 < -m < 0 \to E_{cin} < 0$ → prohibido clásicamente

↳ v_g (veloc. de grupo) $= \frac{dE}{dK'} = \frac{d}{dk'}(V_0 \pm \sqrt{k'^2+m^2}) = \frac{1}{2}\frac{2k'}{\sqrt{k'^2+m^2}} = \frac{K'}{E-V_0}$

Como $E-V_0 < 0$ y v_g debe ser >0 (viene de 0 y sale hacia la dcha), entonces $k' < 0$ (signo negativo de la raíz) La condic. de contorno física.

Por la continuidad, tenemos que:

$$\begin{cases} a+b=c \\ K(a-b)=K'c \end{cases}$$

$$\hookrightarrow a-b=\frac{K'}{K}c = a-(\overset{c-a=b}{c-a}) = 2a-c \rightarrow a=\frac{c}{2}\left(1+\frac{K'}{K}\right)$$

$$\hookrightarrow t\equiv\frac{c}{a}=2\frac{1}{1+\frac{K'}{K}}=\frac{2K}{K'+K} \rightarrow \left(T \equiv \frac{K'}{K}\left(\frac{c}{a}\right)^2 = \frac{4KK'}{(K+K')^2} \right) \rightarrow \text{clásicos si } K,K'\in\mathbb{R}$$

$$r\equiv\frac{b}{a}=\frac{c-a}{a}=\frac{c}{a}-1=\frac{2K-K'-K}{K'+K}=\frac{K-K'}{K'+K} \rightarrow \left(R=\left(\frac{b}{a}\right)^2\frac{K}{K}=\left(\frac{K-K'}{K'+K}\right)^2 \right)$$

ver mejor después

La corriente de probabilidad es:

$$\vec{j}=\frac{i}{2m}\left(\phi^*\overleftrightarrow{\partial}\phi\right)=-\frac{i}{2m}\left(\phi^*\vec{\nabla}\phi-\phi\vec{\nabla}\phi^*\right)$$

$$\hookrightarrow \vec{j}=j_z\hat{z} \rightarrow j_z=-\frac{i}{2m}\left(\phi^*\partial_z\phi-\phi\partial_z\phi^*\right)$$

$$j_{inc}=\frac{-i}{2m}\left(\begin{array}{l} a^*e^{iEt}e^{-iKz}(iK)ae^{-iEt}e^{iKz} \\ -ae^{-iEt}e^{iKz}(-iK)a^*e^{iEt}e^{-iKz}\end{array}\right)=|a|^2\cdot\frac{K}{2m}\cdot 2=\frac{K}{m}|a|^2$$

$$j_{ref}=-\frac{i}{2m}\left[b^*e^{iEt}e^{iKz}(-iK)be^{-iEt}e^{-iKz}-be^{-iEt}e^{-iKz}(+iK)b^*e^{iEt}e^{+iKz}\right)=\frac{-K}{m}|b|^2$$

$$j_{trans}=-\frac{i}{2m}\left[c^*e^{+iEt}e^{-iK'^*z}(+iK')c\cdot e^{-iEt}e^{iK'z}-ce^{-iEt}e^{iK'z}(-iK'^*)c^*e^{+iEt}e^{-iK'^*z}\right]$$

$$=+\frac{1}{2m}|c|^2\left[K'\cdot e^{iz(K'-K'^*)}+K'^*e^{iz(K'-K'^*)}\right]$$

$$=\frac{|c|^2}{2m}\cdot e^{iz(K'-K'^*)}\cdot\left[K'+K'^*\right]=\frac{|c|^2}{2m}\cdot e^{iz\cdot 2i\operatorname{Im}\{K'\}}\cdot 2\operatorname{Re}\{K'\}$$

$$=\frac{|c|^2}{m}e^{-2\cdot z\cdot\operatorname{Im}\{K'\}}\operatorname{Re}\{K'\}=\begin{cases} 0 & \text{pot. intermedio } (K'\in\text{Im puro}) \\ \frac{K'}{m}|c|^2 & \text{pot.} \begin{cases}\text{fuerte}\\ \text{débil}\end{cases} (K'\in\mathbb{R})\end{cases}$$

$$T=\frac{j_{trans}}{j_{inc}}=\begin{cases} 0 & \text{pot. int.} \\ \frac{4KK'}{(K+K')^2} & \text{pot déb + fuerte}\end{cases}$$

$$R=\frac{j_{ref}}{j_{inc}}=\left|\frac{K-K'}{K+K'}\right|^2 \quad \text{para los 3 casos}$$

$$\frac{K^2+K'^2-2KK'+4KK'}{(K+K')^2}=1 ✓$$

- Pot. débil $T+R=1$ ✓
- " Interna $T+R=1$ ✓ $\rightarrow \left|\frac{K-i\operatorname{Im}(K')}{K+i\operatorname{Im}(K')}\right|=\frac{K^2+\operatorname{Im}^2(K')}{K^2+\operatorname{Im}^2(K')}=1$ $\Big\}\hat{=}$ en MQ no relativista
- " fuerte $(K'<0)$ $\hookrightarrow T<0$!!, $R>1$, pero $T+R=1$

$T \rightarrow 0$, $R \rightarrow 1$

se añade a $R=1$ ··· part antip.: contribuyen con signo menos a $T=0$ → LO NUEVO

↳ paradoja de Klein-Gordon. La interpretación es que se crea un par partícula-antipartícula que añade a R la misma cantidad que resta a T. La solución rigurosa se obtiene con campos y operadores. < crea a^+ destrucción a

⑧ Ec. Klein-Gordon:

$$(\Box + m^2)\phi = 0 = (\partial_\mu \partial^\mu + m^2) = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi = 0$$

$\downarrow$

Queremos pasar de una EDO de 2º orden a 2 de 1º. El procedimiento es llamar $\xi = \dot{\phi}$ → te quedan 2:

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) - \vec{\nabla}^2\phi + m^2\phi = 0$$

$\downarrow$

$$\begin{cases} \dfrac{\partial \xi}{\partial t} - \vec{\nabla}^2\phi + m^2\phi = 0 \\ \dfrac{\partial \phi}{\partial t} - \xi = 0 \end{cases}$$

Como $\theta = \frac{1}{2}(\phi + \frac{i}{m}\xi)$; $\chi = \frac{1}{2}(\phi - \frac{i}{m}\xi)$

$$\to \begin{matrix} \theta + \chi = \phi \\ \theta - \chi = \frac{i}{m}\xi \end{matrix} \longrightarrow \begin{cases} \phi = \theta + \chi \\ \xi = im(\chi - \theta) \end{cases}$$ → Cambio de variables

$$\begin{cases} \dfrac{\partial}{\partial t}(\chi - \theta) \cdot im - \vec{\nabla}^2(\theta + \chi) + m^2(\theta + \chi) = 0 & (I) \\ \dfrac{\partial}{\partial t}(\chi + \theta) + im(\theta - \chi) = 0 & | \cdot im \end{cases}$$

$$\hookrightarrow \frac{\partial}{\partial t}(\chi + \theta)\, im + m^2(\chi - \theta) = 0 \qquad (II)$$

$$(I) + (II) = 2im\frac{\partial \chi}{\partial t} + 2m^2\chi - \vec{\nabla}^2(\theta + \chi) = 0$$

$$(II) - (I) = 2im\frac{\partial \theta}{\partial t} + \vec{\nabla}^2(\theta + \chi) - 2m^2\theta = 0$$

$\downarrow$

$$\begin{cases} i\dfrac{\partial \chi}{\partial t} = \dfrac{\vec{\nabla}^2}{2m}(\theta + \chi) - m \cdot \chi \\ i\dfrac{\partial \theta}{\partial t} = -\dfrac{\vec{\nabla}^2}{2m}(\theta + \chi) + m\,\theta \end{cases}$$

$$\hookrightarrow i\frac{\partial}{\partial t}\begin{pmatrix} \theta \\ \chi \end{pmatrix} = \left[\overbrace{-\frac{\vec{\nabla}^2}{2m}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{H_0}\right] \cdot \begin{pmatrix} \theta \\ \chi \end{pmatrix} \quad \checkmark$$

↳ Sigue la forma de la ec. de Schröd. o Hamiltoniano (lineal en derivada temporal) → $i\frac{\partial \Phi}{\partial t} = H_0 \Phi$ con $\Phi = \begin{pmatrix} \theta \\ \chi \end{pmatrix}$

Cambio de variable:

$\phi = e^{-imt}\tilde{\phi}$ (equivale a pasar a imagen de interacción con Ho del reposo: $\tilde{\phi} = U^{\dagger}\phi\,;\ U(t) = e^{-imt}$)

$\downarrow$

$$\dot{\phi} = e^{-imt}\left(\dot{\tilde{\phi}} - im\tilde{\phi}\right)$$

$$\theta = \frac{1}{2}\left(\phi + \frac{i}{m}\dot{\phi}\right) = \frac{e^{-imt}}{2}\left(\tilde{\phi} + \frac{i}{m}\left(\dot{\tilde{\phi}} - im\tilde{\phi}\right)\right) = \frac{e^{-imt}}{2}\left(2\tilde{\phi} + \frac{i}{m}\dot{\tilde{\phi}}\right)$$

$$= e^{-imt}\left(\tilde{\phi} + \frac{i}{2m}\dot{\tilde{\phi}}\right)$$

$$\chi = \frac{1}{2}\left(\phi - \frac{i}{m}\dot{\phi}\right) = \frac{e^{-imt}}{2}\left(\tilde{\phi} - \frac{i}{m}\left(\dot{\tilde{\phi}} - im\tilde{\phi}\right)\right) = -\frac{e^{-imt}}{2m}\cdot i\,\dot{\tilde{\phi}}$$

$\downarrow$

$$i\frac{\partial\chi}{\partial t} = \frac{\nabla^2}{2m}\theta + \chi\left(\frac{\vec{\nabla}^2}{2m} - m\right)$$

$$\hookrightarrow = -\frac{i^2}{2m}e^{-imt}\left(\ddot{\tilde{\phi}} - im\dot{\tilde{\phi}}\right) = -\frac{i}{2m}e^{-imt}\cdot\dot{\tilde{\phi}}\cdot m + \frac{\ddot{\tilde{\phi}}e^{-imt}}{2m}$$

$$= m\chi + \ddot{\tilde{\phi}}\,\frac{e^{-imt}}{2m}$$

$$\hookrightarrow\ 2m\chi = \frac{\vec{\nabla}^2}{2m}\theta + \frac{\vec{\nabla}^2\chi}{2m} + \frac{\ddot{\tilde{\phi}}e^{-imt}}{2m} \quad \ldots\ \frac{\ddot{\tilde{\phi}}e^{-imt}}{2m} \approx$$

$$\chi = \frac{\vec{\nabla}^2\theta}{4m^2} + \frac{\vec{\nabla}^2\chi}{4m^2} + \frac{\ddot{\tilde{\phi}}e^{-imt}}{4m^2}$$

Límite no relativista: $\chi \rightarrow 0$ ya que $\propto \underset{\ldots\text{orden } 1}{\frac{1}{m}}$ mientras que $\theta \sim \underset{\ldots\text{orden } 0}{\left(\frac{1}{m}\right)^0}$

$\downarrow$

$$i\frac{\partial\theta}{\partial t} = -\frac{\vec{\nabla}^2}{2m}\theta + m\theta \quad //$$ (Ec. de Schrödinger)

1ª corrección relativista: $(|\vec{p}| \ll m)$

$\chi \approx \frac{\vec{\nabla}^2\theta}{4m^2}$ → sustituyendo límite anterior en la expresión de χ.

$$\downarrow\ i\frac{\partial\theta}{\partial t} = -\frac{\vec{\nabla}^2}{2m}\left(1 + \frac{\vec{\nabla}^2}{4m^2}\right)\theta + m\theta = \left(-\frac{\vec{\nabla}^2}{2m} - \frac{\vec{\nabla}^2}{8m^3} + m\right)\cdot\theta \quad //$$

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Tª Cuántica de Campos - 4º

⑨

a) $\gamma_5 \sigma^{\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma}$

Dem:

Esta propiedad se demuestra a partir de casos particulares. No obstante, es una propiedad autoevidente, salvo constantes. La antisimetría de $\sigma^{\mu\nu}$ implica un tensor antisimétrico a la derecha, → el símbolo de Levi-Civita, contraído con otra matriz $\sigma_{\rho\sigma}$ que se da debido a las propiedades de paridad de la γ_5 a la izquierda. Además, es necesario que tenga índices abajo para poder contraer con la $\epsilon^{\mu\nu\ldots}$. Por tanto, según este razonamiento, $\gamma_5 \sigma^{\mu\nu} = K \cdot \epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma}$, con $K = \text{cte}$ (escalar)

Para $\mu = \nu \to 0 = 0$ (trivial)

" $\mu \neq \nu$

↳ $\mu = 0, \nu = i$

$\nearrow \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \to \{\gamma^i, \gamma^0\} = 0$

$\gamma_5 \sigma^{0i} = i\gamma^0\gamma^1\gamma^2\gamma^3 \frac{i}{2}(\gamma^0\gamma^i - \gamma^i\gamma^0) = (-1)\gamma^0\gamma^1\gamma^2\underbrace{\gamma^3\gamma^0}_{=-\gamma^0\gamma^3}\gamma^i$

$= +(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^i = \gamma^1\gamma^2\gamma^3\gamma^i$

Para $i = 1 \to \gamma^1\gamma^2\gamma^3\gamma^1 = (\gamma^1)^2\gamma^2\gamma^3 = -\gamma^2\gamma^3$ $\quad \sigma_{ij} = \sigma^{ij} \cdot (-1)^2$

La parte derecha: $K \cdot (\underbrace{\epsilon^{0123}}_{+}\sigma_{23} + \underbrace{\epsilon^{0132}}_{-}\underbrace{\sigma_{32}}_{-\sigma_{23}}) = K \cdot 2\sigma_{23} = 2K \frac{i}{2}[\gamma^2\gamma^3 - \gamma^3\gamma^2]$

$= iK \cdot 2\gamma^2\gamma^3 = \text{p. izqda} = -\gamma^2\gamma^3$

↳ $K = -\frac{1}{2i} = \frac{i}{2}$ ✓

Para $i=2$:

$\gamma^1\gamma^2\gamma^3\gamma^2 = \gamma^1(-)(\gamma^2)^2\gamma^3 = \gamma^1\gamma^3$

$K \cdot (\underbrace{\epsilon^{0213}}\sigma_{13}) \cdot 2 = -2K\frac{i}{2} \cdot 2\gamma^1\gamma^3 = -2iK\gamma^1\gamma^3 \to K = \frac{i}{2}$ ✓

Para $i = 3$:

$\left.\begin{array}{l} \gamma^1\gamma^2(\gamma^3)^2 = -\gamma^1\gamma^2 \\ K \cdot \underbrace{\epsilon^{0312}}_{+}\sigma_{12} \cdot 2 = 2iK\gamma_1\gamma_2 \end{array}\right\} K = \frac{i}{2}$

→ Para $\mu = i, \nu = 0$ es totalmente idéntico, sólo es permutación de índices, como tenemos $\sigma^{\mu\nu}$ y $\epsilon^{\mu\nu\ldots}$, a ambos lados cambia el signo y K no cambia. Sólo falta demostrar $\mu = i, \nu = j$ con $i \neq j$.

$\gamma_5\sigma^{ij} = i\gamma^0\gamma^1\gamma^2\gamma^3\frac{i}{2}2\gamma^i\gamma^j$

$\nearrow \left.\begin{array}{l}\gamma_0 = \gamma^0 \\ \gamma_1 = -\gamma^1\end{array}\right\} \sigma_{0i} = -i\gamma^0\gamma^i$

• $i = 3 \to \gamma_5\sigma^{3j} = +1\gamma^0\gamma^1\gamma^2\gamma^j$

↳ $j = 2 \to = -\gamma^0\gamma^1 \longleftrightarrow K \cdot \underbrace{\epsilon^{3201}}\sigma_{01} \cdot 2 = -2iK\gamma_0\gamma_1 \to K = \frac{i}{2}$ ✓

↳ $j = 1 \to = +\gamma^0\gamma^2 \longleftrightarrow K \cdot \underbrace{\epsilon^{3102}}_{+} 2 \cdot \sigma_{02} = 2iK\gamma_0\gamma_2 = -2iK\gamma^0\gamma^2 \to K = \frac{i}{2}$ ✓

• $i = 2, j = 1$

↳ $i\gamma^0\gamma^1\gamma^2\gamma^3 \cdot i\gamma^2\gamma^1 = -\gamma^0\gamma^1\underbrace{(\gamma_2)^2}_{-}(-)\gamma^3\gamma^1 = -\gamma^0(\gamma_1)^2(-)\gamma^3 = -\gamma^0\gamma^3$

$\left.\longleftrightarrow K \cdot \underbrace{\epsilon^{2103}}_{-} \cdot 2\sigma_{03} = K \cdot (-2)i(-\gamma^0\gamma^3) = 2iK\gamma^0\gamma^3 \right\} \Rightarrow K = \frac{i}{2}$ ✓

Por tanto, se ha comprobado para todos los casos de μ y ν (directamente el caso particular o permutaciones de éste), con lo que queda demostrado en general.

b) $[\sigma^{\alpha\beta}, \gamma^\mu] = -2i(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha)$

Dem:

$\sigma^{\alpha\beta} \equiv \frac{i}{2}[\gamma^\alpha, \gamma^\beta] = \frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$

$[\sigma^{\alpha\beta}, \gamma^\mu] = \sigma^{\alpha\beta}\gamma^\mu - \gamma^\mu\sigma^{\alpha\beta} = \frac{i}{2}\gamma^\alpha\gamma^\beta\gamma^\mu - \frac{i}{2}\gamma^\beta\gamma^\alpha\gamma^\mu$

$-\frac{i}{2}\gamma^\mu\gamma^\alpha\gamma^\beta + \frac{i}{2}\gamma^\mu\gamma^\beta\gamma^\alpha = \frac{i}{2}(\gamma^\alpha\gamma^\beta\gamma^\mu + \gamma^\mu\gamma^\beta\gamma^\alpha)$

$-\frac{i}{2}(\gamma^\beta\gamma^\alpha\gamma^\mu + \gamma^\mu\gamma^\alpha\gamma^\beta)$

Como paso intermedio, es útil demostrar la propied 10 i)

$\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho[\underbrace{\{\gamma^\nu, \gamma^\mu\}}_{2g^{\mu\nu}\text{... simétrico}} - \gamma^\mu\gamma^\nu]$

$= \gamma^\mu(\{\gamma^\nu, \gamma^\rho\} - \gamma^\rho\gamma^\nu) + 2g^{\mu\nu}\gamma^\rho - (\{\gamma^\rho, \gamma^\mu\} - \gamma^\mu\gamma^\rho)\gamma^\nu$

$= \gamma^\mu 2g^{\nu\rho} - \gamma^\mu\gamma^\rho\gamma^\nu + 2g^{\mu\nu}\gamma^\rho - 2g^{\mu\rho}\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu$

$= 2[g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu]$

Aplicamos la propiedad en ambos términos

$[\sigma^{\alpha\beta}, \gamma^\mu] = \frac{i}{\cancel{2}}\cancel{2}(g^{\alpha\beta}\gamma^\mu - g^{\alpha\mu}\gamma^\beta + g^{\beta\mu}\gamma^\alpha) - \frac{i}{\cancel{2}}\cancel{2}[g^{\beta\alpha}\gamma^\mu - g^{\beta\mu}\gamma^\alpha + g^{\alpha\mu}\gamma^\beta]$

$= 2i(g^{\mu\beta}\gamma^\alpha - g^{\alpha\mu}\gamma^\beta) = -2i(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha)$ // c.q.d.

a) $\gamma^\mu \gamma_\mu = 4 \cdot I_4$

Partimos de:

$$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$$
$$\{\gamma^\mu \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \Big\} \quad 2\gamma^\mu\gamma^\nu = [\gamma^\mu, \gamma^\nu] + \{\gamma^\mu, \gamma^\nu\}$$
$$= -2i\sigma^{\mu\nu} + 2g^{\mu\nu}$$

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$$

$$\hookrightarrow \gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = g_{\mu\nu}(g^{\mu\nu} - i\sigma^{\mu\nu})$$

$$= \underbrace{g_{\mu\nu} g^{\mu\nu}}_{} - i \underbrace{\overset{\downarrow}{g_{\mu\nu}}\, \overset{\downarrow}{\sigma^{\mu\nu}}}_{=0} = 4 \cdot I_4 \; /\!/$$

… del espacio de Dirac

$\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix} \cdot \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$; Simétrico ; $g_{\mu\nu} = g_{\nu\mu}$ Antisimétrico $\hookrightarrow \sigma^{\mu\nu} = -\sigma^{\nu\mu}$

b) $$\gamma^\mu\gamma^\nu\gamma_\mu = (\gamma^\mu\gamma^\nu \underbrace{- \gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu}_{0}) \cdot \gamma_\mu = (\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu)\gamma_\mu$$

$$= 2 \cdot g^{\mu\nu}\gamma_\mu - \gamma^\nu(\gamma^\mu\gamma_\mu) \underset{\hookrightarrow \text{por a)}}{=} 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu \; /\!/$$

c) $$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = (\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma_\mu = 2g^{\mu\nu}\gamma^\rho\gamma_\mu - \gamma^\nu(\gamma^\mu\gamma^\rho\gamma_\mu)$$

$$\overset{\to \text{por b)}}{=} 2\gamma^\rho\gamma^\nu - \gamma^\nu \cdot (-2\gamma^\rho) = 2(\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho) = 2 \cdot \{\gamma^\rho, \gamma^\nu\} = 4g^{\rho\nu}$$

$$= 4g^{\nu\rho} \; /\!/$$

d) $$\sigma^{\mu\nu}\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \cdot \frac{i}{2}[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$$

$$= -\frac{1}{4}\Big[\underbrace{\gamma^\mu\gamma^\nu\gamma_\mu}_{(-2\gamma^\nu)\,\dots\,b)}\gamma_\nu - \gamma^\mu\underbrace{\gamma^\nu\gamma_\nu}_{4 \,\to\, a)}\gamma_\mu - \gamma^\nu\underbrace{\gamma^\mu\gamma_\mu}_{4}\gamma_\nu + \underbrace{\gamma^\nu\gamma^\mu\gamma_\nu}_{-2\gamma^\mu\,\dots\,b)}\gamma_\mu\Big]$$

$$= -\frac{1}{4}\Big[(-4) \cdot (\underbrace{\gamma^\mu\gamma_\mu}_{4} + \underbrace{\gamma^\nu\gamma_\nu}_{4}) - 2(\underbrace{\gamma^\nu\gamma_\nu}_{4} + \underbrace{\gamma^\mu\gamma_\mu}_{4})\Big]$$

$$= \frac{1}{4} \cdot [32 + 16] = 12 \cdot I_4 \; /\!/$$

e) $$\gamma^\mu\gamma_5\gamma_\mu = \gamma^\mu \cdot \left(\cancel{\frac{\gamma_5\gamma_\mu}{2} + \frac{\overbrace{\gamma_5\gamma_\mu}^{\gamma_5\gamma_\mu}}{2}} - \frac{\overbrace{\gamma_\mu\gamma_5}^{0}}{2} + \frac{\gamma_\mu\gamma_5}{2}\right)$$

$$= \frac{\gamma^\mu}{2}(\underbrace{\{\gamma_5, \gamma_\mu\}}_{0} + \cancel{[\gamma_5, \gamma_\mu]) = \frac{\gamma^\mu}{2} \cdot [\gamma_5, \gamma_\mu]} \,\cdot$$

$$\gamma^\mu\gamma_5\gamma_\mu = [\underbrace{\{\gamma^\mu, \gamma_5\}}_{=0} - \gamma_5\gamma^\mu] \cdot \gamma_\mu = -\gamma_5\underbrace{\gamma^\mu\gamma_\mu}_{4} = -4\gamma_5 \; /\!/$$

f) $\gamma^\mu\gamma^\nu\gamma_5\gamma_\mu = (\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)\cdot\gamma_5\gamma_\mu$

$= 2g^{\mu\nu}\gamma_5\gamma_\mu - \gamma^\nu(\gamma^\mu\gamma_5\gamma_\mu) = 2\gamma_5\gamma^\nu - \gamma^\nu\cdot(-4\cdot\gamma_5)$

$= 2\gamma_5\gamma^\nu + 4\gamma^\nu\gamma_5 = 2\gamma_5\gamma^\nu + 4(\underbrace{\{\gamma^\nu,\gamma_5\}}_{0} - \gamma_5\gamma^\nu) = -2\gamma_5\gamma^\nu$

$= -2(\{\gamma_5,\gamma^\nu\} - \gamma^\nu\gamma_5) = 2\gamma^\nu\gamma_5$ //

g) $\sigma^{\mu\nu}\gamma_5\sigma_{\mu\nu} = 12\gamma_5$

↳ $\sigma^{\mu\nu}\gamma_5\sigma_{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_5\cdot\frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$= -\frac{1}{4}(\gamma^\mu\gamma^\nu\gamma_5\gamma_\mu\gamma_\nu - \gamma^\nu\gamma^\mu\gamma_5\gamma_\mu\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_5\gamma_\nu\gamma_\mu$
$+ \gamma^\nu\gamma^\mu\gamma_5\gamma_\nu\gamma_\mu)$

por f)

$= -\frac{1}{4}(2\cdot\gamma^\nu\gamma_5\cdot\gamma_\nu - \gamma^\nu\underbrace{(-\gamma_5\gamma^\mu)\gamma_\mu}_{4}\gamma_\nu - \gamma^\mu(-\gamma^5\gamma^\nu)\gamma_\nu\gamma_\mu$
$+ 2\gamma^\mu\gamma_5\gamma_\mu)$

$= -\frac{1}{4}[2\cdot\underbrace{(-\gamma_5\gamma^\nu)\gamma_\nu}_{4} + 4\gamma^\nu(-\gamma_5\gamma_\nu) + 4(-\gamma^5\gamma^\mu)\gamma_\mu - 2\gamma_5\cdot 4]$

$= -\frac{1}{4}[-8\gamma_5 - 16\gamma_5 - 16\gamma_5 - 8\gamma_5] = \frac{1}{4}\cdot 16\cdot 3\gamma_5 = 12\gamma_5$ //

Camino más rápido:

$\sigma^{\mu\nu}\gamma_5\sigma_{\mu\nu} = (\underbrace{[\sigma^{\mu\nu},\gamma_5]}_{=0} + \gamma_5\sigma^{\mu\nu})\sigma_{\mu\nu} = \gamma_5\cdot\underbrace{\sigma^{\mu\nu}\cdot\sigma_{\mu\nu}}_{=12\ (\text{por d}))} = 12\gamma_5$ //

h) $\gamma^\mu\sigma^{\rho\alpha}\gamma_\mu = \cancel{\gamma^\mu}([\sigma^{\rho\alpha},\gamma_\mu] \cancel{+ \gamma_\mu\sigma^{\rho\alpha}) = 4\sigma^{\rho\alpha}} + \gamma^\mu[\sigma^{\rho\alpha},\gamma_\mu]$

$= 4\cdot\sigma^{\rho\alpha} \cancel{+ \gamma^\mu\frac{i}{2}}\cdot\gamma^\mu(\gamma^\rho$

$\gamma^\mu\sigma^{\rho\alpha}\gamma_\mu = \frac{i}{2}\gamma^\mu(\gamma^\rho\gamma^\alpha - \gamma^\alpha\gamma^\rho)\gamma_\mu = \frac{i}{2}(\gamma^\mu\gamma^\rho\gamma^\alpha\gamma_\mu - \gamma^\mu\gamma^\alpha\gamma^\rho\gamma_\mu)$

$\overset{\text{por c)}}{=} \frac{i}{2}\cdot(4g^{\rho\alpha} - 4\underset{\overset{\shortparallel}{g^{\rho\alpha}}}{g^{\alpha\rho}}) = 0$ //

i) $\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho[\underbrace{\{\gamma^\nu,\gamma^\mu\}}_{2g^{\nu\mu}} - \gamma^\mu\gamma^\nu]$

$= \gamma^\mu(\{\gamma^\nu,\gamma^\rho\} - \gamma^\rho\gamma^\nu) + 2g^{\mu\nu}\gamma^\rho - (\{\gamma^\rho,\gamma^\mu\} - \gamma^\mu\gamma^\rho)\gamma^\nu$

$= \gamma^\mu 2g^{\nu\rho} - \gamma^\mu\gamma^\rho\gamma^\nu + 2g^{\mu\nu}\gamma^\rho - 2\underset{\text{simétrico}}{g^{\rho\mu}}\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu$

$= 2[g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu]$ //

$= \{\gamma^\mu,\gamma^\nu)\gamma^\rho\gamma^\sigma\gamma_\mu - \gamma^\nu(\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu) = 2g^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma_\mu$

$-\gamma^\nu\cdot\overset{\text{ver c)}}{(4g^{\rho\sigma})} = 2(\gamma^\rho\gamma^\sigma)\gamma^\nu - 4g^{\rho\sigma}\gamma^\nu = 2(\{\gamma^\rho,\gamma^\sigma\} - \gamma^\sigma\gamma^\rho)\gamma^\nu$

$-4g^{\rho\sigma}\gamma^\nu = 2\,2g^{\rho\sigma}\gamma^\nu - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 4g^{\rho\sigma}\gamma^\nu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$ // ✓

k) $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\mu = 2g^{\mu\nu}\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\mu - \gamma^\nu(\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\mu)$

$= 2\gamma^\rho\gamma^\sigma\gamma^\tau\gamma^\nu - \gamma^\nu\cdot\overset{\text{ver j)}}{(-2)}\gamma^\tau\gamma^\sigma\gamma^\rho$

$= 2(\gamma^\rho\gamma^\sigma\gamma^\tau\gamma^\nu + \gamma^\nu\gamma^\tau\gamma^\sigma\gamma^\rho) = 2([\{\gamma^\rho,\gamma^\sigma\} - \gamma^\sigma\gamma^\rho]\cdot[\{\gamma^\tau,\gamma^\nu\}$
$- \gamma^\nu\gamma^\tau] + [\{\gamma^\nu,\gamma^\tau\} - \gamma^\tau\gamma^\nu]\cdot[\{\gamma^\sigma,\gamma^\rho\} - \gamma^\rho\gamma^\sigma])$

$= 2(2g^{\rho\sigma}\cdot 2g^{\tau\nu} - \gamma^\sigma\gamma^\rho\cdot 2g^{\tau\nu} - 2g^{\rho\sigma}\gamma^\nu\gamma^\tau + \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\tau$
$+ 2g^{\nu\tau}\cdot 2g^{\sigma\rho} - \gamma^\rho\gamma^\sigma\cdot 2g^{\nu\tau} - 2g^{\sigma\rho}\gamma^\tau\gamma^\nu + \gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma)$

$g^{\mu\nu} = g^{\nu\mu}$

$\overset{\searrow}{=} 2(8g^{\rho\sigma}g^{\tau\nu} - 2g^{\nu\tau}\cdot\{\gamma^\sigma,\gamma^\rho\} - 2g^{\rho\sigma}\{\gamma^\nu,\gamma^\tau\} + \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\tau + \gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma)$

$= 2(8g^{\sigma\rho}g^{\tau\nu} \overbrace{- 2g^{\nu\tau}\cdot 2g^{\sigma\rho} - 2g^{\rho\sigma}\cdot 2g^{\nu\tau}}^{=0} + \qquad .. \qquad)$

$= 2\cdot(\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\tau + \gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma)$ // ✓

l)

es un escalar
nº, conmuta con matrices

$\gamma^\mu\not{a}\gamma_\mu = \gamma^\mu\gamma^\nu a_\nu\gamma_\mu = a_\nu\gamma^\mu\gamma^\nu\gamma_\mu \underset{\text{ver b)}}{=} a_\nu\cdot(-2\gamma^\nu)$

$= -2a_\nu\gamma^\nu = -2\not{a}$ // ✓

m)

$\gamma^\mu\not{a}\not{b}\gamma_\mu = \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu a_\nu b_\rho \overset{\text{ver c)}}{=} a_\nu b_\rho\cdot 4g^{\nu\rho} = 4a_\nu b^\nu = 4a_\mu b^\mu$

$= 4a\cdot b$ // ✓

n) $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = a_\nu b_\rho c_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu \overset{\text{ver j)}}{=}$

$= -2a_\nu b_\rho c_\sigma\cdot\gamma^\sigma\gamma^\rho\gamma^\nu \overset{\text{mantener orden }\gamma s,\ a_i, b_i, c_i \text{ sí conmuta}}{=} -2\,c_\sigma\gamma^\sigma b_\rho\gamma^\rho a_\nu\gamma^\nu = -2\not{c}\not{b}\not{a}$ // ✓

ñ) $\gamma^\mu\not{a}\not{b}\not{c}\not{d}\gamma_\mu = a_\nu b_\rho c_\sigma d_\tau\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\mu \overset{\text{ver k)}}{=}$

$= a_\nu b_\rho c_\sigma d_\tau 2(\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\tau + \gamma^\tau\gamma^\nu\gamma^\rho\gamma^\sigma)$

$= 2(c_\sigma\gamma^\sigma b_\rho\gamma^\rho a_\nu\gamma^\nu d_\tau\gamma^\tau + d_\tau\gamma^\tau a_\nu\gamma^\nu b_\rho\gamma^\rho c_\sigma\gamma^\sigma)$

$= 2(\not{c}\not{b}\not{a}\not{d} + \not{d}\not{a}\not{b}\not{c})$ // ó $\overset{\text{línea 3 de k)}}{=} 2a_\nu b_\rho c_\sigma d_\tau\cdot(\gamma^\rho\gamma^\sigma\gamma^\tau\gamma^\nu$

$+ \gamma^\nu\gamma^\tau\gamma^\sigma\gamma^\rho) = 2\cdot(\not{b}\not{c}\not{d}\not{a} + \not{a}\not{d}\not{c}\not{b})$

↙ Repetido

n) $\not{a}$: escalar

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = a_\nu b_\delta c_\sigma \gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\delta\gamma^\nu a_\nu b_\delta c_\sigma$$

$$= -2\not{c}\not{b}\not{a} \; //$$

$\gamma^\mu\gamma_\mu = 4I$

$\not{a} = a_\nu\gamma^\nu$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

---

o) $\not{a}\not{b} = a_\mu\gamma^\mu b_\nu\gamma^\nu = a_\mu b_\nu\gamma^\mu\gamma^\nu = a_\mu b_\nu\cdot\left[\{\gamma^\mu,\gamma^\nu\}-\gamma^\nu\gamma^\mu\right]$

$= a_\mu b_\nu\cdot 2g^{\mu\nu} - \gamma^\nu b_\nu\gamma^\mu a_\mu = 2\cdot a\cdot b - \not{b}\not{a}$ // ✓

p) $\not{a}\not{a} = a_\nu a_\mu\gamma^\nu\gamma^\mu = a_\nu a_\mu\left(2g^{\nu\mu} - \gamma^\mu\gamma^\nu\right) = 2a^2 - a_\nu a_\mu\gamma^\mu\gamma^\nu$

$= 2a^2 - \not{a}\not{a} \rightarrow 2\not{a}\not{a} = 2a^2 \rightarrow \not{a}\not{a} = a^2$ // ✓

ó bien: caso particular de o) con $a = b$

↳ $\not{a}\not{a} = 2a\cdot a - \not{a}\not{a} \rightarrow \not{a}\not{a} = a^2$ // ✓

⑪

a) $\mathrm{Tr}(\gamma^\mu)=0$ → Vía a) → Probar todos los casos particulares

Vía b) → Meter $I=\gamma_5{}^2$ y saber que $\{\gamma_5,\gamma_\mu\}=0$

↓

$$\mathrm{Tr}(\gamma^\mu)=\mathrm{Tr}(\gamma^\mu\gamma_5\gamma_5)\overset{\text{propiedad cíclica de la traza}}{=}\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_5)=\mathrm{Tr}\left[(\underbrace{\{\gamma_5,\gamma_\mu\}}_{0}-\gamma_\mu\gamma_5)\gamma_5\right]$$

$$=-\mathrm{Tr}(\gamma_\mu\gamma_5{}^2)=-\mathrm{Tr}(\gamma^\mu) \quad \rightarrow 2\mathrm{Tr}(\gamma^\mu)=0$$

↳ $\mathrm{Tr}(\gamma^\mu)=0$ ✓

b) $$\mathrm{Tr}(\gamma^\mu\gamma^\nu)=\mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\}-\gamma^\nu\gamma^\mu)=\mathrm{Tr}(2g^{\mu\nu})-\mathrm{Tr}(\gamma^\nu\gamma^\mu)$$

propiedad cíclica de la traza

$$=\mathrm{Tr}(2g^{\mu\nu}\cdot I_4)-\mathrm{Tr}(\gamma^\mu\gamma^\nu)$$

↳ $2\mathrm{Tr}(\gamma^\mu\gamma^\nu)=2g^{\mu\nu}\,\mathrm{Tr}(I_4)=8g^{\mu\nu}\rightarrow\mathrm{Tr}(\gamma^\mu\gamma^\nu)=4g^{\mu\nu}$ ✓

c) $$\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}})=\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\cdot\overset{\equiv I}{\widetilde{\gamma_5\cdot\gamma_5}})$$

Como $\{\gamma_5,\gamma^\mu\}=0\Rightarrow\gamma_5\gamma^\mu=-\gamma^\mu\gamma_5$

$$\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\gamma_5=\gamma^{\mu_1}\dots\gamma^{\mu_{2k}}\gamma_5(-1)\gamma^{\mu_{2k+1}}\gamma_5=\gamma^{\mu_1}\dots\gamma^{\mu_{2k-1}}\gamma_5\gamma^{\mu_{2k}}\gamma^{\mu_{2k+1}}\gamma_5\cdot(-1)^2$$

$$=\dots=\gamma_5\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\gamma_5\cdot(-1)^{2k+1}$$

↳ $$\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}})=\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\gamma_5)\cdot(-1)^{2k+1}=$$ ↗ $k\in\mathbb{N}^+$

$$=-\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\gamma_5)\overset{\text{propied cíclica}}{=}-\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}}\underbrace{\gamma_5\gamma_5}_{I})$$

$$=-\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}})\Rightarrow\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2k+1}})=0$$

d) $$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)=\mathrm{Tr}\left((\{\gamma^\mu,\gamma^\nu\}-\gamma^\nu\gamma^\mu)\cdot(\{\gamma^\rho,\gamma^\sigma\}-\gamma^\sigma\gamma^\rho)\right)$$

$$=2g^{\mu\nu}\cdot2g^{\rho\sigma}+2g^{\rho\sigma}\mathrm{Tr}(-\gamma^\nu\gamma^\mu)-2g^{\mu\nu}\mathrm{Tr}(\gamma^\sigma\gamma^\rho)+\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho)$$

$$=\mathrm{Tr}\left[(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma\right]=2g^{\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma)-\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$$

$$\overset{\text{por b)}}{=}2g^{\mu\nu}\cdot4g^{\rho\sigma}-\mathrm{Tr}(\gamma^\nu(2g^{\mu\rho}-\gamma^\rho\gamma^\mu)\gamma^\sigma)=8g^{\mu\nu}g^{\rho\sigma}-8g^{\mu\rho}g^{\nu\sigma}$$

$$+\mathrm{Tr}(\gamma^\nu\gamma^\rho(2g^{\mu\sigma}-\gamma^\sigma\gamma^\mu))=8g^{\mu\nu}g^{\rho\sigma}-8g^{\mu\rho}g^{\nu\sigma}+8g^{\mu\sigma}g^{\nu\rho}$$

$$-\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)=8(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})-\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$$

prop. cíclica ↳ $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)=4(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})$ ✓

e) $\mathrm{Tr}(\not{a}\not{b}) = \mathrm{Tr}(a_\mu \gamma^\mu b_\nu \gamma^\nu) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu \gamma^\nu) \overset{\text{ver } b)}{=} 4g^{\mu\nu} a_\mu b_\nu = 4\, a\cdot b$

o bien

$\mathrm{Tr}(\not{a}\not{b}) \overset{\text{ver } 10\,b)}{=} \mathrm{Tr}(2ab - \not{b}\not{a}) = \mathrm{Tr}(2ab\cdot I_4) - \mathrm{Tr}(\not{b}\not{a})$

$= 8ab - \overset{\text{prop. cíclica}}{\mathrm{Tr}(\not{a}\not{b})} \rightarrow 2\mathrm{Tr}(\not{a}\not{b}) = 8ab \rightarrow \mathrm{Tr}(\not{a}\not{b}) = 4ab$ ✓

f) $\mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = a_{1\mu} a_{2\nu} a_{3\rho} a_{4\sigma} \cdot \mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \overset{\text{ver } d)}{=}$

↳ g's contraen as: $a_{i\alpha} a_{j\beta}\, g^{\alpha\beta} = a_{i\alpha} a_j^{\alpha} \equiv (a_i \cdot a_j)$

$= 4\left[(a_1a_2)(a_3a_4) - (a_1a_3)(a_2a_4) + (a_1a_4)\cdot(a_2a_3)\right]$ ✓

g) Demostramos por recurrencia: ~o+o-o...

$k = 1 \rightarrow \mathrm{Tr}(\not{a}_1\not{a}_2) = a_1a_2 \cdot \mathrm{Tr}(I_4) = 4a_1a_2$ ✓ → coincide con e)

$k = 2 \rightarrow \mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) \overset{g)}{=} a_1a_2\,\mathrm{Tr}(\not{a}_3\not{a}_4) - a_1a_3\,\mathrm{Tr}(\not{a}_2\not{a}_4) + a_1a_4\,\mathrm{Tr}(\not{a}_2\not{a}_3)$

$\overset{\text{por } e)}{=} (a_1a_2)4(a_3a_4) - (a_1a_3)4(a_2a_4) + (a_1a_4)4(a_2a_3)$ ✓ coincide con f)

$\mathrm{Tr}(\not{a}_1\not{a}_2 \dots \not{a}_{2k}) = \overbrace{a_{1\mu_1} a_{2\mu_2} \dots a_{2k\mu_{2k}}}^{=A} \cdot \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2} \dots \gamma^{\mu_{2k}})$

$\overset{\text{ver } d)}{=} \dots \mathrm{Tr}((2g^{\mu_1\mu_2} - \gamma^{\mu_2}\gamma^{\mu_1})\gamma^{\mu_3} \dots \gamma^{\mu_{2k}}) = 2a_1a_2\,\mathrm{Tr}(\not{a}_3 \dots \not{a}_{2k})$

$- A\,\mathrm{Tr}(\gamma^{\mu_2}(2g^{\mu_1\mu_3} - \gamma^{\mu_3}\gamma^{\mu_1})\gamma^{\mu_4} \dots \gamma^{\mu_{2k}}) = 2a_1a_2\,\mathrm{Tr}(\not{a}_3 \dots \not{a}_{2k}) - 2a_1a_3\mathrm{Tr}(\not{a}_2\not{a}_4 \dots \not{a}_{2k}$

$+ \quad A \cdot \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_3}(2g^{\mu_1\mu_4} - \gamma^{\mu_4})\gamma^{\mu_5} \dots \gamma^{\mu_{2k}}) =$

$= 2a_1a_2\mathrm{Tr}(\not{a}_3 \dots \not{a}_{2k}) - 2a_1a_3\,\mathrm{Tr}(\not{a}_2\not{a}_4 \dots \not{a}_{2k}) + \dots - 2a_1a_{2k-1}\mathrm{Tr}(\not{a}_2\not{a}_3 \dots$

$\dots \not{a}_{2k-2}\not{a}_{2k}) + 2a_1a_{2k}\,\mathrm{Tr}(\not{a}_2 \dots \not{a}_{2k-1})$

h) $\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma_5 \overbrace{\gamma^0\gamma^0}^{I}) \overset{\{\gamma_5\gamma^\mu\}=0}{=} -\mathrm{Tr}(\gamma^0\gamma_5\gamma^0) \overset{\text{prop. cíclica}}{=} -\mathrm{Tr}(\gamma_5\gamma^0\gamma^0) = -\mathrm{Tr}(\gamma_5) = 0$ ✓

i) $\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\ldots\gamma^{\mu_{2k+1}}) = \mathrm{Tr}(\gamma^{\mu_1}\gamma_5(-1)^1\ldots\gamma^{\mu_{2k+1}}) = \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2k+1}}(-1)^{2k+1}\gamma_5)$

$\overset{\text{prop. cíclica}}{=} -\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\ldots\gamma^{\mu_{2k+1}}) = 0$ ✓

~~j) $\mathrm{Tr}(\gamma_5 \not a \not b) = a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) \overset{\{\}=0}{=} \mathrm{Tr}(\gamma^\mu\gamma_5\gamma^\nu)\cdot(-a_\mu b_\nu)$~~

~~$\overset{\text{cíclica}}{=} -a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\nu\gamma^\mu) = -a_\mu b_\nu \mathrm{Tr}(\gamma_5(2g^{\mu\nu} - \gamma^\mu\gamma^\nu))$~~

~~$= -a\cdot b\,\mathrm{Tr}(\overset{0\text{ por h)}}{\gamma_5}) + \mathrm{Tr}(\gamma_5\not a\not b)$~~

~~$\mathrm{Tr}(\gamma_5\not a\not b) = a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = a_\mu b_\nu \mathrm{Tr}(\gamma_5([\gamma^\mu,\gamma^\nu]+\gamma^\nu\gamma^\mu))$~~

~~$\mathrm{Tr}(\gamma_5\not a\not b) = a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) \overset{\text{anticonm.}}{=} a_\mu b_\nu \mathrm{Tr}(-\gamma^\mu\gamma_5\gamma^\nu)$~~

~~$\overset{\text{p. cíclica}}{=} -a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\nu\gamma^\mu) = -a_\mu b_\nu \mathrm{Tr}(\gamma_5(2g^{\nu\mu} + \gamma^\mu$~~

j) $\mathrm{Tr}(\gamma_5\not a\not b) = a_\mu b_\nu \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = a_\mu b_\nu \mathrm{Tr}(\gamma_5(g^{\mu\nu} - i\sigma^{\mu\nu}))$

$= a_\mu b_\nu\left[g^{\mu\nu}\mathrm{Tr}(\overset{0}{\cancel{\gamma_5}}) - i\,\mathrm{Tr}(\gamma_5\sigma^{\mu\nu})\right]$

$= -i a_\mu b_\nu \mathrm{Tr}(\gamma_5\sigma^{\mu\nu}) \overset{\to\text{ ver 9a)}}{=} \frac{a_\mu b_\nu}{2}\mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}) = \frac{a_\mu b_\nu}{2}\epsilon^{\mu\nu\rho\sigma}\underbrace{\mathrm{Tr}(\overset{\text{antis.}}{\sigma_{\rho\sigma}})}_{0}$

$= 0$

k) $\mathrm{Tr}(\gamma_5\not a\not b\not c\not d) = \overbrace{a_\mu b_\nu c_\rho d_\sigma}^{\equiv A}\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$

+ fácil si no desarrollas este término $\gamma^\rho\gamma^\sigma$ ↳ ver 33

$= A\,\mathrm{Tr}(\gamma_5\cdot(g^{\mu\nu} - i\sigma^{\mu\nu})(g^{\rho\sigma} - i\sigma^{\rho\sigma}))$

$= A\cdot(g^{\mu\nu}g^{\rho\sigma}\mathrm{Tr}(\overset{0}{\gamma_5}) - ig^{\mu\nu}\mathrm{Tr}(\overset{0\ \ldots\text{ver j)}}{\gamma_5\sigma^{\rho\sigma}}) - ig^{\rho\sigma}\mathrm{Tr}(\overset{0}{\gamma_5\sigma^{\mu\nu}}) - \overset{\text{ver 9a)}}{\mathrm{Tr}(\gamma_5\sigma^{\mu\nu}\sigma^{\rho\sigma})})$

$= -A\cdot\frac{i}{2}\mathrm{Tr}(\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\sigma^{\rho\sigma}) = -\frac{iA}{2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(\sigma^{\delta\epsilon}\sigma^{\rho\sigma})\cdot g_{\alpha\delta}g_{\beta\epsilon}$

$= \left(\frac{i}{2}\right)^2\left(-\frac{i}{2}\right)A\cdot\epsilon^{\mu\nu\alpha\beta}g_{\alpha\delta}g_{\beta\epsilon}\mathrm{Tr}(\gamma^\delta\gamma^\epsilon\gamma^\rho\gamma^\sigma - \gamma^\epsilon\gamma^\delta\gamma^\rho\gamma^\sigma - \gamma^\delta\gamma^\epsilon\gamma^\sigma\gamma^\rho + \gamma^\epsilon\gamma^\delta\gamma^\sigma\gamma^\rho$

$\overset{\to\text{ 11 a)}}{=} +\frac{i}{8}A\,\epsilon^{\mu\nu\alpha\beta}g_{\alpha\delta}g_{\beta\epsilon}\cdot 4\cdot(\cancel{g^{\delta\epsilon}g^{\rho\sigma}} - g^{\delta\rho}g^{\epsilon\sigma} + g^{\delta\sigma}g^{\epsilon\rho} - \cancel{g^{\epsilon\delta}g^{\rho\sigma}} + g^{\epsilon\rho}g^{\delta\sigma} - g^{\epsilon\sigma}g^{\delta\rho}$

$+ \cancel{g^{\delta\epsilon}g^{\sigma\rho}} + g^{\delta\sigma}g^{\epsilon\rho} - g^{\delta\rho}g^{\epsilon\sigma} + \cancel{g^{\epsilon\delta}g^{\sigma\rho}} - g^{\epsilon\rho}g^{\delta\sigma} + g^{\epsilon\sigma}g^{\delta\rho}) \overset{\to g\text{ simétrica}}{=}$

$= +\frac{i}{2}A\,\epsilon^{\mu\nu\alpha\beta}g_{\alpha\delta}g_{\beta\epsilon}\cdot 4\cdot(g^{\delta\sigma}g^{\epsilon\rho} - g^{\delta\rho}g^{\epsilon\sigma}) = +2iA\,\epsilon^{\mu\nu\alpha\beta}(g_\alpha^{\ \sigma}g_\beta^{\ \rho} - g_\alpha^{\ \rho}g_\beta^{\ \sigma})$

$= 2iA(\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma}) \overset{\to\text{antisim.}}{=} -4i\,a_\mu b_\nu c_\rho d_\sigma\,\epsilon^{\mu\nu\rho\sigma}$ ✓

e)

$\exists \quad \tilde{\gamma}^{\mu} = -\gamma^{\mu T} \quad / \quad \tilde{\gamma}^{\mu\dagger} = \tilde{\gamma}^0 \tilde{\gamma}^{\mu} \tilde{\gamma}^0 \quad ; \{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\} = 2g^{\mu\nu}$

$\exists \; C \; / \; \tilde{\gamma}^{\mu} = C \gamma^{\mu} C^{-1}$, con $CC^{-1} = I$

$$\mathrm{Tr}(\gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{2K}}) = \mathrm{Tr}((C^{-1} \tilde{\gamma}^{\mu_1} \underbrace{C)(C^{-1}}_{I} \tilde{\gamma}^{\mu_2} C) \dots C^{-1} \tilde{\gamma}^{\mu_{2K}} C)$$

→ prop. cíclica

$$= \mathrm{Tr}(\tilde{\gamma}^{\mu_1} \tilde{\gamma}^{\mu_2} \dots \tilde{\gamma}^{\mu_{2K}}) \overset{\to \mathrm{Tr} A = \mathrm{Tr}(A^T)}{=} \mathrm{Tr}(\tilde{\gamma}^{\mu_{2K} T} \dots \tilde{\gamma}^{\mu_2 T} \tilde{\gamma}^{\mu_1 T})$$

$$= (-1)^{2K} \mathrm{Tr}(\gamma^{\mu_{2K}} \dots \gamma^{\mu_1}) = \mathrm{Tr}(\gamma^{\mu_{2K}} \dots \gamma^{\mu_1})$$

C: Conjugación de carga $(\gamma^0 \gamma^2)$?

⑫
$\Gamma^n \equiv (I_4, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu})$: base Dirac 4x4
$\Gamma_n \equiv (I_4, -i\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu})$

a) $\gamma^0 (\Gamma^n)^+ \gamma^0 = \Gamma^n$

- $\gamma^0 I_4 \gamma^0 = I_4(\gamma_0)^2 = I_4$ ✓ $\quad (\gamma_0)^2 = 1$
- $\gamma^0(i\gamma_5)^+\gamma^0 = \gamma^0(-i\gamma_5)\gamma^0 = -i(-\gamma_5\gamma^0)\gamma^0 = i\gamma_5$ ✓ $\quad \gamma_5 = (\gamma_5)^+$, $\{\gamma_5, \gamma^\mu\} = 0$
- $\gamma^0(\gamma^\mu)^+\gamma^0 \rightarrow \mu = 0 \rightarrow (\gamma^0)^+ = \gamma^0 \rightarrow \gamma^0\gamma^0\gamma^0 = \gamma^0$

  $\rightarrow \mu = i = 1,2,3 \rightarrow \{\gamma^0, \gamma^i\} = 2g^{0i} = 0 \rightarrow \gamma^0 \gamma_i^+ \gamma^0 \Big\} \gamma^0\gamma^0\gamma^i\gamma^0\gamma^0 = \gamma^i$ ✓

  Por condición de H hermítico $\rightarrow \gamma^{i+} = \gamma^0\gamma^i\gamma^0$
- $\gamma^0(\gamma_5\gamma^\mu)^+\gamma^0 = \gamma^0(\gamma^\mu)^+(\gamma_5)^+\gamma^0 = \gamma^0(\gamma^0\gamma^\mu\gamma^0)\cdot\gamma_5\gamma^0 = \gamma^\mu\gamma^0\gamma_5\gamma^0$

  $= -\gamma^\mu\gamma_5\gamma^0\gamma^0 = \gamma_5\gamma^\mu$ ✓ $\quad \ldots \gamma^0(\gamma^\mu)^+\gamma^0 = \gamma^\mu \rightarrow \gamma^0 \cdot | \cdot \gamma^0 \rightarrow (\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$
- $\gamma^0(\sigma^{\mu\nu})^+\gamma^0 = \gamma^0\left[\frac{i}{2}[\gamma^\mu, \gamma^\nu]\right]^+\gamma^0 = -\frac{i}{2}\gamma^0((\gamma^\mu\gamma^\nu)^+ - (\gamma^\nu\gamma^\mu)^+)\gamma^0$

  $= -\frac{i}{2}\gamma^0(\gamma^{\nu+}\gamma^{\mu+} - \gamma^{\mu+}\gamma^{\nu+})\gamma^0 = -\frac{i}{2}\gamma^0(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0)\gamma^0$

  $= -\frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \sigma^{\mu\nu}$ ✓

Como $X \equiv \Gamma_{4\times4} = \sum_n c_n \cdot \Gamma^n$ y hay linealidad, $\gamma^0(\Gamma_{4\times4})^+\gamma^0 = \sum_n c_n^* \Gamma^n \neq \Gamma_{4\times4}$ si $\mathrm{Im}\{c_n\} \neq 0$
↳ matriz arbitraria 4x4.

$\mathrm{Tr}(\Gamma^n \Gamma_m) = 4\delta^n{}_m$

- $\mathrm{Tr}(I_4 I_4) = 4$
- $\mathrm{Tr}(i\gamma_5(-i\gamma_5)) = \mathrm{Tr}((\gamma_5)^2) = \mathrm{Tr}(I_4) = 4 \quad \gamma_5^2 = I_4$
- $\mathrm{Tr}(\gamma^\mu\gamma_\alpha) = \mathrm{Tr}(\gamma^\mu\gamma^\nu g_{\alpha\nu}) = \mathrm{Tr}((g^{\mu\nu} + i\sigma^{\mu\nu})g_{\alpha\nu}) = \mathrm{Tr}(g^{\mu\nu}g_{\alpha\nu})$

  $+ i g_{\alpha\nu}\mathrm{Tr}(\sigma^{\mu\nu}) = \mathrm{Tr}(I_4)\delta^\mu_\alpha + 0 = 4\delta^\mu_\alpha$

  ↳ Antisim, Tr(Antis) = 0 ya que $(\sigma^{\mu\nu})$ [illegible] $(\sigma^{\mu\nu})_{\alpha\alpha}$
- $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma_\alpha\gamma_5) = \mathrm{Tr}(-\gamma^\mu\gamma_5\gamma_\alpha\gamma_5) = \mathrm{Tr}(-\gamma^\mu\gamma_5(-\gamma_5\gamma_\alpha)) = \mathrm{Tr}(\gamma^\mu\gamma_\alpha) = 4\delta^\mu_\alpha$
- $\mathrm{Tr}(\sigma^{\mu\nu}\sigma_{\alpha\beta}) = g_{\alpha\gamma}g_{\beta\delta}\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\gamma\delta}) = -\frac{1}{4}g_{\alpha\gamma}g_{\beta\delta}\mathrm{Tr}([\gamma^\mu, \gamma^\nu]\cdot[\gamma^\gamma, \gamma^\delta])$

  $= -\frac{1}{4}g_{\alpha\gamma}g_{\beta\delta}\mathrm{Tr}((\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\cdot(\gamma^\gamma\gamma^\delta - \gamma^\delta\gamma^\gamma))$

  $= -\frac{1}{4}g_{\alpha\gamma}g_{\beta\delta}\cdot\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta - \gamma^\mu\gamma^\nu\gamma^\delta\gamma^\gamma - \gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\delta + \gamma^\nu\gamma^\mu\gamma^\delta\gamma^\gamma)$

  ↗ Ver 11a) $\qquad g^{\delta\gamma} \rightarrow$ simétrico

  $= -g_{\alpha\gamma}g_{\beta\delta}\cdot[\cancel{g^{\mu\nu}}g^{\gamma\delta} - g^{\mu\gamma}g^{\nu\delta} + g^{\mu\delta}g^{\nu\gamma} - (\cancel{g^{\mu\nu}}g^{\gamma\delta} - g^{\mu\delta}g^{\nu\gamma} + g^{\mu\gamma}g^{\nu\delta})$

  $- (\cancel{g^{\nu\mu}}g^{\gamma\delta} - g^{\nu\gamma}g^{\mu\delta} + g^{\nu\delta}g^{\mu\gamma}) + g^{\nu\mu}g^{\delta\gamma} - g^{\nu\delta}g^{\mu\gamma} + g^{\nu\gamma}g^{\mu\delta}]$

  $= -g_{\alpha\gamma}g_{\beta\delta}[-4g^{\mu\gamma}g^{\nu\delta} + 4g^{\mu\delta}g^{\nu\gamma}] = 4g_{\alpha\gamma}g_{\beta\delta}[+g^{\mu\gamma}g^{\nu\delta} - g^{\mu\delta}g^{\nu\gamma}]$.

  $= 4\cdot[g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha] = \begin{cases} 4 & \text{si } \mu=\alpha, \nu=\beta, \text{ con } \mu\neq\nu \\ 0 & \text{si } \mu\neq\alpha, \mu\neq\beta \\ -4 & \text{si } \mu=\beta, \nu=\alpha \text{ con } \mu\neq\nu \end{cases}$ ← permuta, antisimétrico

- $Tr(I_4 \cdot (-i\gamma_5)) = -i\, Tr(\gamma_5) = 0 = Tr(i\gamma_5)$
- $Tr(I_4 (+\gamma^\mu)) = Tr(\gamma^\mu) = 0$
- $Tr(\gamma_\mu \gamma_5) = Tr(-\gamma_5\gamma_\mu) = 0$
- $Tr(\sigma_{\mu\nu}) = 0$ (antis.
- $Tr(i\gamma_5\gamma_\mu\gamma_5) = -i\, Tr(\gamma_\mu) = 0$
- $Tr(i\gamma_5 \sigma_{\mu\nu}) = i g_{\alpha\mu} g_{\beta\nu} Tr(\gamma_5\sigma^{\alpha\beta}) \overset{\text{ver } 9a)}{=} \frac{i}{2} \cdot i\, g_{\alpha\mu} g_{\beta\nu} \epsilon^{\alpha\beta\rho\sigma}\, Tr(\sigma_{\rho\sigma}) = 0$
- $Tr(\gamma^\mu\gamma_\nu\gamma_5) = g_{\nu\alpha} Tr(\gamma^\mu\gamma^\alpha\gamma_5) \overset{11j)}{=} 0$
- $Tr(\gamma^\mu \sigma_{\alpha\nu}) = Tr(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho) \frac{i}{2} \cdot g_{\alpha\rho} g_{\nu\sigma} \overset{11c)}{=} 0$
- $Tr(\gamma_5\gamma^\mu\sigma_{\alpha\nu}) \overset{\text{nº impar matrices, 11c}}{=} 0$

→ Ya hemos comprobado todos los casos, por tanto $Tr(\rho^n \rho_m) = 4\delta^n{}_m$

b) $X = \sum_n c_n \cdot \rho^n$

$Tr(X\rho_m) = \sum_n c_n\, Tr(\rho^n\rho_m) \overset{\text{ver a)}}{=} \sum_n c_n \cdot 4 \cdot \delta^n{}_m = 4c_m$

↳ $X = \sum_n \frac{1}{4} Tr(X\rho_n) \cdot \rho^n$

c) Álgebra multiplicativa, conjunto cerrado si

$X \cdot Y = Z$ con X, Y, Z expandibles como 12 b)

→ Es suficiente (y necesario) que los productos de las matrices de la base $\rho^n \cdot \rho^m$ den ~~otra~~ matriz de la base (salvo una constante) $\rho^n\rho^m = a\rho^j$, $a \in \mathbb{C}$
↳ o una c. lineal $\rho^n\rho^m = a_j\rho^j$

Por tanto, hay que evaluar el cuadro:

direcc. multiplic. ρ^n •→	I_4	$i\gamma_5$	γ^ρ	$\gamma_5\gamma^\rho$	$\sigma^{\alpha\beta}$
I_4	I_4	$i\gamma_5$	γ^ρ	$\gamma_5\gamma^\rho$	$\sigma^{\alpha\beta}$
$i\gamma_5$	$i\gamma_5$	$-I_4$	$i\gamma_5\gamma^\rho$	$i\gamma^\rho$	$-\frac{1}{2}\epsilon^{\alpha\beta}{}_{\delta\epsilon}\sigma^{\delta\epsilon}$
γ^μ	γ^μ	$i\gamma^\mu\gamma_5$	$g^{\mu\rho} \cdot I_4 - i\sigma^{\mu\rho}$	$-g^{\mu\rho}\gamma_5 - \frac{1}{2}\epsilon^{\mu\rho}{}_{\delta\epsilon}\sigma^{\delta\epsilon}$	
$\gamma_5\gamma^\mu$	$\gamma_5\gamma^\mu$	$-i\gamma^\mu$	$g^{\mu\rho}\gamma_5 + \frac{1}{2}\epsilon^{\mu\rho}{}_{\delta\epsilon}\sigma^{\delta\epsilon}$	$-g^{\mu\rho} I_4 + i\sigma^{\mu\rho}$	
$\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$	$-\frac{1}{2}\epsilon^{\mu\nu}{}_{\delta\epsilon} \cdot \sigma^{\delta\epsilon}$			

$\gamma_5\gamma^\mu \cdot i\gamma_5 = -i(\gamma_5)^2\gamma^\mu = -i\gamma^\mu$

$\sigma^{\mu\nu} i\gamma_5 = i\gamma_5\sigma^{\mu\nu} \overset{9a)}{=} -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} = -\frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}\sigma^{\alpha\beta}$

$\gamma^\mu\gamma^\rho = g^{\mu\rho} - i\sigma^{\mu\rho} = \begin{cases} g^{\mu\rho} I_4 & \text{si } \mu = \rho \\ -i\sigma^{\mu\rho} & \text{si } \mu \neq \rho \end{cases}$

$\gamma_5\gamma^\mu\gamma^\rho = \gamma_5 g^{\mu\rho} - i\gamma_5\sigma^{\mu\rho} = \gamma_5 g^{\mu\rho} + \frac{1}{2}\epsilon^{\mu\rho}{}_{\delta\epsilon}\sigma^{\delta\epsilon} = \begin{cases} \gamma_5 g^{\mu\rho} & \text{si } \mu = \rho \\ \frac{\epsilon^{\mu\rho}{}_{\delta\epsilon}}{2}\sigma^{\delta\epsilon} & \text{si } \mu \neq \rho \end{cases}$

$\sigma^{\mu\nu}\gamma^\rho =$

$\gamma^\mu\gamma_5\gamma^\rho = -\gamma_5\gamma^\mu\gamma^\rho$; $\gamma_5\gamma^\mu\gamma_5\gamma^\rho = -\gamma^\mu\gamma^\rho$; $i\gamma_5\sigma^{\alpha\beta} = i\sigma^{\alpha\beta}\gamma_5$

1B) Camino más fácil sin truco: (Viene de página siguiente, salto)

$$2iK\,\Delta\omega_{\alpha\beta}\cdot(g^{\mu\alpha}\gamma^{\beta}-g^{\mu\beta}\gamma^{\alpha}) = 2iK(\Delta\omega^{\mu}{}_{\beta}\gamma^{\beta}+g^{\mu\beta}\gamma^{\alpha}\Delta\omega_{\beta\alpha})$$

$$= 2iK(\Delta\omega^{\mu}{}_{\beta}\gamma^{\beta}+\Delta\omega^{\mu}{}_{\alpha}\gamma^{\alpha}) = 2iK\cdot 2\Delta\omega^{\mu}{}_{\nu}\gamma^{\nu} = \Delta\omega^{\mu}{}_{\nu}\gamma^{\nu}$$

↳ $K = \frac{1}{4i} = -\frac{i}{4}$ ✓

↳ Paso a boost finito. $S(\Lambda) = \lim_{n\to\infty}\left(1-\frac{i}{4}\frac{\Omega_{\alpha\beta}\sigma^{\alpha\beta}}{n}\right)^{n} = \to m = \frac{n}{-\frac{i}{4}\Omega_{\alpha\beta}\sigma^{\alpha\beta}}$

$$= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{m}\right]^{-\frac{i}{4}\Omega_{\alpha\beta}\sigma^{\alpha\beta}} = \exp\left(-\frac{i}{4}\Omega_{\alpha\beta}\sigma^{\alpha\beta}\right)$$

Si boost a lo largo de z: $\Omega^{\mu}{}_{\beta} = \begin{pmatrix}\cosh\omega & 0 & 0 & \text{sh}\,\omega\\ & 1 & & \\ & & 1 & \\ \text{sh}\,\omega & 0 & 0 & \text{ch}\,\omega\end{pmatrix} \to \Omega_{\alpha\beta} = g_{\alpha\mu}\Omega^{\mu}{}_{\beta} = \begin{pmatrix}\text{ch}\,\omega & & & +\text{sh}\,\omega\\ & -1 & & \\ & & 1 & \\ -\text{sh}\,\omega & & & -\text{ch}\,\omega\end{pmatrix}$

↳ $\Omega_{\alpha\beta}\sigma^{\alpha\beta} = \Omega_{03}\sigma^{03} + \Omega_{30}\sigma^{30} = (\Omega_{03}-\Omega_{30})\sigma^{03} = 2\Omega_{03}\sigma^{03}$

$$S = 1-\frac{i}{2}\sinh\omega\cdot\sigma^{03} \simeq 1-\frac{i}{2}\omega\cdot\sigma^{03} = \exp\left(-\frac{i}{2}\omega\sigma^{03}\right) = \exp\left(\frac{\omega}{2}\Sigma\right)$$

$$\Sigma = -i\sigma^{03} = \begin{pmatrix}0 & \sigma^{3}\\ \sigma^{3} & 0\end{pmatrix} \to \Sigma^{2} = 1 \to S = 1 + \frac{\omega}{2}\Sigma + \frac{\omega^{2}}{2!\,4} + \frac{\omega^{3}}{3!}\cdot\Sigma + \dots$$

$$= \cosh\frac{\omega}{2} + \Sigma\sinh\frac{\omega}{2} \;//$$

$$\left.\begin{aligned}\cosh^{2}\frac{\omega}{2} &= \frac{1}{2}(1+\cosh\omega) = \frac{\gamma+1}{2} = \frac{E+m}{2m}\\ \sinh^{2}\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega-1) = \frac{\gamma-1}{2} = \frac{E-m}{2m}\end{aligned}\right\}\tanh^{2}\frac{\omega}{2} = \frac{E-m}{E+m} = \frac{E^{2}-m^{2}}{(E+m)^{2}} = \frac{\vec{k}^{2}}{(E+m)^{2}}$$

↳ $S(\Lambda) = \sqrt{\frac{E+m}{2m}}\left\{1+\frac{K}{E+m}\Sigma\right\} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}1 & 0 & \text{th}\,\omega/2 & 0\\ 0 & 1 & 0 & -\text{th}\,\omega/2\\ \text{th}\frac{\omega}{2} & 0 & 1 & 0\\ 0 & -\text{th}\,\omega/2 & 0 & 1\end{pmatrix}$ //

2d) Ver notas de teoría 2.4.3

$\psi'(x') = S(\Lambda)\psi(x)$

$S(\Lambda)^{-1} = \gamma^{0}S(\Lambda)^{\dagger}\gamma^{0}$ (2.54) ← ya que $S(\Lambda) = 1-\frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu}$

$S(\Lambda)^{\dagger} = 1+\frac{i}{4}\Delta\omega^{\mu\nu}\cdot\gamma^{0}\sigma_{\mu\nu}\gamma^{0}$

$S^{-1}(\Lambda) = S(-\Lambda) = 1+\frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu} = \gamma^{0}S^{\dagger}\gamma^{0}$

$\psi'^{\dagger}(x') = \psi^{\dagger}(x)S(\Lambda)^{\dagger} = \psi^{\dagger}(x)\gamma^{0}S(\Lambda)^{-1}\gamma^{0} \quad |\cdot\gamma^{0}$

$\bar{\psi}' = \bar{\psi}\cdot S(\Lambda)^{-1}$

$\bar{\psi}\Gamma^{n}\psi$ → transformación Lorentz

I_4: $\bar{\psi}'\psi' = \bar{\psi}S(\Lambda)^{-1}\cdot S(\Lambda)\psi(x) = \bar{\psi}\psi$ // escalar invariante

$i\gamma_5$: $i\bar{\psi}'\gamma_5\psi' = i\bar{\psi}S^{-1}\gamma_5 S\psi$

Como $[\gamma_5,\sigma^{\mu\nu}] = 0$ y $\gamma_5\cdot S(\Lambda_{\underset{\text{impropia}}{\overset{\text{propia}}{\pm}}}) = \pm S(\Lambda_{\pm})\gamma_5 = \det(\Lambda\pm)S(\Lambda\pm)\gamma_5$

↳ $i\bar{\psi}S^{-1}S\det(\Lambda)\gamma_5\psi = \det(\Lambda)\cdot\bar{\psi}(i\gamma_5)\psi$ // pseudo escalar

γ^{μ}: $\bar{\psi}'\gamma^{\mu}\psi' = \bar{\psi}S^{-1}\gamma^{\mu}S\psi \overset{\text{por ec. de Dirac}}{=} \Lambda^{\mu}{}_{\nu}\bar{\psi}\gamma^{\nu}\psi$ // Vector (ver 1B, $S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$

$\gamma_5\gamma^{\mu}$: $\bar{\psi}'\gamma_5\gamma^{\mu}\psi' = \bar{\psi}S^{-1}\gamma_5\gamma^{\mu}S\psi = \bar{\psi}S^{-1}\gamma_5\underbrace{SS^{-1}}_{1}\gamma^{\mu}S\psi = \bar{\psi}S^{-1}S\det(\Lambda)\cdot\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\psi$

$= \det(\Lambda)\cdot\Lambda^{\mu}{}_{\nu}(\bar{\psi}\gamma^{\nu}_{5}\psi)$ // Pseudovector (axial)

$\sigma^{\mu\nu}$: $\bar{\psi}'\sigma^{\mu\nu}\psi' = \bar{\psi}S^{-1}\sigma^{\mu\nu}S\psi = \bar{\psi}\frac{i}{2}(S^{-1}\gamma^{\mu}\overset{SS^{-1}}{\vee}\gamma^{\nu}S - S^{-1}\gamma^{\nu}\overset{SS^{-1}}{\vee}\gamma^{\mu}S)\psi$

$= \frac{i}{2}\bar{\psi}(\Lambda^{\mu}{}_{\alpha}\gamma^{\alpha}\cdot\Lambda^{\nu}{}_{\beta}\gamma^{\beta} - \Lambda^{\nu}{}_{\beta}\gamma^{\beta}\Lambda^{\mu}{}_{\alpha}\gamma^{\alpha}) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi)$ // Tensor

⑬ Boost en dirección z ≡ rotac 4D

$$\Lambda(w) = \begin{pmatrix} \cosh w & & \sinh w \\ & I_2 & \\ \sinh w & & \cosh w \end{pmatrix} \qquad \begin{cases} \gamma \equiv \cosh w \\ \gamma\beta = \sinh w \\ \beta = \tanh w \end{cases}$$

Transformación infinitesimal : $\Lambda(w)^\mu{}_\nu = g^\mu{}_\nu + \Delta W^\mu{}_\nu$, $w \to 0$ o bien $w \equiv \delta w$

$\cosh w = 1 + \Theta(w)^2$

$\sinh w = w + \Theta(w)^3$ $\rightarrow \Lambda^0{}_3 = \Lambda^3{}_0 = w$

$$\Lambda^{\mu\nu} := g^{\nu\alpha}\Lambda^\mu{}_\alpha \rightarrow \Lambda^{03} = g^{3\alpha}\Lambda^0{}_\alpha = g^{33}\Lambda^0{}_3 = -\overset{=\Delta W^{03}}{\Lambda^0{}_3} = -w$$

$$\rightarrow \Lambda^{30} = g^{0\alpha}\Lambda^3{}_\alpha = g^{00}\Lambda^3{}_0 = \underset{\Delta W^{30}}{\underset{\shortparallel}{\Lambda^3{}_0}} = w$$

} se cumple que $\Delta W^{\mu\nu}$ es antisimétrico tal que $\Lambda^T g \Lambda = g$

$\Lambda_{03} = \Delta W_{03} = g_{0\alpha}\Delta W^\alpha{}_3 = \Delta W^0{}_3 = w$

$\Lambda_{30} = \Delta W_{30} = g_{3\alpha}\Delta W^\alpha{}_0 = g_{33}\Delta W^3{}_0 = -\Delta W^3{}_0 = -w$

$\Lambda = g + \Delta W$!

Por otro lado, buscamos :

$\psi'(x') = S(\Lambda)\psi(x)$; $\psi(x) = S(\Lambda^{-1})\psi'(x') = S^{-1}(\Lambda)\psi'(x')$

Usamos ecuación de Dirac:

$(i\gamma^\nu\partial_\nu - m)\psi(x) = 0$

$(i\gamma^\nu\partial_\nu - m)S^{-1}(\Lambda)\psi'(x') = 0$ $S\cdot|$

$(iS(\Lambda)\gamma^\nu\partial_\nu S^{-1}(\Lambda) - m)\psi'(x') = 0$

Aparte, $\partial_\nu = \Lambda^\mu{}_\nu\partial'_\mu$

↳ $(i\, S(\Lambda)\gamma^\nu S^{-1}(\Lambda)\Lambda^\mu{}_\nu\partial'_\mu - m)\psi'_{(x')} = 0$

Debe ser igual a la ecuación de Dirac en sistema prima:

$(i\gamma^\mu\partial'_\mu - m)\psi'(x')$

→ Por tanto: $\gamma^\mu = S(\Lambda)\gamma^\nu S^{-1}(\Lambda)\Lambda^\mu{}_\nu$

↳ $\Lambda^\mu{}_\nu\gamma^\nu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda)$

↙ $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \Delta W^\mu{}_\nu$ (infinitesimal)

$S = 1 + X$ (índices Lorentz cerrados)

$X \propto \Delta W$

$\downarrow$

$S^{-1} = 1 - X$ tal que $S \cdot S^{-1} \simeq 1$ (a orden X)

↳ $\gamma^\mu + \Delta W^\mu{}_\nu\gamma^\nu \simeq (1 - X)\gamma^\mu(1 + X)$

$\gamma^\mu + \Delta W^\mu{}_\nu\gamma^\nu = \gamma^\mu - X\gamma^\mu + \gamma^\mu X + \Theta(\Delta W^2)$

↳ $\Delta W^\mu{}_\nu\gamma^\nu = [\gamma^\mu, X]$

Como X debe ser $\propto \Delta W^\alpha{}_\beta$ ó $\Delta W_{\alpha\beta}$ y debe cerrar índices, debe estar multiplicado por otro elemento con dos índices, siendo $\sigma^{\mu\nu}$ el único candidato posible de la base Γ^μ. Por tanto, nuestro Ansatz es que $X \propto \Delta W_{\alpha\beta}\sigma^{\alpha\beta}$ $\Leftrightarrow X = K \cdot$ " "

$[\gamma^\mu, X] = K \cdot \Delta W_{\alpha\beta}[\gamma^\mu, \sigma^{\alpha\beta}] \overset{\text{ver } 9b)}{=} 2i(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha) \cdot K \cdot \Delta W_{\alpha\beta}$

Por otro lado, usamos un truco (idea feliz) para comparar los términos:

↳ " " $\gamma^\nu = \Delta W_{\alpha\nu}\gamma^\nu \cdot g^{\mu\alpha} \overset{\frac{1}{2}, \text{ sumandos}}{=} \frac{1}{2}g^{\mu\alpha}\Delta W_{\alpha\beta}\gamma^\beta + \frac{1}{2}g^{\mu\alpha}\Delta W_{\alpha\beta}\gamma^\beta$ ↳ cambio $\beta \leftrightarrow \alpha$

$= \frac{1}{2}g^{\mu\alpha}\Delta W_{\alpha\beta}\gamma^\beta + \frac{1}{2}g^{\mu\beta}\underset{\Delta W_{\mu\nu} = -\Delta W_{\nu\mu}}{\Delta W_{\beta\alpha}}\gamma^\alpha = \frac{1}{2}g^{\mu\alpha}\Delta W_{\alpha\beta}\gamma^\beta - \frac{1}{2}g^{\mu\beta}\gamma^\alpha\Delta W_{\alpha\beta}$

↳ Por tanto, $K = -\frac{i}{4}$ //

→ $S = 1 - \frac{i}{4}\Delta W_{\alpha\beta}\sigma^{\alpha\beta}$ → Sigue en página anterior (19)

⑭ $u_1(\vec{0}) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad v_1(\vec{0}) = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$u_2(\vec{0}) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\quad v_2(\vec{0}) = \sqrt{2m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$$S(\Lambda) \overset{\nearrow \text{ver } 13}{=} \exp\left(-\frac{i}{4}\Omega_{\alpha\beta}\sigma^{\alpha\beta}\right)$$

$$\Lambda^{\mu}{}_{\beta} \overset{\nearrow \text{ver } 2}{=} \ldots \rightarrow \Lambda_{\alpha\beta} = g_{\mu\alpha}\Lambda^{\mu}{}_{\beta} \Rightarrow \Lambda^{\mu}{}_{\beta} = g^{\mu}{}_{\beta} + \Delta\omega^{\mu}{}_{\beta} \rightarrow \Lambda_{\alpha\beta} = g_{\alpha\beta} + \Delta\omega_{\alpha\beta}$$

$$\Lambda_{\alpha\beta} = \begin{bmatrix} 1+(\gamma-1) & \gamma\beta_x & \gamma\beta_y & \gamma\beta_z \\ -\gamma\beta_x & -1-(\gamma-1)\beta_x^2/\beta^2 & -(\gamma-1)\beta_x\beta_y/\beta^2 & -(\gamma-1)\beta_x\beta_z/\beta^2 \\ -\ \text{idem (ver 2)} & - & - & - \\ - & - & - & - \end{bmatrix}$$

$$\hookrightarrow \Omega_{\alpha\beta}\sigma^{\alpha\beta} = \Omega_{01}\sigma^{01} + \underbrace{\Omega_{10}}_{=-\Omega_{01}}\underbrace{\sigma^{10}}_{=-\sigma^{01}} + \ldots$$

$$= 2\,\Omega_{01}\sigma^{01} + 2\,\Omega_{02}\sigma^{02} + 2\,\Omega_{03}\sigma^{03} + 2\,\Omega_{12}\sigma^{12} + 2\,\Omega_{13}\sigma^{13}$$
$$+ 2\,\Omega_{23}\sigma^{23} = \ldots \text{ demasiado largo}$$

Mejor:

$$S = \exp\left(\frac{\omega}{2}\Sigma\right) = \cosh\frac{\omega}{2} + \sinh\frac{\omega}{2}\,\Sigma \overset{\nearrow \text{ver 2)}}{=} \sqrt{\frac{E+m}{2m}}\cdot\left(1 + \frac{1}{E+m}\tilde{\Sigma}\right)$$

En este caso: $\tilde{\Sigma} = \begin{pmatrix} 0 & \vec{\sigma}\hat{k} \\ \vec{\sigma}\hat{k} & 0 \end{pmatrix}$ (no demuestro)

$$\hookrightarrow S(\Lambda) = \sqrt{\frac{E+m}{2m}}\cdot\begin{bmatrix} 1 & 0 & \frac{k_z}{E+m} & \frac{k_x-ik_y}{E+m} \\ 0 & 1 & \frac{k_x+ik_y}{E+m} & -\frac{k_z}{E+m} \\ \frac{k_z}{E+m} & \frac{k_x-ik_y}{E+m} & 1 & 0 \\ \frac{k_x+ik_y}{E+m} & -\frac{k_z}{E+m} & 0 & 1 \end{bmatrix}$$

$$\psi' = S\,\psi$$

$$\hookrightarrow u_1(\vec{k}) = S\,u_1(\vec{0}) = \sqrt{E+m}\cdot\begin{pmatrix} 1 \\ 0 \\ k_z/E+m \\ \frac{k_x+ik_y}{E+m} \end{pmatrix}$$

$$u_2(\vec{k}) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{k_x-ik_y}{E+m} \\ -\frac{k_z}{E+m} \end{pmatrix}$$

$$v_1(\vec{k}) = \sqrt{E+m}\begin{pmatrix} \frac{k_x-ik_y}{E+m} \\ -\frac{k_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

$$v_2(\vec{k}) = \sqrt{E+m}\begin{pmatrix} k_z/E+m \\ k_x+ik_y/E+m \\ 1 \\ 0 \end{pmatrix}$$

⑮

$$(i\not\partial - m)\psi = 0 \to \cancel{\psi}^{+}(i\gamma^\mu(\partial_\mu\psi) - m\psi) = 0 \qquad |^{+}$$

$$\hookrightarrow -i(\partial_\mu\psi^+)\gamma^{\mu+} - m\psi^+ = 0 \xrightarrow{\cdot(-\gamma^0)} i(\partial_\mu\psi^+)(\gamma^0\gamma^\mu\gamma^0)\cdot\gamma^0 + m\psi^+\gamma^0 = 0$$

$$\hookrightarrow i(\partial_\mu\bar\psi)\gamma^\mu + m\bar\psi = 0 \equiv \bar\psi(i\overleftarrow{\not\partial} + m) = 0$$

$\forall A^\mu, \not A, \psi_i, \bar\psi_i$: (truco) ... procedimiento de idea feliz

$$\bar\psi_2\not A\cdot(i\overrightarrow{\not\partial} - m)\psi_1 - \bar\psi_2(i\overleftarrow{\not\partial} + m)\not A\psi_1 = 0$$

$$\hookrightarrow -2m\bar\psi_2\not A\psi_1 + i\bar\psi_2(\not A\not\partial - \overleftarrow{\not\partial}\not A)\psi_1 = 0$$

¡Por otro lado:

$$\not a\not b = a_\mu b_\nu\gamma^\mu\gamma^\nu = a_\mu b_\nu(g^{\mu\nu} - i\sigma^{\mu\nu}) = a_\mu b^\mu - i a_\mu b_\nu\sigma^{\mu\nu}$$

$$\bar\psi_2\not A\psi_1 = \frac{i}{2m}\bar\psi_2(A_\mu\partial^\mu - iA_\mu\partial_\nu\sigma^{\mu\nu} - A_\mu\overleftarrow{\partial}^\mu + i\overleftarrow{\partial}_\mu A_\nu\sigma^{\mu\nu})\psi_1$$

$$= \frac{i\bar\psi_2}{2m}(A_\mu\overleftrightarrow{\partial}^\mu)\psi_1 + \frac{1}{2m}\bar\psi_2 A_\mu\overset{\sigma^{\mu\nu}}{\partial_\nu}(\bar\psi_1)\cancel{\sigma^{\mu\nu}} - \frac{1}{2m}\partial_\mu(\bar\psi_2)A_\nu\sigma^{\mu\nu}\psi_1$$

$\not A = A_\nu\gamma^\nu \to$ Si $A_\nu = \delta_{\mu\nu} \to \not A = \gamma^\mu$ $\qquad / \ \sigma^{\mu\nu} = -\sigma^{\nu\mu}$

$$\bar\psi_2\gamma^\mu\psi_1 = \frac{i}{2m}(\bar\psi_2\overleftrightarrow{\partial}^\mu\psi_1) + \frac{1}{2m}\bar\psi_2\overset{\sigma^{\mu\nu}}{\partial_\nu}(\bar\psi_1) + \frac{1}{2m}\partial_\mu(\bar\psi_2)\sigma^{\nu\mu}\psi_1$$

$\mu \leftrightarrow \nu$

$$= \frac{i}{2m}(\bar\psi_2\overleftrightarrow{\partial}^\mu\psi_1) + \frac{1}{2m}(\bar\psi_2\sigma^{\mu\nu}\partial_\nu(\psi_1) + \partial_\nu(\bar\psi_2)\sigma^{\mu\nu}\psi_1)$$

$$= \frac{i}{2m}(\bar\psi_2\overleftrightarrow{\partial}^\mu\psi_1) + \frac{1}{2m}\partial_\nu(\bar\psi_2\sigma^{\mu\nu}\psi_1) \qquad |\cdot g_{\alpha\mu}$$

$$\bar\psi_2\gamma_\alpha\psi_1 = \frac{i}{2m}(\bar\psi_2\overleftrightarrow{\partial}_\alpha\psi_1) + \frac{1}{2m}\partial_\nu(\bar\psi_2\sigma_\alpha{}^\nu\psi_1)$$

$\hookrightarrow = \sigma_{\alpha\beta}\cdot g^{\nu\beta}$

$$= \frac{i}{2m}(\bar\psi_2\overleftrightarrow{\partial}_\alpha\psi_1) + \frac{1}{2m}\partial^\beta(\bar\psi_2\sigma_{\alpha\beta}\psi_1) \quad //$$

descomposición de Gordon

⑯

$$\psi = \sum_{p,r}\left(u_r(\vec{p})\, a_r(\vec{p})\, e^{-ipx} + \bar{v}_r(\vec{p})\, b_r^+(\vec{p})\, e^{ipx}\right)$$

Queremos usar el resultado de problema 15

ψ_1, ψ_2 cumplen la ecuación de Dirac, pero no hace falta que sean el campo completo, sino la parte de u ó v que nos interesa.

$\psi_1 \sim u_r(\vec{k}_2)\, e^{-ik_2x}$ ↳ lo que también cumple la ec. de Dirac

$\psi_2 \sim u_s(\vec{k}_1)\, e^{ik_1x}$

$$\bar{u}_s(\vec{k}_1) e^{ik_1x}\gamma_\mu u_r(\vec{k}_2)e^{-ik_2x} = \frac{i}{2m}\left(\bar{u}_s(\vec{k}_1)e^{ik_1x}\, u_r(\vec{k}_2)(-ik_{2\mu})\, e^{-ik_2x} - \bar{u}_s(\vec{k}_1)(ik_{2\mu})\, u_r(\vec{k}_2)\overset{?}{e^{-ik_2x}}\right)$$

$$+\frac{1}{2m}\left(\bar{u}_s(\vec{k}_1)\, ik_1^\nu\, e^{ik_1x}\sigma_{\mu\nu}\, u_r(\vec{k}_2)e^{-ik_2x} + \bar{u}_s(\vec{k}_1)e^{ik_1x}\sigma_{\mu\nu}\, u_r(\vec{k}_2)e^{-ik_2x}(-ik_2^\nu)\right)$$

↓

$$\bar{u}_s(\vec{k}_1)\gamma_\mu u_r(\vec{k}_2) = \frac{1}{2m}\bar{u}_s(\vec{k}_1)(k_1+k_2)_\mu u_r(\vec{k}_2)$$

$$+\frac{i}{2m}\bar{u}_s(\vec{k}_1)(k_1-k_2)^\nu\sigma_{\mu\nu}\, u_r(\vec{k}_2)$$

$$= \frac{1}{2m}\bar{u}_s(\vec{k}_1)\cdot\left[(k_1+k_2)_\mu + i(k_1-k_2)^\nu\sigma_{\mu\nu}\right]\cdot u_r(\vec{k}_2)$$

Idem para:

$u_{r'}(\vec{k}')\gamma^\mu v_r(\vec{k}) = \ldots = \frac{1}{2m}\bar{u}_{r'}(\vec{k}')\cdot\left[(k'-k)^\mu + i(k'+k)_\nu\sigma^{\mu\nu}\right]v_r(\vec{k})$

$\bar{v}_{r'}(\vec{k}')\gamma^\mu u_r(\vec{k}) = \ldots = \frac{1}{2m}\bar{v}_{r'}(\vec{k}')\cdot\left[(k-k')^\mu - i(k'+k)_\nu\sigma^{\mu\nu}\right]u_r(\vec{k})$

$\bar{v}_{r'}(\vec{k}')\gamma^\mu v_r(\vec{k}) = \ldots = \frac{1}{2m}\bar{v}_{r'}(\vec{k}')\cdot\left[-(k+k')^\mu - i(k'-k)_\nu\sigma^{\mu\nu}\right]v_r(\vec{k})$

El resto de combinaciones se pueden obtener tomando el adjunto.

- Si $\vec{k} = \vec{k}'$, $k^0 = k^0 \to k'^\mu = k^\mu$

  $\bar{u}_{r'}(\vec{k})\gamma^\mu u_r(\vec{k}) = k^\mu/m\ \bar{u}_r(\vec{k})\, u_r(\vec{k}) \overset{*}{=} 2k^\mu\delta_{rr'}$

  $\bar{v}_{r'}(\vec{k})\gamma^\mu v_r(\vec{k}) = -k^\mu/m\ v_{r'}(\vec{k})\, v_r(\vec{k}) = 2k^\mu\delta_{rr'}$

  $\bar{u}_{r'}(\vec{k})\gamma^\mu v_r(\vec{k}) = ik_\nu/m\ u_{r'}(\vec{k})\sigma^{\mu\nu}v_r(\vec{k})$

  $\bar{v}_{r'}(\vec{k})\gamma^\mu u_r(\vec{k}) = -ik_\nu/m\, v_{r'}(\vec{k})\sigma^{\mu\nu}u_r(\vec{k})$

- Si $\vec{k} = -\vec{k}'$, $k^0 = k^0 \to (k+k')^\mu = 2k^0\delta^{\mu 0}$; $(k-k')^\mu = 2k^i\delta^{\mu i}$

  $\bar{u}_{r'}(-\vec{k})\gamma^\mu u_r(\vec{k}) = \frac{1}{m}\bar{u}_{r'}(-\vec{k})\cdot\left(k^0\delta^{\mu 0} - ik_j\delta^{\mu j}\right)u_r(\vec{k})$

  $\bar{v}_{r'}(-\vec{k})\gamma^\mu v_r(\vec{k}) = 1/m\cdot\bar{v}_{r'}(-\vec{k})\left(-k^0\delta^{\mu 0} + ik_j\delta^{\mu j}\right)v_r(\vec{k})$

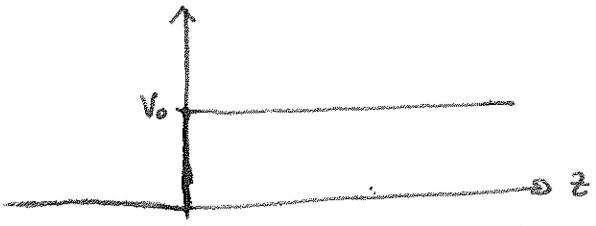
  $\bar{u}_{r'}(-\vec{k})\gamma^\mu v_r(\vec{k}) = 1/m\ \bar{u}_{r'}(-\vec{k})\cdot\left(-k^j\delta^{\mu j} + ik^0\delta^{\mu 0}\right)v_r(\vec{k})$

  $\bar{v}_{r'}(-\vec{k})\gamma^\mu u_r(\vec{k}) = 1/m\ \bar{v}_{r'}(-\vec{k})\cdot\left(k^j\delta^{\mu j} - ik^0\delta^{\mu 0}\right)u_r(\vec{k})$

  $\bar{u}_{r'}(-\vec{k})\gamma^0 v_r(\vec{k}) = 0 = \bar{v}_{r'}(-\vec{k})\gamma^0 u_r(\vec{k})$

⑰

$V(z) = V_0 \theta(z)$



$$(E-V)^2 = |\vec{k}|^2 + m^2$$

$z \equiv (t, 0, 0, z)$

$$\psi = \sum_r a_r(\vec{k}) \cdot u_r(\vec{k}) \cdot e^{-ikz} + b_r^{+}(\vec{k}) v_r(\vec{k}) \cdot e^{ikz}$$

↳ Supongamos que a y b no son operadores sino constantes

$\vec{k} = K_z \hat{z}$

$$u_1(K_z) \overset{\text{ver 14}}{=} \begin{pmatrix} 1 \\ 0 \\ K_z/E+m \\ 0 \end{pmatrix} \cdot \sqrt{E+m} \qquad v_1(K_z) = \begin{pmatrix} 0 \\ -K_z/E+m \\ 0 \\ 1 \end{pmatrix} \sqrt{E+m}$$

$$u_2(K_z) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -K_z/E+m \end{pmatrix} \cdot \sqrt{E+m} \qquad V_2(K_z) = \begin{pmatrix} K_z/E+m \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \sqrt{E+m}$$

↓ Analizaremos qué pasa si no se sigue el formalismo de Vs, a, b (como se hace en FQ)

Para $z < 0$

viene de $-\infty$, $\nexists\, e^{-ikz}$, única polarizac $s = +\frac{1}{2}$

$$\psi_{inc} = a_1 V_1(K_z) e^{-iEt} e^{+iKz} \doteq a \cdot \begin{pmatrix} 1 \\ 0 \\ K_z/E+m \\ 0 \end{pmatrix} \cdot e^{-iEt+iK_z z} \quad (a \text{ engloba las ctes})$$

puede aparecer nuevas polarizacs, va a la izqda → e^{-ikz}

$$\psi_{ref} = b_1 u_1 e^{-iEt} e^{-ik_z z} + c_2 u_2 e^{-iEt} e^{ik_z z} = e^{-iEt-ik_z z} \cdot \left(b \begin{pmatrix} 1 \\ 0 \\ -K_z/E+m \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 0 \\ K_z/E+m \end{pmatrix} \right)$$

con $E = \sqrt{K_z^2 + m^2}$

Para $z > 0$

$$K_z'^2 = (E-V_0)^2 - m^2 = (E - V_0 - m)(E - V_0 + m)$$

$$\psi_{trans} = e^{-iEt} e^{-iK_z' z} \cdot \left\{ d \begin{pmatrix} 1 \\ K_z'/(E-V_0+m) \\ [illegible] \\ 1 \end{pmatrix} + e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -K_z'/(E-V_0+m) \end{pmatrix} \right\}$$

Continuidad en $z=0$, $\forall t$

↳ $a + b = d$
$c = e$

$ik_z(a - b) = -ik_z' \cdot d$

(18) $\mathcal{L}(\phi,\partial\phi) \to \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

a)

Ec. de Schrödinger: $E = \frac{\vec{p}^2}{2m} + V$

$i\partial^0\psi = \left(-\frac{\vec\nabla^2}{2m} + V\right)\psi \Rightarrow \left(i\frac{\partial}{\partial t} + \frac{\vec\nabla^2}{2m} - V\right)\psi = 0 \xrightarrow{V=V^*} \left(-i\frac{\partial}{\partial t} + \frac{\vec\nabla^2}{2m} - V\right)\psi^* = 0$

$\mathcal{L}(\psi, \psi^*, \partial_\mu\psi, \partial_\mu\psi^*)$ → supones que aparece ψ con ψ^* (inv. fase)

$-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} + \frac{\partial\mathcal{L}}{\partial\psi^*} = \left(i\frac{\partial}{\partial t} + \frac{\vec\nabla^2}{2m} - V\right)\psi$ → las segundas derivadas ($\vec\nabla^2$) tienen que ir con $\partial_\mu\frac{\partial\mathcal{L}}{\partial_\mu\psi^*}$

↳ $\frac{\partial\mathcal{L}}{\partial\psi^*} = i\dot\psi - V\psi \to \mathcal{L} = (i\dot\psi - V\psi)\psi^* + f(\partial_\mu\psi^*)$

↳ $\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\psi^*)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^*)} = -\frac{\partial_i\partial_i}{2m}\psi \to \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^*)} = 0$

↳ $\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^*)} = -\frac{\partial_i}{2m}\psi = \frac{\partial f}{\partial(\partial_i\psi^*)} \to f = -\frac{\partial_i\psi}{2m}\cdot\partial_i\psi^* = -\frac{\vec\nabla\psi\cdot\vec\nabla\psi^*}{2m}$

$\Rightarrow \mathcal{L} = i\psi^*\dot\psi - \frac{1}{2m}\vec\nabla\psi\cdot\vec\nabla\psi^* - V\cdot\psi\cdot\psi^*$ // $\doteq \psi^*\left(i\frac{\partial}{\partial t} + \frac{\vec\nabla^2}{2m} - V\right)\psi$

se diferencian en una derivada total, equivalente

$\vec\nabla(\psi^*\vec\nabla\psi) - \psi^*\vec\nabla^2\psi = \vec\nabla\psi\cdot\vec\nabla\psi^*$

b) • Ec. de Klein-Gordon real: $(\Box + m^2)\phi = (\partial_\mu\partial^\mu + m^2)\phi = 0$; $\phi = \phi^+$

↳ $\partial_\mu(\partial^\mu\phi) + m^2\phi = 0 \longleftrightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

Identificamos

$m^2\phi = -\frac{\partial\mathcal{L}}{\partial\phi} \to \mathcal{L} = -\frac{m^2\phi^2}{2} + f(\partial^\alpha\phi)$

$\frac{\partial f}{\partial_0\phi} = \partial_0\phi = \partial^0\phi \to f = \frac{1}{2}(\partial^0\phi)^2 = \frac{1}{2}\partial^0\phi\partial_0\phi$

$\frac{\partial f}{\partial_i\phi} = -\partial_i\phi \to f = -\frac{1}{2}(\partial_i\phi)^2 = \frac{1}{2}\partial_i\phi\partial^i\phi$

$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial f(\partial^\alpha\phi)}{\partial(\partial_\mu\phi)} = \partial^\mu\phi =$ [illegible] $\to f = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi$

$\Rightarrow \mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2$ //

• Ec. de K-G compleja: $(\Box + m^2)\phi = 0$; $(\Box + m^2)\phi^+ = 0$; $\phi \neq \phi^+$

2 ecs. de Euler-Lagrange: → Supones $\mathcal{L}(\phi, \phi^+, \partial_\mu\phi, \partial_\mu\phi^+)$

$\partial_\mu(\partial^\mu\phi) + m^2\phi = 0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^+)} - \frac{\partial\mathcal{L}}{\partial\phi^+}$ ‐ ‐ ‐ → y que es invariante de fase (aparece ϕ con ϕ^+)

$\partial_\mu(\partial^\mu\phi^+) + m^2\phi^+ = 0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi}$

derivas respecto a ϕ^+, te queda ϕ

↳ $\frac{\partial\mathcal{L}}{\partial\phi^+} = -m^2\phi \to \mathcal{L} = -m^2\phi\phi^+ + f(\partial\phi)$

$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^+)} = \partial^\mu\phi = \frac{\partial f}{\partial(\partial_\mu\phi^+)} \to f = \partial^\mu\phi\,\partial_\mu\phi^+ \Rightarrow \mathcal{L} = \partial^\mu\phi\,\partial_\mu\phi^+ - m^2\phi\phi^+$ //

↳ se cumple también en la 2ª ecuac. de E-L.

c) Ecua de Dirac : $(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$ $\qquad \psi \neq \bar{\psi}$

$\bar{\psi}(i\overleftarrow{\partial}_{\mu}\gamma^{\mu} + m) = 0$

↳ Hay dependencia lineal en derivadas : $\mathcal{L}(\psi, \bar{\psi}, \partial_{\mu}\psi)$

$$-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial_{\mu}\bar{\psi}}\right) + \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = i\gamma^{\mu}\partial_{\mu}\psi - m\psi$$

→ suponemos tb que ψ aparezca con $\bar{\psi}$

↳ Si $\frac{\partial\mathcal{L}}{\partial_{\mu}\bar{\psi}} = 0$ ↳ $\underline{\underline{\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi}}$

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial_{\mu}\psi}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = \partial_{\mu}(\bar{\psi}i\gamma^{\mu}) + m\bar{\psi} = 0 \rightarrow \bar{\psi}(i\overleftarrow{\partial}_{\mu}\gamma^{\mu} + m) = 0 \quad ✓$$

Se ha obtenido suponiendo asimetría de las derivadas de los campos en $\mathcal{L}$. Se puede obtener un lagrangiano simétrico (hermítico), descripción equivalente pues sólo difiere en una derivada total. ¡Noether

$$\rightarrow \mathcal{L} = \frac{1}{2}\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - \frac{1}{2}\bar{\psi}(i\gamma^{\mu}\overleftarrow{\partial}_{\mu} + m)\psi$$

d) Ec de Maxwell

$$\partial_{\mu}\tilde{F}^{\mu\nu} = 0$$
$$\partial_{\mu}F^{\mu\nu} = J^{\nu} \longrightarrow ; \quad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$
$$\partial_{\mu}\partial^{\mu}A^{\nu} - \partial_{\mu}\partial^{\nu}A^{\mu} - J^{\nu} = 0$$

$$\mathcal{L}(A^{\nu}, \partial^{\mu}A^{\nu})$$

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}^{\mu}A_{\nu}^{\mu})}\right) + \frac{\partial\mathcal{L}}{\partial A_{\nu}^{\mu}} = \partial_{\mu}\partial^{\mu}A^{\nu} - \partial_{\mu}\partial^{\nu}A^{\mu} - J^{\nu} \quad \rightarrow \frac{\partial\mathcal{L}}{\partial A_{\nu}} = -J^{\nu}$$

↳ $\mathcal{L} = -A_{\nu}J^{\nu} + f(\partial_{\mu}A_{\nu})$

no demuestro

↳ $$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} = -\partial^{\mu}A^{\nu} + \partial^{\nu}A^{\mu} = -F^{\mu\nu} = \frac{\partial f}{\partial(\partial_{\mu}A_{\nu})} \rightarrow f = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

⑲

$$Q = i \sum_j Q_j \int d^3x \left\{ \pi_j^+(x)\, \phi_j^+(x) - \pi_j(x)\, \phi_j(x) \right\}$$

Calculamos previamente los conmutadores que aparecerán:

$$[Q, \phi_i(y)]_{x^0=y^0} = i \int d^3x\, Q_j\, [\pi_j^+ \phi_j^+(x) - \pi_j \phi_j(x),\ \phi_i(y)]_{x^0=y^0}$$

$\hookrightarrow [AB, C] = ABC - CAB = A[B,C] + [A,C]B$
$\quad\quad - ACB + ACB$

$$= i\, Q_j \int d^3x \cdot \{ \pi_j^+ \cdot \underset{0}{[\phi_j^+(x), \phi_i(y)]_{x^0=y^0}} + \underset{0}{[\pi_j^+(x), \phi_i(y)]_{x^0=y^0}} \cdot \phi_j^+(x) - \pi_j(x) \cdot \underset{0}{[\phi_j(x), \phi_i(y)]_{x^0=y^0}}$$

$$- \underbrace{[\pi_j(x), \phi_i(y)]_{x^0=y^0}}_{= -i\delta^{(3)}(\vec{x}-\vec{y}) \cdot \delta_{ij}} \cdot \phi_j(x) = + i \int d^3x\, Q_j \cdot i\, \delta_{ij} \cdot \delta^{(3)}(\vec{x}-\vec{y})\, \phi_j(x)$$

$$= - Q_i\, \phi_i(y) \quad //$$

$Q^+ = Q$

$$[Q, \phi_i^+(y)] = Q \cdot \phi_i^+(y) - \phi_i^+(y)\, Q = (\phi_i Q^+)^+ - (Q^+ \phi_i)^+ = [\phi_i(y), Q]^+$$

$$= - [Q, \phi_i(y)]^+ = - (- Q_i \phi_i^+) = Q_i\, \phi_i^+(y) \quad //$$

$$\phi_i'(x) = e^{i\theta Q} \phi_i(x) e^{-i\theta Q} \overset{1^{\text{er}}\text{ orden}}{\simeq} (1 + i\theta Q)\, \phi_i(x)\, (1 - i\theta Q)$$

$$= \phi_i(x) + i\theta [Q, \phi_i(x)] + O(\theta^2) = \phi_i(x) \cdot (1 - i\theta Q_i)$$

$$\hookrightarrow \phi_i'(x) = \lim_{n\to\infty} \phi_i(x) \cdot \left(1 - \frac{\theta Q_i}{n}\right)^n = \phi_i(x) \left[\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m \right]^{-\theta Q_i} = \phi_i(x)\, e^{-i\theta Q_i} \quad //$$

Alternativamente, se puede demostrar con el teorema de Hadamard:

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \dots$$

En nuestro caso, $\lambda = i\theta$, $A \equiv Q$, $B = \phi_i(x)$

$[Q, \phi_i] = - Q_i \phi_i$

$[Q, [Q, \phi_i]] = - Q_i [Q, \phi_i] = (-Q_i)^2 \phi_i$

$\underbrace{[Q, [Q, \dots [Q, Q_i]]]}_{n} = (-Q_i)^n \phi_i \quad \to$ Para $n+1$: $\underbrace{[Q, [\dots]]}_{n} = (-Q_i)^n [Q, \phi_i] = (-Q_i)^{n+1} \phi_i$ ✓

$$\hookrightarrow e^{i\theta Q} \phi_i(x) e^{-i\theta Q} = \phi_i + i\theta(-Q_i)\phi_i + \frac{(i\theta)^2 (-Q_i)^2}{2!} \phi_i + \dots + \frac{(i\theta)^n}{n!} \cdot (-Q_i)^n \phi_i + \dots$$

$$= \phi_i(x) e^{-i\theta Q_i} \quad //$$

$\downarrow$ idem, por mismo procedimiento, o conjugando rela anterior

$$\phi_i'^+ = \phi_i^+(x) \cdot e^{i\theta Q_i} \quad //$$

⑳

$$\mathcal{M}^{ij} = \int d^3x \{ x^i T^{0j} - x^j T^{0i} + \pi_m \Sigma^{ij} \phi_m \}$$

$$[\mathcal{M}^{ij}, \phi_m(y)] = \int d^3x\, [x^i \pi_K \partial^j \phi_K(x) - x^j \pi_K \partial^i \phi_K(x) + \pi_m \Sigma^{ij} \phi_m(x), \phi_m(y)]$$

$$[\pi_K(x) \partial^j \phi_K(x), \phi_m(y)] = \pi_K(x) \cdot [\partial^j \phi_K(x), \phi_m(y)] + [\pi_K(x), \phi_m(y)] \partial^j \phi_K(x)$$

$$= \pi_K(x) \partial^j_x \underset{0}{[\phi_K(x), \phi_m(y)]}^{x^0=y^0} + (-i)\delta^{(3)}(\vec{x}-\vec{y}) \partial^j \phi_K(x) \cdot \delta_{Km}$$

$$= -i\delta^{(3)}(\vec{x}-\vec{y}) \partial^j \phi_K(x) \delta_{Km} = -i\delta^{(3)}(\vec{x}-\vec{y}) \partial^j \phi_m(x)$$

nota de índices

$$\hookrightarrow [\pi_m(x) \Sigma^{ij} \phi_m(x), \phi_m(y)] = \begin{cases} 0 \quad \text{en Klein-Gordon porque } \Sigma^{ij} = 0 \\ [\pi_{m\alpha}(x) \Sigma^{ij}_{\alpha\beta} \phi_m(x)_\beta, \phi_m(y) I_{4\gamma\delta}] \end{cases}$$

$$\hookrightarrow [AB, C] = \begin{matrix} ABC - CAB \\ +ACB - ACB \end{matrix} = A\{B,C\} - \underset{i\delta^{(3)}(\vec{x}-\vec{y})}{\overset{}{\{A,C\}}} B$$

$$= \begin{cases} 0 \\ -i\delta^{(3)}(\vec{x}-\vec{y}) \overset{\delta_{\alpha\gamma}}{\gamma} \Sigma^{ij}_{\alpha\beta} \phi_m(x)_\beta = -i\delta^{(3)}(\vec{x}-\vec{y}) \Sigma^{ij} \phi_m(x) \end{cases}$$

$$[\mathcal{M}^{ij}, \phi_m(y)] = \int d^3x\, \delta^{(3)}(\vec{x}-\vec{y}) \cdot \{ x^i (-i) \partial^j \phi_m(x) - x^j (-i) \partial^i \phi_m(x) - i \Sigma^{ij} \phi_m(x) \}$$

$$= -i \int d^3x\, \delta^{(3)}(\vec{x}-\vec{y}) \{ x^i \partial^j - x^j \partial^i + \Sigma^{ij} \} \phi_m(x)$$

$$= -i \{ y^i \partial^j - y^j \partial^i + \Sigma^{ij} \} \phi_m(y) \quad //$$

㉑ Demostrar que:

$$P^\mu = \sum_K K^\mu \{ N(\vec{K}) + \frac{1}{2} [a(\vec{K}), a^+(\vec{K})] \}$$

Partimos de: $P^\mu = \int d^3x (\pi_i \partial^\mu \phi_i - g^{0\mu} \mathcal{L})$; $\mathcal{L}_{KG\,\text{real}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

$$P^\mu = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial^\mu \phi - g^{0\mu} \frac{1}{2} \dot{\phi}^2 + \frac{(\vec{\nabla}\phi)^2}{2} g^{0\mu} + \frac{1}{2} m^2 \phi^2 g^{0\mu} \right)$$

$$= \int d^3x \left(\dot{\phi} \partial^\mu \phi - \quad '' \quad\quad '' \quad\quad '' \right)$$

$$P^0 = \int d^3x \left(\frac{\dot{\phi}^2}{2} + \frac{m^2 \phi^2}{2} - \frac{1}{2} \partial^i \phi \partial_i \phi \right) = \int d^3x \left(\dot{\phi}^2 + \frac{m^2 \phi^2}{2} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi \right)$$

$$\phi = \sum_K (a(\vec{K}) e^{-iKx} + a^+(\vec{K}) e^{iKx})$$

$$\dot{\phi} = \sum_K (-iK^0) (a(\vec{K}) e^{-iKx} - a^+(\vec{K}) e^{iKx})$$

$$\dot{\phi}^2 = \sum_{K,K'} (-K^0 K'^0) (a(\vec{K}) a(\vec{K}') e^{-i(K+K')x} - a(\vec{K}) a^+(\vec{K}') e^{-i(K-K')x} - a^+(\vec{K}) a(\vec{K}') e^{i(K-K')x} + a^+(\vec{K}) a^+(\vec{K}') e^{i(K+K')x}$$

$(2\pi)^3 \delta^{(3)}(\vec{K} \pm \vec{K}') \ldots \quad \sum_{K'} = \int \frac{d^3K'}{2K'^0} (2\pi)^{-3}$

$$\int d^3x\, \dot{\phi}^2 = \sum_K \frac{(2\pi)^3 (-K^0 K^0)}{(2\pi)^3\, 2K^0} \cdot (a(\vec{K}) a(-\vec{K}) e^{-i2K^0x^0} - a(\vec{K}) a^+(\vec{K}) - a^+(\vec{K}) a(\vec{K}) + a^+(\vec{K}) a^+(-\vec{K}) e^{i2K^0x^0})$$

↳ El $\frac{1}{(2\pi)^3 K^0}$ sale del $\sum_{K'}$

$$\phi^2 = \sum_{K,K'} (a(\vec{K}) a(\vec{K}') \cdot e^{-i(K+K')x} + a(\vec{K}) a^+(\vec{K}') e^{-i(K-K')x} + a^+(\vec{K}) a(\vec{K}') e^{i(K-K')x} + a^+(\vec{K}) a^+(\vec{K}') e^{i(K+K')x}$$

$$\int d^3x\, \phi^2 = (2\pi)^3 \sum_K [a(\vec{K}) a(-\vec{K}) e^{-2iK^0x^0} + a(\vec{K}) a^+(\vec{K}) + a^+(\vec{K}) a(\vec{K}) + a^+(\vec{K}) a^+(-\vec{K}) e^{2iK^0x^0}] / 2K^0 (2\pi)^3$$

$$\partial^\mu\phi\,\partial_\mu\phi = \sum_K [a(\vec{k})\,(-ik^\mu)e^{-ikx} + a^+(\vec{k})\,(ik^\mu)\,e^{ikx}]\cdot\sum_{K'}(-ik'_\mu)\,[a(\vec{k}')e^{-ik'x} - a^+(\vec{k}')e^{ik'x}]$$

$$= \sum_{K,K'}(-K^\mu K'_\mu)\,[a(\vec{k})\,a(\vec{k}')\,e^{-i(k+k')x} - a(\vec{k})a^+(\vec{k}')e^{-i(k-k')x} - a^+(\vec{k})\,a(\vec{k}')\cdot e^{i(k-k')x} + a^+(\vec{k})\,a^+(\vec{k}')\,e^{i(k+k')x}]$$

$$\int d^3x\,\partial^\mu\phi\,\partial_\mu\phi \overset{\delta(\vec{k}\pm\vec{k}')}{=} (2\pi)^3\sum_K \frac{1}{(2\pi)^3 K^0 2}\cdot[\,a(\vec{k})a(-\vec{k})e^{-i2k^0x^0}\cdot(-K^{0\,2}-\vec{k}^2) - a(\vec{k})\,a^+(\vec{k})\cdot(-K^{0\,2}+\vec{k}^2) - a^+(\vec{k})\,a(\vec{k}) + a^+(\vec{k})\,a^+(-\vec{k})\,e^{i2k^0x^0}\cdot(-K^{0\,2}-\vec{k}^2)\,]$$

$$\to \mathbb{P}^0 = \int d^3x\,\left(\dot\phi^2 + \frac{m^2\phi^2}{2} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi\right)$$

$$= \frac{\cancel{(2\pi)^3}}{(2\pi)^3}\sum_K\{\,a(\vec{k})a(-\vec{k})\cdot e^{-i2K^0x^0}\cdot\Big[-(K^0)^2 + \frac{\overset{(k^0)^2-\vec{k}^2}{m^2}}{2} + \frac{\vec{k}^2}{2} + \frac{(K^0)^2}{2}\Big] \to =0$$

lógico, no debe depender de x^0.

$$+ a^+(\vec{k})a^+(-\vec{k})e^{2iK^0x^0}\cdot\Big[-(k^0)^2 + \frac{m^2}{2} + \frac{\vec{k}^2}{2} + \frac{(k^0)^2}{2}\Big] \to =0$$

$$+\big[a(\vec{k})a^+(\vec{k}) + a^+(\vec{k})a(\vec{k})\big]\cdot\Big[(k^0)^2 + \frac{m^2}{2} + \underbrace{\frac{\vec{k}^2}{2} - \frac{(k^0)^2}{2}}_{=0}\Big]\Big\}\cdot\frac{1}{2k^0}$$

$$= \sum_K (K^0)^2\,[a^+a + [a,a^+] + a^+a]$$

$$= \sum_K \frac{1}{2k^0}\cdot(k^0)^2\,[2\,N(\vec{k}) + [a(\vec{k}),a^+(\vec{k})]]$$

$$\mathbb{P}^0 = \sum_K k^0\,[N(\vec{k}) + \frac{1}{2}[a,a^+]]$$

Para $\mu = i$:

$$\mathbb{P}^i = \int d^3x\,\dot\phi\cdot\partial^i\phi$$

$$\partial^i\phi = \sum_K (a(\vec{k})\cdot(-ik^i)e^{-ikx} + a^+(\vec{k})\,(ik^i)e^{ikx})$$

$$\dot\phi\cdot\partial^i\phi = \sum_{K,K'}(-iK^0)(-ik'^i)\,[a(\vec{k})\,a(\vec{k}')\,e^{-i(k+k')x} - a(\vec{k})a^+(\vec{k}')e^{-i(k-k')x} - a^+(\vec{k})\,a(\vec{k}')\,e^{i(k-k')x} + a^+(\vec{k})a^+(\vec{k}')\,e^{i(k+k')x}]$$

$$\int d^3x\,\dot\phi\,\partial^i\phi = (2\pi)^3\sum_K\frac{(-)\,K^0}{(2\pi)^3\cdot 2k^0}[\,a(\vec{k})\,a(-\vec{k})\cdot(-k^i)e^{-2ik^0x^0} - a(\vec{k})\,a^+(\vec{k})\cdot K^i - a^+(\vec{k})a(\vec{k})K^i + a^+(\vec{k})\,a^+(-\vec{k})(-k^i)\cdot e^{2ik^0x^0}\,]$$

$$\mathbb{P}^i = \sum_K\frac{1}{2}\cdot K^i\cdot\{\,aa^+ + a^+a + a(\vec{k})a(-\vec{k})e^{-2ik^0x^0} + a^+(\vec{k})a^+(-\vec{k})\,e^{2iK^0x^0}\}$$

$\to$ $K^i\cdot a(\vec{k})\cdot a(-\vec{k})$ es una función impar en K^i, ya que $[a(\vec{k}),a(-\vec{k})]=0$

$\to$ Por tanto, por simetría, esa integral se anula. Idem para a^+.

$\to f(-\vec{k}) = -f(\vec{k})$

$$\Rightarrow \mathbb{P}^i = \sum_K k^i\,\{N(\vec{k}) + \frac{1}{2}[a,a^+]\}$$

$\Downarrow$ juntando con $\mathbb{P}^0$

$$\mathbb{P}^\mu = \sum_K\cdot K^\mu\,[N(\vec{k}) + \frac{1}{2}[a(\vec{k}),a^+(\vec{k})]]\}$$

- $\mathcal{M}^{\mu\nu} = \int d^3x \cdot (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \equiv J^{0,\mu\nu}$ (tensor totalmente antisimétrico)

$$= \int d^3x \left(x^\mu \{ \pi_K \partial^\nu \phi_K - g^{0\nu} \mathcal{L} \} - x^\nu \{ \pi_K \partial^\mu \phi_K - g^{0\mu} \mathcal{L} \} \right)$$

$$\mathcal{M}^{00} = 0$$

$$\mathcal{M}^{0i} = -\mathcal{M}^{i0} = \int d^3x \left(x^0 T^{0i} - x^i T^{00}\right) \overset{\text{ver } P^i}{=} \int d^3x \left(x^0 \dot{\phi}\, \partial^i \phi - x^i (\dot{\phi}^2 + \frac{m}{2}\phi^2 - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi)\right)\Big)$$

$$= x^0 \underbrace{\int d^3x\, \dot{\phi}\, \partial^i \phi}_{P^i} - \int d^3x \cdot \underset{\frac{\partial}{\partial K_i} e^{iKx}}{\underset{\equiv}{x^i}}$$

$$\int e^{i(k-\vec{k}')} x^i d^3x = (2\pi)^3 \frac{\partial}{\partial k_i} \delta(\vec{k} - \vec{k}')$$

$$\int dk' f(k') \partial_{k'} = \partial_{k_i'} f(k')$$

㉒

$$\phi(x) = \sum_{K} \{ a(\vec{K}) e^{-iKx} + a^{+}(\vec{k}) e^{iKx} \}$$

$$[\phi(x), \phi(y)] = \sum_{K,K'} \{ [a(\vec{k}), a(\vec{k}')] \cdot e^{-ikx} e^{-ik'y} + [a(\vec{k}), a^{+}(\vec{k}')] \cdot e^{-ikx} e^{ik'y}$$
$$+ [a^{+}(\vec{k}), a(\vec{k}')] e^{ikx} e^{-ik'y} + [a^{+}(\vec{k}), a^{+}(\vec{k}')] \cdot e^{ikx} e^{ik'y} \}$$

$$[A, B] = AB - BA + BA - BA = \{A, B\} - 2BA$$

$$[\phi(x), \phi(y)] = \sum_{K,K'} \{ \overset{0}{\cancel{\{a, a\}}} \cdots - 2a(\vec{k}') a(\vec{k}) e^{-ikx} e^{-ik'y}$$
$$+ \underbrace{\{a(\vec{k}), a^{+}(\vec{k}')\}}_{\Delta \vec{k}\vec{k}'} \cdot e^{-ikx} e^{ik'y} - 2a^{+}(\vec{k}') a(\vec{k}) e^{-ikx} e^{ik'y}$$

$$\sum_{k'} \Delta \vec{k}\vec{k}' = 1$$

$$+ \overbrace{\{a^{+}(\vec{k}), a(\vec{k}')\}} e^{ikx} e^{-ik'y} - 2a(\vec{k}') a^{+}(\vec{k}) e^{ikx} e^{-ik'y}$$
$$+ \{a^{+}, \underset{0}{\overset{\shortparallel}{a^{+}}}\} \cdots \qquad - 2a^{+}(\vec{k}') a^{+}(\vec{k}) e^{ikx} e^{ik'y}]$$

$$= \sum_{K} \underbrace{\left(e^{-iK(x-y)} + e^{iK(x-y)} \right)}_{= \{\phi(x), \phi(y)\}} - 2\phi(y)\phi(x)$$

Para analizar microcausalidad hay que analizar el caso en que $x^0 - y^0 = 0$

$$[\phi(x), \phi(y)]_{x^0 = y^0} = \sum_{K} \underbrace{\left(e^{-i\vec{k}(\vec{x}-\vec{y})} + e^{i\vec{k}(\vec{x}-\vec{y})} \right)}_{2\cos(\vec{k}(\vec{x}-\vec{y}))} - 2\phi(0, \vec{y}) \cdot \phi(0, \vec{x})$$

lo cual es distinto de cero en general, pues no aparece la función $\Delta(x-y)$. → No satisface microcausalidad.

↓

Idem para

$$i(\Delta^{+}(x-y) - \Delta^{-}(x-y)) = i\Delta_1(x-y)$$

$$\{\phi(x), \phi(y)\} = \sum_{K} \left[e^{-ik(x-y)} + e^{ik(x-y)} \right] \neq i\Delta(x-y) = \sum_{K} \left[e^{-ik(x-y)} - e^{ik(x-y)} \right]$$

$$\hookrightarrow = i\Delta(x-y) + 2 \underbrace{\sum_{K} e^{iK(x-y)}}_{i\Delta^{-}(x-y)}$$

↳ Término extra que rompe la microcausalidad

↳ no se cancela al no haber simetría ± notando,

㉓

$$\psi(x) = \sum_{K,r}\left\{a_r(\vec{k})\,u_r(\vec{k})\,e^{-ikx} + b_r^+(\vec{k})\,V_r(\vec{k})\,e^{ikx}\right\}$$

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x)$$

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \bar{\psi}(x)\,i\gamma^0 = i\,\psi^+(x)$$

por ec. de Dirac ; $\mathcal{L}=0$ si campos son solución de la ecuación de Dirac

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi} - \overbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}^{=0} = i\psi^+\dot{\psi}$$

$$\psi^+(x) = \sum_{K,r}\left\{a_r^+(\vec{k})\,u_r^+(\vec{k})\,e^{ikx} + b_r(\vec{k})\,V_r^+(\vec{k})\,e^{-ikx}\right\}$$

$$\dot{\psi}(x) = \sum_{K,r}(-iK^0)\left\{a_r(\vec{k})\,u_r(\vec{k})\,e^{-ikx} - b_r^+(\vec{k})\,V_r(\vec{k})\,e^{ikx}\right\}$$

$$\begin{aligned}\mathcal{H} = i\psi^+\dot{\psi} = \sum_{K,K',r,r'}(K^{0\prime})\cdot\{ & a_r^+(\vec{k})a_{r'}(\vec{k}')\,u_r^+(\vec{k})u_{r'}(\vec{k}')\,e^{i(k-k')x}\\ & - a_r^+(\vec{k})b_{r'}^+(\vec{k}')\,u_r^+(\vec{k})V_{r'}(\vec{k}')\,e^{i(k+k')x}\\ & + b_r(\vec{k})a_{r'}(\vec{k}')\,V_r^+(\vec{k})u_{r'}(\vec{k}')\,e^{-i(k+k')x}\\ & - b_r(\vec{k})b_{r'}^+(\vec{k}')\,V_r^+(\vec{k})V_{r'}(\vec{k}')\cdot e^{-i(k-k')x}\}\end{aligned}$$

$\frac{1}{(2\pi)^3}\sum_{K'}K^{0\prime}$; $(2\pi)^3\,\delta^{(3)}(\vec{k}\pm\vec{k}') \sim 1/2$

$$\begin{aligned}H = \int d^3x\,\mathcal{H} = \sum_{K,r,r'}\frac{1}{2}\{ & a_r^+(\vec{k})a_{r'}(\vec{k})\,u_r^+(\vec{k})\overbrace{u_{r'}(\vec{k})}^{2K^0\delta_{rr'}} - b_r(\vec{k})b_{r'}^+(\vec{k})\,\overbrace{V_r^+(\vec{k})V_{r'}(\vec{k})}^{2K^0\delta_{rr'}}\\ & - \ldots\,\underbrace{u_r^+(\vec{k})V_{r'}(-\vec{k})}_{=0}\ldots + \ldots\underbrace{V_r^+(\vec{k})u_{r'}(-\vec{k})}_{0}\ldots\}\end{aligned}$$

$$= \sum_{K,r}K^0\left\{a_r^+(\vec{k})a_r(\vec{k}) - b_r(\vec{k})\,b_r^+(\vec{k}')\right\}$$

Generalizable a P^μ , con $H = P^0$

$\mathcal{L}=0$ si campo sol. ec. Dirac

$$P^\mu = \int d^3x\,(\pi\partial^\mu\psi - g^{0\mu}\mathcal{L}) = \int d^3x\;\pi\,\partial^\mu\psi$$

$$\partial^\mu\psi = \sum_{K,r}(-iK^\mu)\left\{a_r(\vec{k})\,u_r(\vec{k})\,e^{-ikx} - b_r^+(\vec{k})\,V_r(\vec{k})\,e^{ikx}\right\}$$

$\sum_{K'} \equiv \int\frac{1}{2K'^0}\,d^3k'\cdot\frac{1}{(2\pi)^3}$

$$P^\mu = \int d^3x\sum_{\substack{K,K'\\r,r'}}(K^{\mu\prime})\cdot\{\ldots\} = \sum_{K,r,r'}\frac{K^\mu}{2K^0}\cdot\left\{a_r^+a_{r'}\cdot 2K^0\delta_{rr'} - b_r b_{r'}^+\,2K^0\delta_{rr'}\right\}$$

$$= \sum_{K,r}K^\mu\left[a_r^+(\vec{k})a_r(\vec{k}) - b_r(\vec{k})\,b_r^+(\vec{k})\right]$$

$Q \Rightarrow \exists$ por invariancia de fase del Lagrangiano $\sim e^{iQ_i\theta}$ (ver 3.36 Teoría)

La corriente de Noether $j^\mu = i\sum_i Q_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i^+)}\psi_i^+ - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\psi_i\right)$

1 sólo campo ; 0 ← no se deriva ψ^+ sino ψ

$$\hookrightarrow Q = \int d^3x\,j^0 = i\int d^3x\,Q\cdot(\underset{0}{\cancel{\Pi^+}}\psi^+ - \pi\psi) = -iQ\int d^3x\;i\,\psi^+\psi$$

$$\begin{aligned}= Q\int d^3x\,\psi^+\psi = \sum_{\substack{K,K'\\r,r'}}\int d^3x\{ & a_r^+(\vec{k})a_{r'}(\vec{k}')\,u_r^+(\vec{k})u_{r'}(\vec{k}')\,e^{i(k-k')x}\\ & + a_r^+(\vec{k})b_{r'}^+(\vec{k}')\,u_r^+(\vec{k})V_{r'}(\vec{k}')\,e^{i(k+k')x}\\ & + b_r(\vec{k})a_{r'}(\vec{k}')\,V_r^+(\vec{k})u_{r'}(\vec{k}')\,e^{-i(k+k')x}\\ & + b_r(\vec{k})b_{r'}^+(\vec{k}')\,V_r^+(\vec{k})V_{r'}(\vec{k}')\,e^{-i(k-k')x}\}\end{aligned}$$

$$= \dots = \sum_{k,r} \left(a_r^+(\vec{k})\, a_r(\vec{k}) + b_r(\vec{k})\, b_r^+(\vec{k}') \right) \quad //$$

$$\downarrow$$

$$\frac{1}{2K^0}(2\pi)^3 \delta^{(3)}(\vec{k} \pm \vec{k}') \cdot \underbrace{u^{\pm\vec{k}} v^{\pm\vec{k}}}_{2K^0 \delta_{rr'} \text{ ó } 0}$$

a) Cuantizamos con conmutadores

$$[a_r(\vec{k}), a_{r'}^+(\vec{k}')] = \delta_{rr'} \Delta_{\vec{k}\vec{k}'} = [b_r(\vec{k}), b_{r'}^+(\vec{k}')]$$

El resto de combinaciones son cero. $\quad \hookrightarrow bb^+ = [b, b^+] + b^+ b$

$$P^\mu = \sum_{k,r} K^\mu \cdot \left(a^+ a - ([b, b^+] + b^+ b) \right) = \sum_{k,r} K^\mu \left(a^+ a - b^+ b - [b, b^+] \right)$$

$$:P^\mu: = P^\mu - \langle 0 | P^\mu | 0 \rangle = \sum_{k,r} K^\mu \left(a^+ a - b^+ b \right)$$

"redefines origen E, sólo mides diferencias respecto al vacío.

$\downarrow$

$\langle {a^+ a \atop b^+ b} \rangle = 0$

$\langle [b, b^+] \rangle = [b, b^+]$

$\hookrightarrow b^+ |0\rangle$ crea partículas con energía negativa

$P^0 < 0$

$\hookrightarrow \nexists$ límite inferior de E

$\hookrightarrow \mathcal{H}$ no está acotado inferiormente ni es definido positivo.

$$Q = Q \sum_{k,r} \left(a^+ a + [b, b^+] + b^+ b \right)$$

$$:Q: = Q \sum_{k,r} \left(a^+ a + b^+ b \right) \quad //$$

$\longrightarrow$ La carga siempre es positiva

$\hookrightarrow \nexists$ interpretación partícula-antipartícula

b) Con anticonmutadores

$$\{a_r(\vec{k}), a_{r'}^+(\vec{k}')\} = \delta_{rr'} \Delta_{\vec{k}\vec{k}'} = \{b_r(\vec{k}), b_{r'}^+(\vec{k}')\}$$

$\hookrightarrow bb^+ = \{b, b^+\} - b^+ b$

$$P^\mu = \sum_{k,r} K^\mu \left(a^+ a - (\{b, b^+\} - b^+ b) \right) = \sum_{k,r} \left(a^+ a + b^+ b - \{b, b^+\} \right)$$

$$:P^\mu: = \sum_{k,r} K^\mu \left(a^+ a + b^+ b \right) \longrightarrow$$

$\mathcal{H}$ definido positivo, todas partícula con $E \geq 0$; $K^0 > 0$

$\mathcal{H}$ acotado inferiormente, $\exists$ vacío $|0\rangle$ = estado de mínima E. ...

$$Q = Q \sum_{k,r} \left(a^+ a + \{b, b^+\} - b^+ b \right)$$

$$:Q: = Q \sum_{k,r} \left(a^+ a - b^+ b \right) \longrightarrow Q \gtrless 0 \rightarrow \exists \text{ interpretación partícula antipartícula}$$

$$a_r^+(\vec{k}) |0\rangle = |m, \vec{K}, r, +Q\rangle = \text{Partícula}$$

$$b_r^+(\vec{k}) |0\rangle = |m, \vec{K}, r, -Q\rangle = \text{Antipartícula} \quad //$$

㉔ $[j^\mu(x), j^\nu(y)] = 0$ si $(x-y)^2 < 0$

$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)$

En notación de índice spinorial:

$j^\mu(x) = \bar\psi_k(x)(\gamma^\mu)_{k\ell}\,\psi_\ell(x)$ → ya son todo "nºs", no matrices ni vectores, sino sus componentes, aunque en ψ_i hay operadores

↳ $[j^\mu(x), j^\nu(y)] = [\underbrace{\bar\psi_k(x)}\,\underbrace{(\gamma^\mu)_{k\ell}\psi_\ell(x)}, \underbrace{\bar\psi_m(y)}\,\underbrace{(\gamma^\nu)_{mn}\psi_n(y)}]$

Es una estructura tipo $[A \cdot B \;,\; C \quad D\;]$

↓

$[AB, CD] = ABCD - CDAB = ABCD + (ACBD - ACBD) - (ACDB - ACDB)$

$+ (CADB - CADB) - CDAB = A\{B,C\}D - AC\{B,D\} + \{A,C\}DB - C\{A,D\}B$ //

↓

En nuestro caso:

nº, no hay operador, sale del conmutador → Lo que no conmuta/anticonmuta son ψ_i porque incluyen operadores de creación-destrucción

$$[j^\mu(x), j^\nu(y)] = \bar\psi_k(x)\{(\gamma^\mu)_{k\ell}\psi_\ell(x), \bar\psi_m(y)\}(\gamma^\nu)_{mn}\psi_n(y)$$
$$- \bar\psi_k(x)\bar\psi_m(y)\{(\gamma^\mu)_{k\ell}\psi_\ell(x), (\gamma^\nu)_{mn}\psi_n(y)\}$$
$$+ \{\bar\psi_k(x), \bar\psi_m(y)\}(\gamma^\nu)_{mn}\psi_n(y)(\gamma^\mu)_{k\ell}\psi_\ell(x)$$
$$- \bar\psi_m(y)\{\bar\psi_k(x), (\gamma^\nu)_{mn}\psi_n(y)\}(\gamma^\mu)_{k\ell}\psi_\ell(x)$$

→ $\{A,B\} = \{B,A\}$

Las γ^μ_{ij} salen del anticonmutador. Utilizamos las propiedades dadas en teoría:

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\bar\psi_\alpha(x), \bar\psi_\beta(y)\} = 0$$
$$\{\psi_\alpha(x), \bar\psi_\beta(y)\} = i\,(i\not\partial_x + m)_{\alpha\beta}\,\Delta(x-y)$$

↳ $[j^\mu(x), j^\nu(y)] = \bar\psi_k(x)(\gamma^\mu)_{k\ell}\cdot i\,(i\not\partial_x + m)_{\ell m}(\gamma^\nu)_{mn}\psi_n(y)\cdot\Delta(x-y)$

$- 0 + 0$

$- \bar\psi_m(y)(\gamma^\nu)_{mn}\, i\,(i\not\partial_y + m)_{nk}(\gamma^\mu)_{k\ell}\psi_\ell(x)\,\Delta(y-x)$

Reagrupo términos para recuperar el producto y contraer índices

$$= \bar\psi(x)\gamma^\mu(i\not\partial_x + m)\gamma^\nu\psi(y)\cdot i\Delta(x-y)$$
$$- \bar\psi(y)\gamma^\nu(i\not\partial_y + m)\gamma^\mu\psi(x)\cdot i\Delta(y-x)$$

→ $\Delta(x-y)$ y $\Delta(y-x)$ se anulan si $(x-y)^2 < 0$

↳ Por tanto $[j^\mu(x), j^\nu(y)] = 0$ " "

↳ Se cumple microcausalidad.

Esto se puede generalizar a cualquier observable, donde por invariancia Lorentz los campos fermiónicos aparecen a pares (bilineales de Dirac): $O_i(x) \equiv \overline{\psi}(x)\Gamma^i \psi(x)$

donde Γ^i es una matriz 4x4 arbitraria, combinación lineal de la base Γ de 16 matrices: $\Gamma = \{I_4, \gamma^\mu, i\gamma_5, \gamma_5\gamma^\mu, \sigma^{\mu\nu}\}$

La demostración es equivalente, pues no hemos usado ninguna propiedad de γ^μ que la distinga de las restantes de Γ.

↳ Hay que sustituir simplemente $(\gamma^\mu)_{kl}$ por $(\Gamma^i)_{kl}$, con $i=1,\dots 16$ o una combinación lineal de éstas. (por linealidad conmutador, también se cumple)

Por completitud demuestro las propiedades utilizadas para calcular los anticonmutadores:

$$\psi_\alpha(x) \equiv \sum_{k,r} \{a_r(\vec{k}) u_{r\alpha}(\vec{k}) e^{-ikx} + b_r^+(\vec{k}) v_{r\alpha}(\vec{k}) e^{ikx}\}$$

$$\overline{\psi}_\beta(x) = \sum_{k',r'} \{a_{r'}^+(\vec{k}') \overline{u}_{r'\beta}(\vec{k}') e^{ik'x} + b_{r'}(\vec{k}') \overline{v}_{r'\beta}(\vec{k}') e^{-ik'x}\}$$

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = \sum_{\substack{k,k'\\ r,r'}} u_{r\alpha}(\vec{k}) \overline{u}_{r'\beta}(\vec{k}') e^{-ikx} e^{ik'y} \cdot \{a_r(\vec{k}), a_{r'}^+(\vec{k}')\}$$
$$+ \text{''}\; u_{r\alpha}(\vec{k}) \overline{v}_{r'\beta}(\vec{k}') e^{-ikx} e^{-ik'y} \{a_r(\vec{k}), b_{r'}(\vec{k}')\}$$
$$+ \text{''}\; v_{r\alpha}(\vec{k}) \overline{u}_{r'\beta}(\vec{k}') e^{ikx} e^{ik'y} \{b_r^+(\vec{k}), a_{r'}^+(\vec{k}')\}$$
$$+ \text{''}\; v_{r\alpha}(\vec{k}) \overline{v}_{r'\beta}(\vec{k}') e^{ikx} e^{-ik'y} \{b_r^+(\vec{k}), b_{r'}(\vec{k}')\}$$

$= \{b_{r'}(\vec{k}'), b_r^+(\vec{k})\}$

Las condiciones canónicas de cuantización implican:

$\{a_r(\vec{k}), a_s^+(\vec{k}')\} = \{b_r(\vec{k}), b_s^+(\vec{k}')\} = \delta_{rs}\Delta_{\vec{k}\vec{k}'}$, resto de combinds $= 0$

donde $\sum_{k,k'} f(\vec{k}')\Delta_{\vec{k}\vec{k}'} = \sum_k f(\vec{k})$

↳

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = \sum_{k,r} u_{r\alpha}(\vec{k}) \overline{u}_{r\beta}(\vec{k}) e^{-ik(x-y)} + 0 + 0$$
$$+ \sum_{k,r} v_{r\alpha}(\vec{k}) \overline{v}_{r\beta}(\vec{k}) \cdot e^{ik(x-y)}$$

Aparte, los espinores cumplen: $\left.\begin{array}{l} \sum_r u_r(\vec{k})\overline{u}_r(\vec{k}) = \not{k}+m \\ \sum_r v_r(\vec{k})\overline{v}_r(\vec{k}) = \not{k}-m \end{array}\right\}$ Identidad matricial

↳ $(\not{k}+m)_{\alpha\beta} = \sum_r u_{r\alpha}(\vec{k})\overline{u}_{r\beta}(\vec{k})$

$(\not{k}-m)_{\alpha\beta} = \sum_r v_{r\alpha}(\vec{k})\overline{v}_{r\beta}(\vec{k})$

↳ $$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = \sum_k \left[(\not{k}+m)_{\alpha\beta} e^{-ik(x-y)} + (\not{k}-m)_{\alpha\beta} e^{ik(x-y)}\right]$$

$\not{k} \equiv k_\mu\gamma^\mu$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ → $\partial_\mu e^{-ik(x-y)} = \partial_\mu e^{-ik_\mu(x^\mu - y^\mu)} = -ik_\mu e^{-ik(x-y)}$

→ $\partial_\mu e^{ik(x-y)} = + \text{''} = ik_\mu e^{ik(x-y)}$

$$\to \not\partial_x \equiv \gamma^\mu \partial_\mu \to \gamma^\mu \partial_\mu e^{\pm ik(x-y)} = \pm i \not k \, e^{\pm ik(x-y)} = \not\partial_x e^{\pm ik(x-y)}$$

$$\hookrightarrow \{ \psi_\alpha(x), \bar{\psi}_\beta(y) \} = \sum_K \left[(i\not\partial_x + m)_{\alpha\beta} e^{-ik(x-y)} + (-i\not\partial_x - m)_{\alpha\beta} e^{ik(x-y)} \right]$$

$$= (i\not\partial_x + m)_{\alpha\beta} \sum_K \left[e^{-ik(x-y)} - e^{ik(x-y)} \right] \equiv i S(x-y) \equiv i\,(i\not\partial_x + m)_{\alpha\beta} \Delta(x-y)$$

…Klein-Gordon

$$\hookrightarrow \Delta(x-y) \equiv i \sum_K \left[e^{ik(x-y)} - e^{-ik(x-y)} \right] \quad (= 0 \text{ si } (x-y)^2 < 0)$$

Aparte, $\{ \psi_\alpha(x), \psi_\beta(y) \}_{x^0 = y^0} = 0$ por condic. canónicas de cuantiz.

$$\psi_\alpha(x)\psi_\beta(y) + \psi_\beta(y)\psi_\alpha(x) = 0 \quad |^{+}$$

$$\psi_\beta^+(y)\psi_\alpha^+(x) + \psi_\alpha^+(x)\psi_\beta^+(y) = \{ \psi_\alpha^+(x), \psi_\beta^+(y) \} = 0$$

$$\bar{\psi}_\alpha = (\psi^+ \gamma^0)_\alpha = (\psi^+)_\beta (\gamma^0)_{\beta\alpha}$$

$$[(\gamma^0)_{\beta\alpha}]^2 = \delta_{\beta\alpha}$$

…nºs, no operadores, salen del anticonmutador

$$\hookrightarrow \{ \bar{\psi}_\alpha(x), \bar{\psi}_\beta(y) \} = \{ (\psi^+(x))_\delta (\gamma^0)_{\delta\alpha} , (\psi^+(y))_\epsilon (\gamma^0)_{\epsilon\beta} \}$$

$$= (\gamma^0)_{\delta\alpha} \gamma^0_{\epsilon\beta} \{ \psi^+_\delta(x), \psi^+_\epsilon(y) \} = 0$$

㉕

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi$$
$$\psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi$$

a) $\gamma_5 \psi_L = \frac{1}{2}(\gamma_5 - \overset{I}{\gamma_5^2})\psi = \frac{1}{2}(\gamma_5 - 1)\cdot\psi = -\frac{1}{2}(1-\gamma_5)\psi = -\psi_L$ ✓

$\gamma_5 \psi_R = \frac{1}{2}(\gamma_5 + \gamma_5^2)\psi = \frac{1}{2}(\gamma_5 + 1)\cdot\psi = \frac{1}{2}(1+\gamma_5)\psi = \psi_R$ ✓

b) $\psi' = S(\Lambda)\psi$

$\gamma_5 S(\Lambda) = +S(\Lambda)\gamma_5$ si $\mathcal{L}_+^\uparrow$ (ver 12 d))

$\psi_L' = \frac{1}{2}(1-\gamma_5)\psi' = \frac{1}{2}(1-\gamma_5)S(\Lambda)\psi = \frac{1}{2}(S(\Lambda)-\gamma_5 S(\Lambda))\psi$

$= \frac{1}{2}(S(\Lambda)-S(\Lambda)\gamma_5)\psi = S(\Lambda)\cdot\frac{1}{2}(1-\gamma_5)\psi = S(\Lambda)\psi_L$ ✓

↳ Por tanto se transforma independientemente de ψ_R.

Idem para R:

$\psi_R' = \frac{1}{2}(1+\gamma_5)S(\Lambda)\psi = S(\Lambda)\frac{1}{2}(1+\gamma_5)\psi = S(\Lambda)\psi_R$ ✓

c) $\mathcal{L} = \bar\psi(i\not\partial - m)\psi$

$$\left.\begin{array}{l}\psi_L + \psi_R = \psi \\ \overline{\psi_L} + \overline{\psi_R} = \bar\psi\end{array}\right\} \rightarrow \mathcal{L} = (\overline{\psi_L} + \overline{\psi_R})(i\not\partial - m)(\psi_L + \psi_R)$$

$\mathcal{L} = i\,\overline{\psi_L}\not\partial\psi_L + i\,\overline{\psi_R}\not\partial\psi_R - m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)$

$+ i\,\overline{\psi_L}\not\partial\psi_R + i\,\overline{\psi_R}\not\partial\psi_L - m(\overline{\psi_L}\psi_L + \overline{\psi_R}\psi_R)$

↳ $\overline{\psi_L}\psi_L = \frac{1}{4}\psi^+(1-\gamma_5)\gamma^0(1-\gamma_5)\psi = \frac{1}{4}\psi^+(\gamma^0 - \overset{=-\gamma^0\gamma_5}{\gamma_5\gamma^0})(1-\gamma_5)\psi$

$= \frac{1}{4}\bar\psi(1+\gamma_5)(1-\gamma_5)\psi = \frac{1}{4}\bar\psi(1-\underset{1}{\gamma_5^2})\psi = 0$

Idem con R:

$\overline{\psi_R}\psi_R = \frac{1}{4}\psi^+(1+\gamma_5)\gamma^0(1+\gamma_5)\psi = \frac{1}{4}\bar\psi(1-\gamma_5)(1+\gamma_5)\psi = 0$

↳ anticonmutan

$\not\partial\psi_R = \partial_\mu\gamma^\mu\frac{1}{2}(1+\gamma_5)\psi = \frac{1}{2}(\gamma^\mu + \gamma^\mu\gamma_5)\partial_\mu\psi = \frac{1}{2}(\gamma^\mu - \gamma_5\gamma^\mu)\partial_\mu\psi$

$= \frac{1}{2}(1-\gamma_5)\not\partial\psi$

↳ $\overline{\psi_L}\not\partial\psi_R = \psi^+\frac{1}{2}(1-\gamma_5)\gamma^0\frac{1}{2}(1-\gamma_5)\not\partial\psi = \frac{1}{4}\bar\psi\underbrace{(1+\gamma_5)(1-\gamma_5)}_{=0}\not\partial\psi = 0$

$\not\partial\psi_L = \gamma^\mu\partial_\mu\frac{1}{2}(1-\gamma_5)\psi = \frac{1}{2}(1+\gamma_5)\not\partial\psi$

$\overline{\psi_R}\not\partial\psi_L = \psi^+\frac{1}{2}(1+\gamma_5)\gamma^0\frac{1}{2}(1+\gamma_5)\not\partial\psi = \frac{1}{4}\bar\psi\overbrace{(1-\gamma_5)(1+\gamma_5)}^{=0}\not\partial\psi = 0$

⟹ Por tanto, $\mathcal{L} = i(\overline{\psi_L}\not\partial\psi_L + \overline{\psi_R}\not\partial\psi_R) - m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)$ ✓

d) $m=0$

$\hookrightarrow \mathcal{L} = i\bar{\psi}_L \not{\partial} \psi_L + i\bar{\psi}_R \not{\partial} \psi_R \rightarrow \mathcal{L} = 0$ si campos son solución de la ecuación de Dirac

$\mathcal{H} = \pi_L \cdot \dot{\psi}_L + \pi_R \dot{\psi}_R - \mathcal{L} = \pi_L \dot{\psi}_L + \pi_R \dot{\psi}_R$

$\pi_L = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_L} = i\bar{\psi}_L\gamma^0 = i\psi_L^+$

$\pi_R = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_R} = i\bar{\psi}_R\gamma^0 = i\psi_R^+$

$\hookrightarrow \mathcal{H} = i\psi_L^+\dot{\psi}_L + i\psi_R^+\dot{\psi}_R$

Alternativamente, $\mathcal{H} = i\psi^+\dot{\psi} = i(\psi_L^+ + \psi_R^+)(\dot{\psi}_L + \dot{\psi}_R)$

$= i\psi_L^+\dot{\psi}_L + i\psi_R^+\dot{\psi}_R + \frac{i}{4}\bar{\psi}\underbrace{(1-\gamma_5)(1+\gamma_5)}_{0}\dot{\psi} + \frac{i}{4}\bar{\psi}\underbrace{(1+\gamma_5)(1-\gamma_5)}_{0}\dot{\psi}$

$= i\psi_L^+\dot{\psi}_L + i\psi_R^+\dot{\psi}_R$ ✓ coincide!

Aparte, $\mathcal{L} = i\bar{\psi}_L\gamma^0\dot{\psi}_L + i\bar{\psi}_R\gamma^0\dot{\psi}_R + i\bar{\psi}_L\gamma^i\partial_i\psi_L + i\bar{\psi}_R\gamma^i\partial_i\psi_R \overset{\text{sol. ec. Dirac}}{=} 0$

$\Rightarrow i\psi_L^+\dot{\psi}_L + i\psi_R^+\dot{\psi}_R = -i(\bar{\psi}_L\gamma^i\partial_i\psi_L + \bar{\psi}_R\gamma^i\partial_i\psi_R)$

$\hookrightarrow \mathcal{H} = -i(\bar{\psi}_L\gamma^i\partial_i\psi_L + \bar{\psi}_R\gamma^i\partial_i\psi_R) =$ por ppo. de correspondencia, $p_i = i\partial_i$, $p^i = i\partial^i = -i\partial_i$

$= (\bar{\psi}_L\gamma^i p^i\psi_L + \bar{\psi}_R\gamma^i p^i\psi_R) = \bar{\psi}_L\vec{\gamma}\cdot\vec{p}\,\psi_L + \bar{\psi}_R\vec{\gamma}\cdot\vec{p}\,\psi_R$

$= \psi_L^+\gamma^0\cdot\vec{\gamma}\cdot\vec{p}\,\psi_L + \psi_R^+\gamma^0\vec{\gamma}\cdot\vec{p}\,\psi_R$

$\hookrightarrow \gamma^0\cdot\vec{\gamma} = \begin{pmatrix} I_2 & \\ & -I_2\end{pmatrix}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{pmatrix}$

$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix} \rightarrow \gamma_5\cdot\vec{\Sigma} \overset{?}{=}$

$\gamma_5\Sigma^k = i\gamma^0\gamma^1\gamma^2\gamma^3\Sigma^k$

$\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0\end{pmatrix} = \begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3\end{pmatrix}$

$\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0\end{pmatrix}\gamma^2\gamma^3 = \begin{pmatrix} 0 & -\sigma^1\sigma^2\sigma^3 \\ \sigma^1\sigma^2\sigma^3 & 0\end{pmatrix}$

$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & -\sigma^1\sigma^2\sigma^3 \\ -\sigma^1\sigma^2\sigma^3 & 0\end{pmatrix}$

$\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ +i & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0\end{pmatrix}$

$\hookrightarrow \gamma_5\cdot\vec{\Sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{pmatrix} = \gamma^0\vec{\gamma}$

$$\mathcal{H} = \psi_L^+ \gamma_5 \vec{\Sigma}\cdot\vec{p}\, \psi_L + \psi_R^+ \gamma_5 \vec{\Sigma}\cdot\vec{p}\, \psi_R$$

El operador helicidad es $\frac{1}{2}\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}$

Analicemos la expresión de H:

$(i\gamma^\mu\partial_\mu - m)\psi = 0 \xrightarrow[m=0]{} i\gamma^0\partial_0\psi = -i\gamma^i\partial_i\psi \to i\partial_0\psi = -i\gamma^0\gamma^i\partial_i\psi$

$\Vert$

$i|\dot{\psi}\rangle = H|\psi\rangle$

$\hookrightarrow H = -i\gamma^0\gamma^i\partial_i = \gamma^0\vec{\gamma}\cdot\vec{p} = \gamma_5\cdot\vec{\Sigma}\cdot\vec{p}$

$= \gamma_5\cdot|\vec{p}|\cdot\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} = |\vec{p}|\,\gamma_5\cdot 2\cdot\frac{\vec{\Sigma}}{2}\cdot\frac{\vec{p}}{|\vec{p}|}$

Cuando $m = 0 \to H = E = |\vec{p}|$; Si $|\psi_\pm\rangle$ es propio de helicidad con autovalor $\pm\frac{1}{2}$:

$H|\psi_\pm\rangle = |\vec{p}|\cdot\gamma_5\cdot(\pm 1)|\psi_\pm\rangle \overset{\checkmark}{=} |\vec{p}|\cdot|\psi_\pm\rangle$

¡idénticos salvo factor 2

Por tanto también es propia de quiralidad (γ_5) con autovalor ± 1

La quiralidad es invariante relativista, por lo que si $m = 0$, al compartir subespacios propios, no se podrá cambiar el signo de la helicidad (será también invariante).

Como ψ_L y ψ_R son propios de quiralidad, lo serán también de helicidad con el mismo signo en el autovalor y con factor 1/2.

$\hookrightarrow \gamma_5\psi_L = -\psi_L \quad\Longrightarrow\quad \frac{1}{2}\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\cdot\psi_L = -\frac{1}{2}\psi_L$

$\hookrightarrow \gamma_5\psi_R = +\psi_R \qquad\qquad \frac{1}{2}\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\psi_R = +\frac{1}{2}\psi_R$

$/\ H|\psi_{\substack{L\\R}}\rangle = |\vec{p}|\,\psi_{\substack{L\\R}}\rangle$

$\Longrightarrow$ ψ_L quiralidad y helicidad negativas. → representan fermiones
ψ_R " " " positivas.

El neutrino → quiralidad negativa $\xrightarrow{\text{si su } m=0 \text{ fuese}}$ → left-handed
antineutrino → " positiva → idem para su helicidad → right-handed

Si $m \neq 0$ → ∃ neutrino right-handed si aplicas un boost adecuado.
anti " left- "

$\Longrightarrow$ No observado! salvo en 10^{-10} de proporción, dentro del error

↳ quiralidad no comparte autoestados con helicidad y no es cte. de movimiento

e) Buscamos:

$$\mathcal{L} = (\bar\psi_L \ \ \bar\psi_R)\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \bar\psi_L(a\psi_L + b\psi_R) + \bar\psi_R(c\psi_L + d\psi_R)$$

$$= \bar\psi_L a \psi_L + \bar\psi_L b\psi_R + \bar\psi_R c\psi_L + \bar\psi_R d\psi_R \overset{\text{ver c)}}{=} i\bar\psi_L \not\partial \psi_L + i\bar\psi_R\not\partial\psi_R - m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)$$

$\hookrightarrow a = d = i\not\partial$

$b = c = -m$

$$\Rightarrow \mathcal{L} = (\bar\psi_L \ \ \bar\psi_R)\cdot\begin{pmatrix} i\not\partial & -m \\ -m & i\not\partial\end{pmatrix}\cdot\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \ //$$

Ecuac de movimiento:

$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \leftarrow \psi = \psi_L + \psi_R$

$$(1\ 1)\begin{pmatrix} i\gamma^\mu\partial_\mu - m & 0 \\ 0 & i\gamma^\mu\partial_\mu - m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0$$

(26)

$$\mathcal{L}_M = i\,\bar{\psi}_L \not{\partial}\, \psi_L - \frac{m}{2}\left(\overline{\psi_L^c}\, \psi_L + \bar{\psi}_L\, \psi_L^c\right)$$

$$\psi_L^c \equiv C\, \bar{\psi}_L{}^T$$

$$C\psi C = -i(\bar{\psi}\gamma^0\gamma^2)^T = -i\gamma^2(\psi^+)^T \;;\quad C\bar{\psi}C = -i(\gamma^0\gamma^2\psi)^T$$

$$C\gamma^\mu = (-\gamma^\mu)^T C \quad \text{ver M l)}$$

a)

$$\gamma_5\, \psi_L{}^c = \gamma_5 \cdot C \cdot \bar{\psi}_L{}^T = \gamma_5\, C \cdot (\psi^+_L\gamma^0)^T = \gamma_5\, C(\gamma^0)^T(\psi^+)^T \overset{\nearrow \gamma_5 = \gamma_5{}^T}{=}$$

$$= \gamma_5{}^T C (\gamma^0)^T (\psi_L^+)^T = i\,\gamma_3{}^T\gamma_2{}^T\gamma_1{}^T\gamma_0{}^T\, C\gamma^0 (\psi_L^+)^T = i\gamma_3{}^T\gamma_2{}^T\gamma_1{}^T(-C\gamma^0)\gamma^0(\psi_L^+)^T$$

$$= \ldots = C\, i\, \gamma_3\gamma_2\gamma_1\gamma^0\gamma^0(\psi_L^+)^T = C\gamma_5\gamma^0(\psi_L^*) = -C\gamma^0\gamma^5\psi_L^* \overset{\nearrow \text{ver 25a)}}{=} C\gamma^0\psi_L^*$$

$$= C(\gamma^0)^T(\psi_L^+)^T = C\,\bar{\psi}_L{}^T = \psi_L{}^c \;//\quad \rightarrow \text{spinor dextrógiro}$$

b) $\psi' = S(\Lambda)\cdot\psi$, con $\det(\Lambda) = +1$

$$\bar{\psi}_L'\not{\partial}'\psi_L' \overset{\nearrow \text{ver 12d)}}{=} \bar{\psi}_L'\, \partial'_\mu\gamma^\mu\, \psi_L' \overset{\nearrow \partial \text{ no afecta a } \gamma,\, S(\Lambda)}{=} \bar{\psi}_L\, S^{-1}\partial'_\mu\gamma^\mu S\,\psi \overset{\nearrow S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu}{=}$$

$$\bar{\psi}_L\,\partial'_\mu\,\Lambda^\mu{}_\nu\gamma^\nu\,\psi_L = \bar{\psi}_L\,\partial_\nu\gamma^\nu\,\psi_L = \bar{\psi}_L\not{\partial}\,\psi_L$$

$$\overline{\psi_L'^c}\,\psi_L' = C\,\bar{\psi}_L'^T\,\psi_L' = C(\bar{\psi}_L S^{-1})^T\cdot S\psi_L = C\gamma^0(\psi_L'^+)^T S\psi_L$$

$$\bar{\psi}_L'\,\psi_L'^c = \bar{\psi}_L S^{-1}\cdot C\,\bar{\psi}_L'^T = \bar{\psi}_L S^{-1} C (S^{-1}\bar{\psi}_L)^T$$

c) Análogo a 25

EXTRA (NO está en boletín)

Calcular $j^\mu(x)$, $\int d^3x\, j^\mu(x)$

$$\psi(x) = \sum_{k,r} a_r(\vec{k})\, u_r(\vec{k})\, e^{-ikx} + b_r^+(\vec{k})\, v_r(\vec{k})\, e^{ikx}$$

$$\bar\psi(x) = \psi^+\gamma^0 = \sum_{k',r'} a_{r'}^+(\vec{k}')\, \bar u_{r'}(\vec{k}')\, e^{ikx} + b_{r'}(\vec{k}')\, \bar v_{r'}(\vec{k}')\, e^{-ik'x}$$

$$j^\mu = \bar\psi\gamma^\mu\psi = \sum_{\substack{k,k'\\ r,r'}} \Big\{ a_{r'}^+(\vec{k}')\, a_r(\vec{k})\, \bar u_{r'}(\vec{k}')\gamma^\mu u_r(\vec{k})\, e^{-i(k-k')x} + a_{r'}^+(\vec{k}')\, b_r^+(\vec{k})\, \bar u_{r'}(\vec{k}')\gamma^\mu v_r(\vec{k})\, e^{i(k+k')x} + b_{r'}(\vec{k}')\, a_r(\vec{k})\, \bar v_{r'}(\vec{k}')\gamma^\mu u_r(\vec{k})\cdot e^{-i(k+k')x} + b_{r'}(\vec{k}')\, b_r^+(\vec{k})\, \bar v_{r'}(\vec{k}')\gamma^\mu v_r(\vec{k})\cdot e^{i(k-k')x} \Big\}$$

$\downarrow$

$\int d^3x \to$ saca $\delta^{(3)}(2\pi)^3$

$$\int d^3x\, j^\mu(x) = (2\pi)^3 \sum_{\substack{k,k'\\ r,r'}} \Big\{ T_1\cdot\delta^{(3)}(\vec{k}-\vec{k}') + T_2\,\delta^{(3)}(\vec{k}+\vec{k}') + T_3\cdot\delta^{(3)}(\vec{k}+\vec{k}') + T_4\,\delta^{(3)}(\vec{k}-\vec{k}') \Big\}$$

$\sum_{k'} = \left(\frac{1}{2\pi}\right)^3 \int \frac{d^3k'}{2k'^0}$ ✓ $k^0 = k^{0\prime}$ en todos casos

$$= \sum_{k,r,r'} \frac{1}{2k^0} \Big\{ a_{r'}^+(\vec{k})\, a_r(\vec{k})\, \bar u_{r'}(\vec{k})\gamma^\mu u_r(\vec{k})\cdot + a_{r'}^+(-\vec{k})\, b_r^+(\vec{k})\, \bar u_{r'}(-\vec{k})\gamma^\mu v_r(\vec{k})\, e^{i2k^0x^0} + b_{r'}(-\vec{k})\, a_r(\vec{k})\, \bar v_{r'}(-\vec{k})\gamma^\mu u_r(\vec{k})\cdot e^{-2ik^0x^0} + b_{r'}(\vec{k})\, b_r^+(\vec{k})\, \bar v_{r'}(\vec{k})\gamma^\mu v_r(\vec{k}) \Big\}$$

Usamos resultados de 16):

$$\int d^3x\, j^\mu(x) = \sum_{k,r,r'} \frac{1}{2k^0}\cdot\Big[a_{r'}^+(\vec{k})\, a_r(\vec{k})\cdot 2k^\mu\cdot\delta_{rr'} + b_{r'}(\vec{k})\, b_r^+(\vec{k})\cdot 2k^\mu\,\delta_{rr'} + b_{r'}(-\vec{k})\, a_r(\vec{k})\cdot \frac{1}{m}\, \bar v_{r'}(-\vec{k})\,(k^j\delta^{\mu j} - ik^0\delta^{\mu 0})\, u_r(\vec{k})\, e^{-2ik^0x^0} + a_{r'}^+(-\vec{k})\, b_r^+(\vec{k})\, \frac{1}{m}\, \bar u_{r'}(-\vec{k})\,(-k^j\delta^{\mu j} + ik^0\delta^{\mu 0})\, v_r(\vec{k})\, e^{2ik^0x^0} \Big]$$

$$= \sum_{k,r} \frac{k^\mu}{k^0}\Big(a_r^+(\vec{k})\, a_r(\vec{k}) - b_r^+(\vec{k})\, b_r(\vec{k}) + \{ b_r(\vec{k}), b_r^+(\vec{k}) \} \Big) + \sum_{k,r,r'} \frac{1}{2k^0 m}\Big(b_{r'}(-\vec{k})\, a_r(\vec{k})\, \bar v_{r'}(-\vec{k})\, u_r(\vec{k})\, e^{-2ik^0x^0} - a_{r'}^+(-\vec{k})\, b_r^+(\vec{k})\cdot \bar u_{r'}(-\vec{k})\, v_r(\vec{k})\cdot e^{2ik^0x^0} \Big)\cdot (k^j\delta^{\mu j} - ik^0\delta^{\mu 0})$$

//

$$Q = \int d^3x\, j^0(x) = \sum_{k,r} \Big(N_a(\vec{k}) - N_b(\vec{k}) + \{ b_r(\vec{k}), b_r^+(\vec{k}) \} \Big)$$

$$:Q: = \sum_{k,r} \big(N_a(\vec{k}) - N_b(\vec{k}) \big)$$

//

El segundo sumando. $\sum_{k,r,r'} \left(-\frac{i}{2m}\right)\cdot(\ldots)$ se anula por simetría al $\sum_k$

Es más fácil verlo con: $\bar u_{r'}(-\vec{k})\gamma^0 v_r(\vec{k}) = u_{r'}^+(-\vec{k})\, v_r(\vec{k}) \overset{\text{x relaciones entre spinors}}{=} 0$

$\bar v_{r'}(-\vec{k})\gamma^0 u_r(\vec{k}) = v_{r'}^+(-\vec{k})\, u_r(\vec{k}) = 0$

$$\int d^3x\, j^i(x) = \sum_{k\, r} \frac{k^i}{k^0}\big(N_a - N_b + \{\,\} \big) + \sum_{k,r,r'} (k^i)\cdot(\ldots)$$

//

(27) $\mathcal{L}_I = -\frac{g}{2}\Phi\varphi^2$ → campos reales → a, a^+

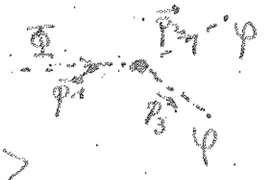
$$S = T\exp\left\{ i\int d^4x\,\mathcal{L}_I \right\} \simeq I - \frac{ig}{2}T\int d^4x\,\Phi\varphi^2(x) + O(g^2)$$

↓

Amplitud del proceso: $\langle f | : S - I : | i \rangle = -\frac{ig}{2}T\int d^4x : \Phi\varphi^2(x) : + \left(\frac{-ig}{2}\right)^2 \cdot \frac{1}{2!}T\int d^4x \int d^4y : \mathcal{L}_I(x) :: \mathcal{L}_I(y) :$

$= -i(2\pi)^4\delta^{(4)}(p_i^\mu - p_f^\mu)\mathcal{M} \equiv A$

a) $\Phi(\vec{p_1}) \to \varphi(\vec{p_2})\varphi(\vec{p_3})$



$i\langle 0 | T a_3^\varphi a_2^\varphi \left(-\frac{ig}{2}\right) : \Phi\varphi^2 : a_1^{\Phi +} | 0 \rangle$

$= \frac{-ig}{2}\int d^4x \langle 0 | a_3^\varphi a_2^\varphi : \Phi\varphi\varphi(x) : a_1^{\Phi +} | 0 \rangle \cdot 2$ ··· 3 y 2 podría contraerlos al revés con los 2 φs
↓
como son bosones, da igual el cruce de líneas.

$a(\vec{k})\varphi \doteq [a(\vec{k}), \varphi] = \sum_{k'} \Delta_{\vec{k}\vec{k}'} e^{ik'x} = e^{ikx}$ ··· salen

$a^+(\vec{k})\varphi \doteq [a^+(\vec{k}); \varphi] = \sum_{k'} \Delta_{\vec{k}\vec{k}'} e^{-ik'x} = e^{-ikx}$ ···> entran

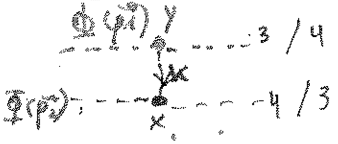
actuando sobre $|0\rangle$
$\langle 0|$

$\langle 0|0\rangle = 1$

↳ $A = -ig\int d^4x\, e^{-i(k_3 + k_2 - k_1)x} = -ig\,\delta^{(4)}(k_3^\mu + k_2^\mu - k_1^\mu)\cdot(2\pi)^4$

$\mathcal{M} = g$ //

b) $\Phi(\vec{p_1})\Phi(\vec{p_2}) \to \varphi(\vec{p_3})\varphi(\vec{p_4})$



$p \equiv k$

ir a 2º orden

$A = -\frac{g^2}{4}\iint \frac{1}{2!} T \langle 0 | a_4^\varphi a_3^\varphi : \Phi\varphi\varphi(x) :: \Phi\varphi\varphi(y) : a_2^{\Phi +} a_1^{\Phi +} | 0 \rangle\, d^4x\, d^4y$

fijo 1 en y

$= -\iint \frac{g^2}{8}\cdot 2 \cdot \langle 0 | a_4^\varphi a_3^\varphi : \Phi\varphi\varphi(x) :: \Phi\varphi\varphi(y) : a_2^{\Phi +} a_1^{\Phi +} | 0 \rangle \cdot 2 \cdot 2\, d^4x d^4y$ ··· coger propag. en x e y entre φs

$= -g^2\iint \langle 0 | a_4^\varphi a_3^\varphi : \varphi(x) :: \varphi(y) : \left(i\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \right) e^{-i(k_1 y + k_2 x)} \, 0 \rangle d^4x d^4y$

··· 2 topologías distintas

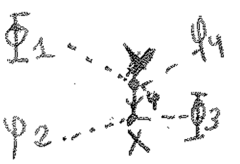
$$= -ig^2 \int \frac{d^4k}{(2\pi)^4} \int d^4x \int d^4y \frac{1}{k^2 - m^2 + i\epsilon}\left[e^{-i(k_2 - k_4 + k)x} e^{-i(k_1 - k_3 - k)y} + e^{-i(k_2 - k_3 + k)x} e^{-i(k_1 - k_4 - k)y} \right]$$

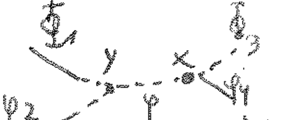
$$= -ig^2\int d^4k\,(2\pi)^4\,\frac{\delta^{(4)}(k + k_2 - k_4)\,\delta^{(4)}(k + k_3 - k_1) + \delta^{(4)}(k + k_2 - k_3)\,\delta^{(4)}(k + k_4 - k_1)}{k^2 - m^2 + i\epsilon}$$

$$= -ig^2\delta^{(4)}(k_3 + k_4 - k_1 - k_2)\cdot(2\pi)^4 \cdot \left[\frac{1}{(k_4 - k_2)^2 - m^2 + i\epsilon} + \frac{1}{(k_1 - k_4)^2 - m^2 + i\epsilon} \right]$$ //

↳ $\mathcal{M} = g^2\left[\cdots\right]$ // → Coincide con reglas de Feynman

c) $\Phi(\vec{p_1})\,\varphi(\vec{p_2}) \longrightarrow \Phi(\vec{p_3})\,\varphi(\vec{p_4})$





↳ 2º orden

$$\mathcal{A} = -\iint \frac{g^2}{4}\cdot\frac{1}{2!} \langle 0 | a_4^{\varphi} a_3^{\Phi} : \Phi\varphi\varphi(x) : : \Phi\varphi\varphi(y) : a_1^{+\Phi} a_2^{+\varphi} | 0 \rangle \cdot \overset{\text{fijo 1 a y}}{2} \cdot \underbrace{2 \cdot 2}_{\text{fijo propagador}} d^4x\, d^4y$$

↙ 2 topologías

$$A = -ig^2 \cdot \delta^{(4)}(p_3+p_4-p_1-p_2)\cdot\left[\frac{1}{(k_1-k_4)^2-m^2+i\epsilon} + \frac{1}{(k_1+k_2)^2-m^2}\right](2\pi)^4$$

↳ Uso directamente reglas de Feynman

d)

$$\mathcal{A} = -\iint \frac{g^2}{8} \langle 0 | a_4 a_3 : \Phi\varphi\varphi(x) : : \Phi\varphi\varphi(y) : a_1^{+} a_2^{+} | 0 \rangle\, d^4x\, d^4y \cdot \overset{\text{fijo 1}}{4} \cdot d^4x\, d^4y$$

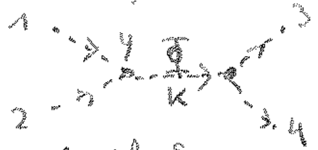
$$= -\frac{g^2}{2}\iint d^4x\, d^4y\, \Phi\Phi \cdot \{ \langle 0 | a_4 a_3 : \varphi(x)\varphi(x) : : \varphi(y)\varphi(y)\, a_1^{+} a_2^{+} | 0 \rangle \cdot \overset{\text{a}_3,\text{a}_4\text{ eligen entre 2(y)}}{2}$$

$$+ \qquad\qquad \cdot \overset{\text{a}_2\text{ elige entre 2(y)}}{2}$$

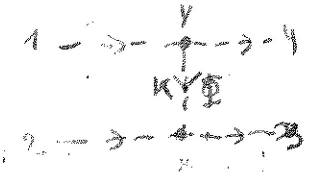
$$+ \qquad\qquad \} \cdot 2$$

$$= -ig^2\, \delta^{(4)}(\underset{K \equiv p}{K_3+K_4-K_1-K_2}) \cdot \left[\frac{1}{\underset{s}{(k_1+k_2)^2} - \underset{M \text{ de } \Phi}{M^2} + i\epsilon} + \frac{1}{\underset{u}{(K_1-K_4)^2} - M^2 + i\epsilon} + \frac{1}{\underset{t}{(K_3-K_4)^2} - M^2 + i\epsilon}\right](2\pi)^4$$

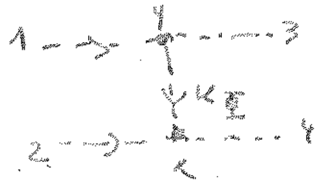
3 topologías:



Canal s



Canal u

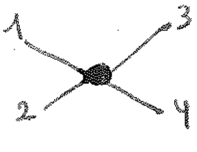


Canal t

(28)

E_i [illegible]$K_i \ll M$

↳ $\mathcal{A} = \dfrac{ig^2}{3M^2}\, \delta^{(4)}(K_3+K_4-K_1-K_2)(2\pi)^4 \longrightarrow \mathcal{M} = -\dfrac{g^2}{3M^2}$



↳ El propagador no es "observable" a esas energías

$\mathcal{L}_{ef_I} \rightarrow$ tipo φ^4

$$\mathcal{L}_{ef_I} = -\frac{\lambda}{4!}\cdot\varphi^4 \rightarrow \mathcal{A} = -i\frac{\lambda}{4!}\cdot \overset{\text{cómo contraer 1-4}}{4!}(2\pi)^4\delta^{(4)}(K_4+K_3-K_2-K_1) \rightarrow \mathcal{M} = \lambda = -\frac{g^2}{3M^2}$$

↳ $\mathcal{L}_{ef_I} = \dfrac{g^2}{3M^2}\varphi^4$

$[g] = [E]$
$[\lambda] = 1$

㉙ $\mathcal{L}_I = -g\,\Phi_M \varphi_m^+ \varphi_m$ $\quad \Phi \to a, a^+ \quad \to \overline{a\Phi} = e^{ikx},\ \overline{a^+\Phi} = e^{-ikx}$

$\varphi \to a, b^+ \quad \to \overline{a\varphi} = 0 = \overline{b\varphi^+} = \overline{\varphi b^+} = \overline{\varphi^+ a^+}$

$\to \overline{\varphi a^+} = e^{-ikx} = \overline{\varphi^+ b^+}$ ~~$= e^{ikx}$~~ ; $\overline{a\varphi^+} = \overline{b\varphi} = e^{ikx}$

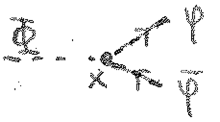
$\overline{\varphi\varphi} = 0$; $\varphi^+\varphi \neq 0$

a) $\Phi_1 \to \varphi_2 \overline{\varphi}_3$

$\mathcal{A} = -igT\langle 0|b_3 a_2 :\Phi\,\varphi^+\varphi: \overset{\Phi}{a_1^+}|0\rangle d^4x = -ig\,\delta^{(4)}(K_3+K_2-K_1)(2\pi)^4$

$\mathcal{M} = g$ //

Única topología

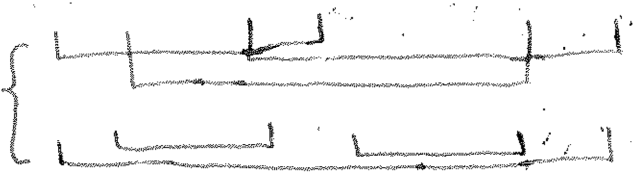


→ flechas marcan corriente cargada $(Q_\varphi > 0)$

b) $\Phi_1 \Phi_2 \to \varphi_3 \overline{\varphi}_4$

$\mathcal{A} = -\frac{g^2}{2!}\iint \langle 0|b_4 a_3 :\Phi\varphi^+\varphi(x)::\Phi\varphi^+\varphi(y): \overset{\Phi}{a_1^+}\overset{\Phi}{a_2^+}|0\rangle \cdot 2\, d^4x\, d^4y$ ···fijo 1

2 topologías



$\to T\langle 0|\overline{\varphi^+(x)\varphi(y)}|0\rangle = \int \frac{d^4K}{(2\pi)^4}\frac{e^{-iK(x-y)}}{K^2-m^2+i\epsilon}$

$\to T\langle 0|\overline{\varphi(x)\varphi^+(y)}|0\rangle = \frac{e^{-iK(x-y)}}{K^2-m^2+i\epsilon}$

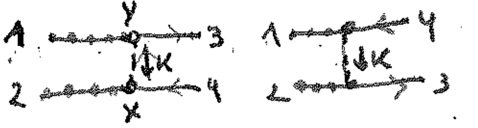
↓

Como está T-ordenado, ~~sólo hay que cambiar~~ son equivalentes ~~x por y en la exponencial.~~

↳ cambiar signo flecha dibujo

$= -ig^2(2\pi)^4\delta^{(4)}(K_4+K_3-K_2-K_1)\,\cdot$

$\cdot\left\{\frac{1}{\underset{\underset{t}{\parallel}}{(K_1-K_3)^2}-m^2+i\epsilon} + \frac{1}{\underset{\underset{u}{\parallel}}{(K_1-K_4)^2}-m^2+i\epsilon}\right\}$ //

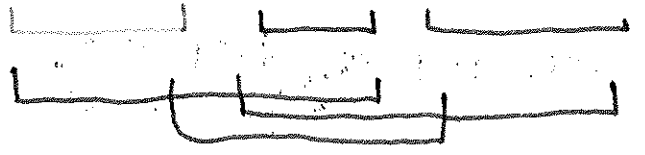


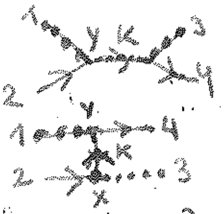
→ No hay canal s, porque Φ neutro:

c) $\Phi_1 \varphi_2 \to \Phi_3 \varphi_4$

$\mathcal{A} = -\frac{ig^2}{2!}\iint d^4x\,d^4y\ \langle 0|a_4 \overset{\Phi}{a_3} :\Phi\varphi^+\varphi(x)::\Phi\varphi^+\varphi(y): \overset{\Phi}{a_1^+} a_2^+|0\rangle \cdot 2$ ···fijo 1

2 topolog.





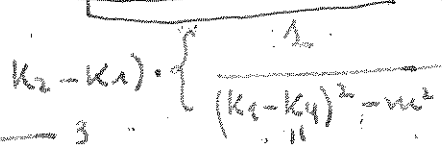
$= -ig^2(2\pi)^4\delta^{(4)}(K_4+K_3-K_2-K_1)\cdot\left\{\frac{1}{\underset{\underset{s}{\parallel}}{(K_1+K_2)^2}-m^2+i\epsilon} + \frac{1}{\underset{\underset{u}{\parallel}}{(K_1-K_4)^2}-m^2+i\epsilon}\right\}$ //

↳ No hay canal t porque $\varphi^+\varphi$ cargado.

d) $\varphi_1\varphi_2 \to \varphi_3\varphi_4$

$\mathcal{A} = -\frac{g^2}{2!}\iint d^4x\,d^4y \langle 0|a_4 a_3 :\Phi\varphi^+\varphi(x)::\Phi\varphi^+\varphi(y): a_1^+ a_2^+|0\rangle \cdot 2$ ···fijo 1

2 topologías



$\mathcal{A} = -ig^2\,\delta^{(4)}(K_4+K_3-K_2-K_1)\cdot\left\{\frac{1}{\underset{\underset{u}{\parallel}}{(K_1-K_4)^2}-m^2+i\epsilon} + \frac{1}{\underset{\underset{t}{\parallel}}{(K_1-K_3)^2}}\right\}$ //

→ No hay canal s porque Φ es neutro.

(30)

$\mathcal{L}_I = -ig(\bar\psi\gamma_5\psi)\Phi$

$\psi(x) a_r^\dagger(\vec p) = u_r(\vec p)e^{-ipx}$ $\qquad a_r(\vec p)\bar\psi(x) = \bar u_r e^{ipx}$

$\bar\psi(x) b_r^\dagger(\vec p) = \bar v_r(\vec p)e^{ipx}$ $\qquad b_r(\vec p)\psi(x) = v_r(\vec p)e^{ipx}$

$$\psi(x)\bar\psi(y) = i\int\frac{d^4k}{(2\pi)^4}\frac{\not k+m}{k^2-m^2+i\epsilon}e^{-ik(x-y)}$$

Para fermiones
$a_i \equiv a_r(k_i)$

a) $\Phi(\vec p_1) \to \psi(\vec p_2)\bar\psi(\vec p_3)$

$$A = -ig\int\langle 0|b_3\, a_2 :(\bar\psi\gamma_5\psi)\Phi(x): a_1^{\Phi\dagger}|0\rangle d^4x$$

$$= -ig\int(\bar u_{r_2}e^{ip_2x}\gamma_5 v_{r_3}e^{ip_3x}\cdot e^{-ip_1x})d^4x$$

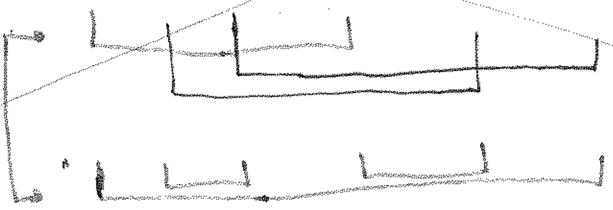
$$= -i(2\pi)^4\delta^{(4)}(p_3+p_2-p_1)\cdot(\bar u_{r_2}\gamma_5 v_{r_3})\cdot g$$

~~b) $\Phi_1\Phi_2 \to \bar\psi_3\psi_4$~~

~~fijo 1~~

~~$A = -\frac{g^2}{2!}\int d^4x\int d^4y\langle 0|b_4 a_3 :\bar\psi\gamma_5\psi(x)\Phi: :\bar\psi\gamma_5\psi(y)\Phi(y): a_1^{\Phi\dagger}\ a_2^{\Phi\dagger}|0\rangle\cdot 2$~~

~~↳ 2 topologías~~



~~→ 3 cortes: signo −~~
~~además está el propagador al revés: otro signo −~~
~~⇒ global +~~

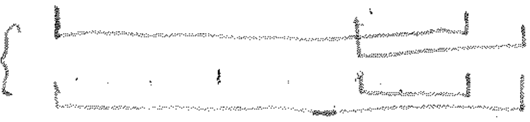
~~$= -ig^2(2\pi)^4\delta^{(4)}(p_4+p_3-p_2-p_1)\left\{\frac{\not p_1-\not p_3+m}{(p_1-p_3)^2-m^2+i\epsilon}+\frac{\not p_2-\not p_4+m}{(p_2-p_4)^2-m^2+i\epsilon}\right\}$~~

b) $\Phi_1\bar\Phi_2 \to \psi_3\bar\psi_4$

fijo 3

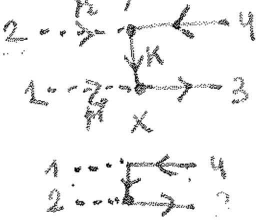
$$A = -\frac{g^2}{2!}\int d^4x\int d^4y\langle 0|b_4 a_3 :\bar\psi\gamma_5\psi\Phi(x): :\bar\psi\gamma_5\psi\Phi(y): a_1^{\Phi\dagger} a_2^{\Phi\dagger}|0\rangle\cdot 2$$

2 topologías (bosones, signos =)



$$= -i(2\pi)^4\delta^{(4)}(p_4+p_3-p_2-p_1)\cdot\mathcal{M}$$

$$\mathcal{M} = g^2\bar u_3\gamma_5\cdot\left\{\frac{\not p_3-\not p_1+m}{(p_3-p_1)^2-m^2+i\epsilon}+\frac{\not p_3-\not p_2+m}{(p_3-p_2)^2-m^2+i\epsilon}\right\}\gamma_5 v_4$$

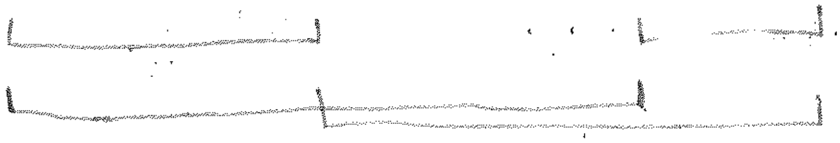


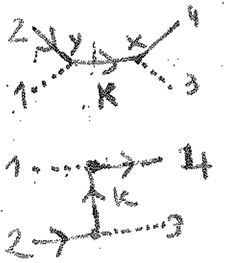
$$= g^2\bar u_3\underbrace{(\gamma_5)^2}_{=1}\left\{\frac{\not p_1-\not p_3+m}{(p_1-p_3)^2-m^2+i\epsilon}+\frac{\not p_2-\not p_3+m}{(p_2-p_3)^2-m^2+i\epsilon}\right\}$$

c) $\Phi_1 \, \psi_2 \to \Phi_3 \, \psi_4$

$$A = -\frac{g^2}{2!}\iint d^4x\, d^4y < 0 | a_3^{\Phi}\, a_4 : \bar\psi \gamma_5 \psi \Phi(x) : : \bar\psi \gamma_5 \psi \Phi(y)\, a_2^{+} \cdot a_1^{\Phi +} | 0 > \cdot 2 \quad \cdots \text{fijo } 2$$

2 topologías:





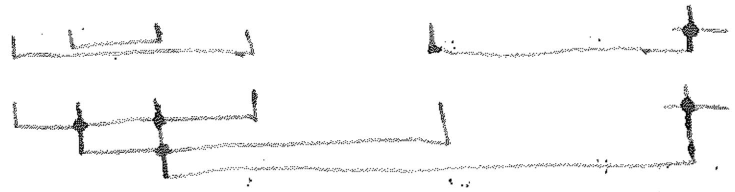
$$M = g^2\, \overline{u_4} \gamma_5 \left\{ \frac{k_2 + k_1 + m}{(K_2+K_1)^2 - M^2 + i\epsilon} + \frac{k_2 - k_3 + m}{(K_2-K_3)^2 - m^2 + i\epsilon} \right\} \cdot \gamma_5 u_2$$

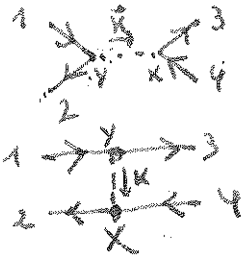
d)

$\psi_1 \overline{\psi_2} \to \psi_3 \overline{\psi_4}$

$$A = -\frac{g^2}{2!}\iint d^4x\, d^4y < 0 | b_4\, a_3 : \bar\psi \gamma_5 \psi \Phi : (x) : \bar\psi \gamma_5 \psi \Phi(y) : b_2^{+}\, a_1^{+} | 0 > \cdot 2 \quad \cdots \text{fijo } 1$$

2 topologías





$$M = g^2 \left\{ - \frac{(\bar u_3 \gamma_5 V_4) \cdot (\bar V_2 \gamma_5 u_1)}{\underset{s}{\underset{\shortparallel}{(p_1+p_2)^2}} - M^2 + i\epsilon} + \frac{(\bar V_2 \gamma_5 V_4)(\bar u_3 \gamma_5 u_1)}{\underset{t}{\underset{\shortparallel}{(p_2-p_3)^2}} - M^2 + i\epsilon} \right\}$$

(31)

a) $\Gamma(\Phi \to \psi\bar\psi)$

$\Gamma \sim |M|^2$

$$|M|^2 = M^+ M = g^2\, (\bar u_2 \gamma_5 V_3)^+ (\bar u_2 \gamma_5 V_3) = g^2 (V_3^+ \gamma_5 \gamma^0 u_2)(\bar u_2 \gamma_5 V_3)$$

$$= -g^2 (\bar V_3 \gamma_5 u_2)(\bar u_2 \gamma_5 V_3) \overset{\text{índices}}{=} -g^2 (\bar V_{3\alpha} (\gamma_5)_{\alpha\beta} u_{2\beta} \bar u_{2\delta} (\gamma_5)_{\delta\epsilon} V_{3\epsilon})$$

$$\sum_{\lambda} u_\alpha \bar u_\beta = (\not p + m)_{\alpha\beta};\ V_\alpha \bar V_\beta = (\not p - m)_{\alpha\beta} \to = -g^2\, V_{3\epsilon} \bar V_{3\alpha} (\gamma_5)_{\alpha\beta} (\not p + m)_{\delta\epsilon}$$

$$\sum |M|^2 = -g^2\, Tr\, ((\not p_3 - m)\gamma_5 (\not p_2 + m)\gamma_5) \overset{\text{cíclica}}{=} -g^2 Tr\, ((-\gamma_5 \not p_3 - m\gamma_5)(\not p_2 + m)\gamma_5)$$

$$= g^2\, Tr\, (\gamma_5^2 (\not p_3 + m)(\not p_2 + m)) \overset{\text{nº impar } \gamma \to Tr = 0}{=} g^2\, Tr\, ((\not p_3 \not p_2 + m^2))$$

$$= ~~4g^2(p_2 p_3 + m^2)~~ = g^2\, (Tr(\not p_2 \not p_3) + 4m^2) \overset{\text{(1e)}}{=} 4g^2\, (p_2 p_3 + m^2)$$

→ Sumamos sobre polarizaciones finales: si no, arrastrámos términos:

$u\bar u = (\not p + m)(1 + \gamma_5 \not s)/2$ siendo s el vector de polarización

$V\bar V = (\not p - m)(1 + \gamma_5 \not s)/2$

$$\frac{d\Gamma}{d\Omega_{cm}} = \overset{\text{ver teoría, 7.17}}{\frac{|\vec p_f|^{cm}}{32\pi^2 M_\Phi^2}}\cdot\sum_{\lambda_f}|\mathcal{M}|^2 = \frac{|\vec p_4^{\,cm}|}{32\pi^2 M_\Phi^2}\cdot(p_2p_3+m^2)$$

Si estamos en CM: $|\vec p| = |\vec p_2^{\,cm}| = |\vec p_3^{\,cm}|$; $E = E_2^{cm} = E_3^{cm}$ $(m_2 = m_3)$; $\vec p_2^{\,cm} = -\vec p_3^{\,cm}$

$$\frac{d\Gamma}{d\Omega_{cm}} = \frac{|\vec p|}{32\pi^2 M_\Phi^2}\cdot(E^2 + |\vec p|^2 + m^2) = \frac{|\vec p|\cdot E^2}{16\pi^2 M_\Phi^2}$$ $\Omega_{cm} \to$ aleatorio's

$$\hookrightarrow \Gamma = \frac{|\vec p|\cdot E^2}{16\pi^2 M_\Phi^2}\cdot 4\pi = \frac{|\vec p|\cdot E^2}{4\pi M_\Phi^2} \;//$$

b) $\mathcal{M}^\dagger = \overset{\text{ver 3 b)}}{g^2}\, v_4^\dagger \;\{\ \}^\dagger\; \gamma u_3 \xrightarrow{\text{muevo } \gamma^0 \text{ a la izquierda}} \cancel{-g^2}\,\cancel{v_4^\dagger}\,\cancel{\gamma_5}\,\cancel{\{\ \}}\,\gamma^0 u_3$

Aparte, $(\not{p})^\dagger = (p_\mu\gamma^\mu)^\dagger = p_\mu(\gamma^\mu)^\dagger = p_\mu\gamma^0\gamma^\mu\gamma^0 = \gamma^0\not{p}\gamma^0$

$$\mathcal{M}^\dagger = g^2 v_4^\dagger\gamma^0\left(\frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2+i\epsilon} + \frac{\not{p}_2-\not{p}_3+m}{(p_2-p_3)^2-m^2+i\epsilon}\right)\cdot\gamma^0\cdot\gamma^0 u_3$$

$$= g^2\,\bar v_4\,\left(\quad '' \quad + \quad '' \quad\right)\cdot u_3$$

$$\sum|\mathcal{M}|^2 = \sum\mathcal{M}^\dagger\mathcal{M} = g^4\sum_{\lambda_f}\bar v_4\{\ \}u_3\,\bar u_3\{\ \}v_4$$

$$= g^4\,\mathrm{Tr}\left(\{\ \}(\not{p}_3+m)\{\ \}(\not{p}_4-m)\right) =$$

$$= g^4\,\mathrm{Tr}\left(\left(\frac{\not{p}_1\not{p}_3-\not{p}_3^2+m\not{p}_3+m\not{p}_1-m\not{p}_3+m^2}{t-m^2+i\epsilon} + \frac{\not{p}_2\not{p}_3-\not{p}_3^2+m\not{p}_3+m\not{p}_2-m\not{p}_3+m^2}{u\,\cancel{s}-m^2+i\epsilon}\right)\{\ \}\right.$$

$$= g^4\cdot\mathrm{Tr}\left(\left(\swarrow\right)\cdot\left(\frac{\not{p}_1\not{p}_4-\not{p}_3\not{p}_4+m\not{p}_4-m\not{p}_1+m\not{p}_3-m^2}{t-m^2+i\epsilon} + \frac{\not{p}_2\not{p}_4-\not{p}_3\not{p}_4+m\not{p}_4\overset{-m\not{p}_2+m\not{p}_3-m^2}{}}{u-m^2+i\epsilon}\right)\right)$$

$$= g^4\,\mathrm{Tr}\left(\frac{T_1}{(t-m^2)^2} + \frac{T_2}{(u-m^2)^2} + \frac{T_3}{(t-m^2)(u-m^2)}\right)$$ Al tomar traza

$$\mathrm{Tr}(T_1) \overset{\text{quitamos nº impar de gammas}}{=} \mathrm{Tr}\big(\not{p}_3\not{p}_1\not{p}_1\not{p}_4 - \not{p}_1\not{p}_3^2\not{p}_4 - \cancel{m^2\not{p}_1\not{p}_3} - \not{p}_3^2\not{p}_1\not{p}_4 + \not{p}_3^3\not{p}_4 + \cancel{m^2\not{p}_3^2}$$
$$+ m^2\not{p}_3\not{p}_4 - \cancel{m^2\not{p}_3\not{p}_1} + \cancel{m^2\not{p}_3^2} - m^2\not{p}_1^2 + \cancel{m^2\not{p}_1\not{p}_3} + m\not{p}_1\not{p}_4$$
$$- m\not{p}_3\not{p}_4 + \cancel{m^2\not{p}_3\not{p}_1} - \cancel{m^2\not{p}_3^2} + m^2\not{p}_1\not{p}_4 - m^2\not{p}_3\not{p}_4 - m^4\big)$$

$$\text{¡muy largo!} = \mathrm{Tr}\big(\not{p}_3\not{p}_1^2\not{p}_4 - \not{p}_3^2\not{p}_4\not{p}_1 - \not{p}_3^2(\not{p}_1\not{p}_4) + \not{p}_3^3\not{p}_4 + 2m^2\not{p}_1\not{p}_4$$
$$- m^2\not{p}_3\not{p}_4 - m^4\big)$$

$$= \mathrm{Tr}\big(\not{p}_3\not{p}_1^2\not{p}_4 - \not{p}_3^2(\not{p}_4\not{p}_1 + \not{p}_1\not{p}_4) + \not{p}_3^3\not{p}_4\big) + 8m^2p_1p_4 - 4m^2p_3p_4 - 4m^4$$

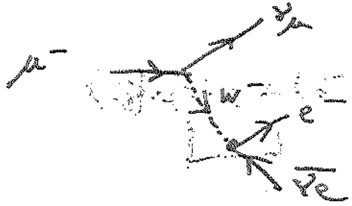
$$= 4\big((p_3p_1)(p_1p_4) - (p_3p_1)(p_1p_4) + p_3p_4p_3^2 - p_3^2p_4p_1 + \cancel{(p_3p_4)(p_3p_1)} - (p_3p_1)(p_3p_4)$$
$$- p_3^2p_1p_4 + \cancel{(p_3p_1)}(p_3p_4) - (p_3p_4)\cancel{(p_3p_1)} + p_3^2\,p_3p_4 \;=\; p_3^2p_3p_4 + (p_3p_4)(p_3p_4)\big) + \ldots$$

$= 4(\ldots)$

$$= 4\left(\cancel{m^2 p_3 p_4} \cancel{- 2m^2 p_4 p_1} + (p_3 p_4)^2 \cancel{+ 2m^2 p_1 p_4} \cancel{- m^2 p_3 p_4} - m^4 \right)$$

$$= 4\left((p_3 p_4)^2 - m^4 \right) //$$

Idem con T_2, T_3



$$\mathcal{L}_W = -\frac{g}{2\sqrt{2}}\left\{ W_\alpha \sum_{l=\mu,e,\tau} \left[\bar{l}\, \gamma^\alpha (1-\gamma_5) \nu_e \right] + W_\alpha^+ \sum_{l=e,\mu,\tau} \left[\bar{\nu}_e \gamma_\alpha (1-\gamma_5)\, l \right] \right\}$$

campo W, índice Lorentz

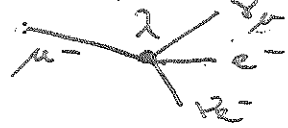
$\bar{l} \equiv \overline{\psi_l}$

$\nu_e \equiv \psi_{\nu e}$

Como $m_\mu^2 << M_W^2$

$$y \quad W \propto \frac{1}{q^2 - M_W^2}\left(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{M_W^2}\right) \overset{q^2 << M_W^2}{\tilde{\propto}} \frac{g^{\alpha\beta}}{M_W^2} \equiv \lambda$$

↳ Si q no es lo suficientemente grande, no ves el W, sino



→ Teoría efectiva tipo ϕ^3

Fermi, a final de 1930, usando ec. Dirac, predicción ν de Pauli, escribió $\mathcal{L}_{ef}$: (prueba combinas con bilineales Dirac, cierra índices Lorentz, es lo que da más ajustado a distribución angular de Energía medido experimentalmente)

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\left[\bar{l}\gamma^\alpha(1-\gamma_5)\nu_e\right]\left[\overline{\nu_\mu}\gamma_\alpha(1-\gamma_5)\mu\right]$$

↳ sin conocer Modelo estándar

campos → $E^{3/2}$

4 campos → E^6 ⟺ $G_F \propto E^{-2}$

↳ Por comparación: $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$

Ej: • Calcular Γ (~Q3, reglas Fermiones que no ha dado)

- Dimensionalmente: $\Gamma = \frac{G_F^2 m_\mu^5}{128\pi^3 \cdots d Q_3}$ → falta factor $\frac{3}{2}$ → $\frac{1}{192\pi^3}$
- Analíticamente

- Calcular si $m_e \neq 0$ → $\Gamma = \Gamma_0 \cdot f\left(\frac{m_e^2}{m_\mu^2}\right)$

+ pensar notas (ahí explicado)

- τ_μ → permite medir G_F, sabes M (orden) ~ 100 GeV (electrodébil)
- τ_τ → calcular con argumentos dimensionales con una precisión del 2%

$$\langle 0| a_e\, a_{\bar\nu_e}\, a_{\nu_\mu} (-)\frac{G_F}{\sqrt{2}} [\bar e \gamma^\alpha (1-\gamma_5)\nu_e][\bar\nu_\mu \gamma_\alpha (1-\gamma_5)\mu]\, a^+_\mu(\vec p_\mu)|0\rangle$$

$$= i\int d^4x \langle 0|\, a_e \overleftrightarrow{a_{\bar\nu_e}}\, a_{\nu_\mu} [\bar e \gamma^\alpha (1-\gamma_5)\nu_e][\bar\nu_\mu \gamma_\alpha (1-\gamma_5)\mu]\, a^+_\mu(p)\rangle \quad \cdot -G_F$$

$$= i\int d^4x \langle 0|\, \bar u_e\, \bar u_{\nu_\mu}\, v_{\bar\nu_e}\, e^{i(p_e+p_{\nu_\mu}+p_{\bar\nu_e})x} \cdot \underline{-G_F} \qquad u_\mu e^{-ip_\mu x}$$

$$= i(2\pi)^4 \delta^{(4)}(p_e+p_{\nu_\mu}+p_{\bar\nu_e}-p_\mu)\cdot \qquad u_\mu$$
$$\cdot -\frac{G_F}{\sqrt{2}} \quad \bar u_e \gamma^\alpha (1-\gamma_5) v_{\bar\nu_e}\, \bar u_{\nu_\mu} \gamma_\alpha (1-\gamma_5) u_\mu$$

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}\left[\bar u_e \gamma^\alpha (1-\gamma_5) v_{\bar\nu_e}\right]\left[\bar u_{\nu_\mu}\gamma_\alpha (1-\gamma_5) u_\mu\right]$$

$$\mathcal{M}^+ = -\frac{G_E}{\sqrt{2}}\, u^+_\mu (1-\gamma_5)\gamma_\alpha^+ \gamma^0 u_{\nu_\mu}\, v^+_{\bar\nu_e}(1-\gamma_5)(\gamma^\alpha)^+\gamma^0 u_e$$

$$= -\frac{G_F}{\sqrt{2}}\left[\bar u_\mu (1+\gamma_5)\gamma_\alpha u_{\nu_\mu}\right]\cdot\left[\bar v_{\bar\nu_e}(1+\gamma_5)\gamma^\alpha u_e\right]$$

$$\mathcal{M}\mathcal{M}^+ = \frac{G_F^2}{2}\, \bar u_e \gamma^\alpha (1-\gamma_5) v_{\bar\nu_e}\, \bar u_{\nu_\mu}\gamma_\alpha (1-\gamma_5) u_\mu\, \bar u_\mu (1+\gamma_5)\gamma_\beta u_{\nu_\mu}\, \bar v_{\bar\nu_e}(1+\gamma_5)\gamma^\beta u_e)$$

$$= \frac{G_F^2}{2}\left[\bar u_{\nu_\mu}\gamma_\alpha (1-\gamma_5)(\not p_\mu + m)(1+\gamma_5)\gamma_\beta u_{\nu_\mu}\right]\cdot\left[\bar u_e \gamma^\alpha (1-\gamma_5)(\not p_{\bar\nu_e} - m)\cdot(1+\gamma_5)\gamma^\beta u_e\right]$$

$\swarrow$ indices

$$= \frac{G_F^2}{2}\mathrm{Tr}\left[\gamma_\alpha (1-\gamma_5)(\not p_\mu + m)(1+\gamma_5)\gamma_\beta(\not p_{\nu_\mu} + \cancel{m})\right]\cdot \mathrm{Tr}\left[\gamma^\alpha (1-\gamma_5)(\not p_{\bar\nu_e} - m)\cdot(1+\gamma_5)\gamma^\beta(\not p_e + m)\right]$$

$$\mathrm{Tr}\left(\gamma_\alpha (\not p_\mu + m_\mu)\gamma_\beta(\not p_{\nu_\mu} + \cancel{m})\right) = \mathrm{Tr}\left(\gamma_\alpha \not p_\mu \gamma_\beta \not p_{\nu_\mu}\right)$$
$$+\, 0 + 0 + \mathrm{Tr}(\gamma_\alpha \cancel{m}\gamma_\beta m) = \mathrm{Tr}(\gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma)\, p^\rho p^\sigma$$
$$+\, \cancel{m^2 T} = 4(g_{\alpha\rho} g_{\beta\sigma} - g_{\alpha\beta} g_{\rho\sigma} + g_{\alpha\sigma} g_{\rho\beta})\, p^\rho p^\sigma$$
$$= 4(p_{\mu\alpha} p_{\nu_\mu\beta} - g_{\alpha\beta}(p_\mu\cdot p_{\nu_\mu}) + p_{\nu_\mu\alpha}\cdot p_{\mu\beta})$$

$$\mathrm{tr}\left(\gamma_\alpha (1-\gamma_5)(\not p_\mu + m_\mu)(1+\gamma_5)\gamma_\beta \not p_{\nu_\mu}\right)$$

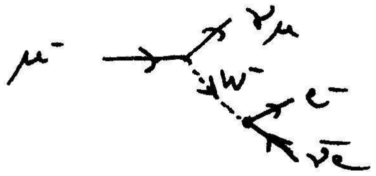
$$\mathrm{Tr}\left(-\gamma_\alpha \gamma_5 (\not p_\mu + m)\gamma_\beta \not p_{\nu_\mu}\right) = -\mathrm{Tr}\left(\gamma_\alpha \gamma_5 \not p_\mu \gamma_\beta \not p_{\nu_\mu}\right)$$
$$= \mathrm{Tr}(\gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma)\, p^\rho p^\sigma$$

Fernando Hueso González - 4º Grado Física

Teoría Cuántica de Campos

Parecido a ejc. (33)

$$\mu^- \longrightarrow e^- + \bar{\nu}_e + \nu_\mu$$



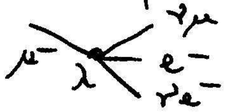
campo W, índice Lorentz

$$\mathcal{L}_W = -\frac{g}{2\sqrt{2}}\Big\{W_\alpha \sum_{l=\mu,l,\tau}[\bar{e}\gamma^\alpha(1-\gamma_5)\nu_e] + W_\alpha^\dagger \sum_{l=e,\mu,\tau}[\bar{\nu}_e\gamma_\alpha(1-\gamma_5)e]\Big\}.$$

$e \equiv \psi_e$; $\nu_e \equiv \psi_{\nu_e}$

Como $m_\mu^2 << M_W^2$ y $W^{\alpha\beta} \propto \frac{1}{q^2-M_W^2}\left(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{M_W^2}\right) \overset{q^2<<M_W^2}{\simeq} \frac{g^{\alpha\beta}}{M_W^2} \equiv \lambda$

Cuando q no es lo suficientemente grande, no ves el W, sino:



→ Teoría efectiva tipo ϕ^3

Fermi, a final de 1930, usando ec. Dirac, predicción ν de Pauli escribió $\mathcal{L}_{ef}$, probando combinaciones con los bilineales de Dirac, armando todos los índices Lorentz; se queda con lo que más ajusta a datos experimentales en cuanto a distribución angular de energía.

↳ $\mathcal{L}_F = -\frac{G_F}{\sqrt{2}}[\bar{e}\gamma^\alpha(1-\gamma_5)\nu_e]\cdot[\bar{\nu}_\mu\gamma_\alpha(1-\gamma_5)\mu]$

↳ Lo escribió sin conocer modelo estándar

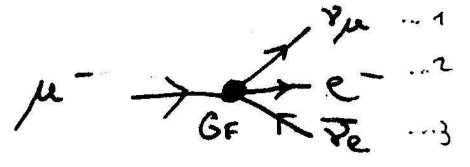
Ejercicio:

Calcular

- Amplitud del proceso $\langle f|S-I|i\rangle$ (probabilidad de transición)
- Elemento de matriz reducido $\mathcal{M}$
- $|\mathcal{M}|^2$
- Γ_μ (anchura de desintegración)
- τ_μ (vida media del muón)
  ↳ A partir del valor experimental: calcular G_F → saber orden de M_W

↳ τ_τ

- Γ con argumentos dimensionales + comparar con el cálculo exacto



→ Descripción efectiva, 1^{er} orden G_F (1 vértice).

→ Diagrama único

Amplitud de probabilidad de transición:

$$<f|S-I|i> = -i(2\pi)^4 \delta^{(4)}(p_f^\mu - p_i^\mu)\cdot \mathcal{M}_{i\to f}$$

↳ $\mu^- \equiv A$; $\nu_\mu \equiv 1$; $e^- \equiv 2$; $\overline{\nu_e} \equiv 3$

mismo tiempo todo

$$S = T\{\exp[\, i\int d^4x :\mathcal{L}_F(x):]\} \simeq I + i\int d^4x :\mathcal{L}_F(x):$$

$$|i> \equiv |\vec{p}_A> = a_A^+|0>$$

$$|f> = |\vec{p}_1\vec{p}_2\vec{p}_3> = a_1^+ a_2^+ b_3^+|0> \quad ; \quad a_i^+ \equiv a_{i_r}^+(\vec{p}_i)$$

r: polarizac

→ Todos $\psi_i \equiv \psi_i(x)$

$$<f|S-I|i> = \int_{d^4x} <0|\underbrace{b_3\, a_2\, a_1}_{\text{conmutan por ser partículas dttas}} \cdot i\frac{G_F}{\sqrt{2}}(-):[\overline{\psi_2}\gamma^\alpha(1-\gamma_5)\psi_3]\cdot[\overline{\psi_1}\gamma_\alpha(1-\gamma_5)\psi_A]:a_A^+|0>$$

$$= -\frac{iG_F}{\sqrt{2}}\int d^4x <0|a_1 b_3 a_2 :(\overline{\psi_2}\gamma^\alpha(1-\gamma_5)\psi_3)\cdot(\overline{\psi_1}\gamma_\alpha(1-\gamma_5)\psi_A): a_A^+|0>$$

↳ Única topología

→ Dirac, libre

$$\psi_i(x) = \sum_{p_i,r}\left(a_{i_r}(\vec{p})u_r(\vec{p}_i)e^{-ipx} + b_{i_r}^+(\vec{p}_i)v_r(\vec{p}_i)e^{ip_ix}\right) \qquad i \equiv A,1,2,3 \text{ (denota partícula)}$$

Utilizamos las propiedades:

$$a_r(\vec{p})\overline{\psi}(x) = \overline{u_r}(\vec{p})e^{ipx}$$

$$b_r(\vec{p})\psi(x) = v_r(\vec{p})e^{ipx}$$

$$\psi(x)a_r^+(\vec{p}) = u_r(\vec{p})e^{-ipx}$$

$$\overline{\psi}(x)b_r^+(\vec{p}) = \overline{v_r}(\vec{p})e^{-ipx}$$

→ Llevar a_i hasta sus ψ_s correspondientes, conmutan con γ_s

$$<f|S-I|i> = -\frac{iG_F}{\sqrt{2}}\int^{d^4x} <0|(\overline{u_2}e^{ip_2x}\gamma^\alpha(1-\gamma_5)v_3e^{ip_3x})\cdot(\overline{u_1}e^{ip_1x}\gamma_\alpha(1-\gamma_5)u_Ae^{-ip_Ax})|0>$$

, $<0|0> = 1$, ya no quedan operadores

$$= -i\frac{G_F}{\sqrt{2}}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_A)\cdot(\overline{u_2}\gamma^\alpha(1-\gamma_5)v_3)\cdot(\overline{u_1}\gamma_\alpha(1-\gamma_5)u_A)$$

$$= -i(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_A)\cdot\underbrace{\frac{G_F}{\sqrt{2}}[\overline{u_2}\gamma^\alpha(1-\gamma_5)v_3]\cdot[\overline{u_1}\gamma_\alpha(1-\gamma_5)u_A]}_{\mathcal{M}}$$

$\mathcal{M}$ (elem de matriz reducido)

$$\mathcal{M}^+ = \frac{G_F}{\sqrt{2}}[u_A^+(1-\gamma_5)\overset{(\gamma^0)^2=I}{\vee}\gamma_\alpha^+\gamma^0u_1]\cdot[v_3^+(1-\gamma_5)\overset{(\gamma^0)^2=I}{\vee}\gamma^{\alpha+}\gamma^0u_2]$$

$$= \frac{G_F}{\sqrt{2}}[u_A^+(\gamma^0-\overset{-\gamma^0\gamma_5}{\gamma_5\gamma^0})\gamma_\alpha u_1]\cdot[v_3^+(\gamma^0-\gamma_5\gamma^0)\gamma^\alpha u_2] \longrightarrow \gamma^0\gamma_\alpha^+\gamma^0 = \gamma_\alpha$$

$$= \frac{G_F}{\sqrt{2}}\cdot[\overline{u}_A(1+\gamma_5)\gamma_\alpha u_1]\cdot[\overline{v_3}(1+\gamma_5)\gamma^\alpha u_2]$$

→ α: mudo
lo cambio por β
para no mezclar índices de multiplicar $\mathcal{M}^+\mathcal{M}$

$$|\mathcal{M}|^2 = \mathcal{M}^+ \mathcal{M} = \frac{G_F^2}{2} \cdot [\bar{u}_A (1+\gamma_5)\gamma_\beta u_1] \cdot [\bar{v_3} (1+\gamma_5)\gamma^\beta v_2] \cdot [\bar{v_2}\gamma^\alpha (1-\gamma_5) v_3]$$

$$\cdot [\bar{u}_1 \gamma_\alpha (1-\gamma_5) u_A] =$$

→ Con notación de índices, se ve que se obtienen trazas si hacemos la suma sobre polarizaciones finales y promediamos sobre iniciales: $\frac{1}{2S_A+1}\sum_{r_i} \equiv \bar{\sum}$

↳ espín A → $S_A = \frac{1}{2}$

$$\bar{\sum} |\mathcal{M}|^2 = \frac{G_F^2}{2} \cdot \frac{1}{2} \cdot$$

$$\cdot \operatorname{Tr}[(1+\gamma_5)\gamma_\beta (\not{p}_1 + m_1)\gamma_\alpha (1-\gamma_5)(\not{p}_A + m_A)]$$

$$\cdot \operatorname{Tr}[(1+\gamma_5)\gamma^\beta (\not{p}_2 + m_2)\gamma^\alpha (1-\gamma_5)(\not{p}_3 - m_3)] \overset{}{=}$$

→ Supongo $m_1, m_3 \to 0$ (neutrinos con masa despreciable frente al momento)

$$= \frac{G_F^2}{4} \cdot \operatorname{Tr}[(1+\gamma_5)\gamma_\beta \not{p}_1 \gamma_\alpha (1-\gamma_5)(\not{p}_A + m_A)]$$

$$\cdot \operatorname{Tr}[(1+\gamma_5)\gamma^\beta (\not{p}_2 + m_2)\gamma^\alpha (1-\gamma_5)\not{p}_3]$$

↙ $\operatorname{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n+1}}) = 0$

$$= \frac{G_F^2}{4} \cdot \operatorname{Tr}[(1+\gamma_5)\gamma_\beta \not{p}_1 \gamma_\alpha (1-\gamma_5)\not{p}_A] \cdot \operatorname{Tr}[(1+\gamma_5)\gamma^\beta \not{p}_2 \gamma^\alpha (1-\gamma_5)\not{p}_3]$$

$$\hookrightarrow \operatorname{Tr}[(1+\gamma_5)\gamma_\beta \not{p}_1 \gamma_\alpha (1-\gamma_5)\not{p}_A] \overset{\text{cíclica}}{=} \operatorname{Tr}[\gamma_\beta \not{p}_1 \gamma_\alpha (1-\gamma_5)\not{p}_A (1+\gamma_5)] \overset{\{\gamma_5,\gamma^\mu\}=0}{=}$$

$$= \operatorname{Tr}[\gamma_\beta \not{p}_1 \gamma_\alpha \not{p}_A (1+\gamma_5)^2] \overset{(1+\gamma_5)^2 = 2(1+\gamma_5)}{=} 2\operatorname{Tr}[\gamma_\beta \not{p}_1 \gamma_\alpha \not{p}_A (1+\gamma_5)]$$

$$= 2 p_1^\mu p_A^\nu \operatorname{Tr}[\gamma_\beta \gamma_\mu \gamma_\alpha \gamma_\nu (1+\gamma_5)]$$

$$\operatorname{Tr}(\gamma_\beta \gamma_\mu \gamma_\alpha \gamma_\nu) \overset{\text{ver ejercicio 11d) y bajar índices}}{=} 4(g_{\beta\mu}g_{\alpha\nu} - g_{\beta\alpha}g_{\mu\nu} + g_{\beta\nu}g_{\mu\alpha})$$

$$\operatorname{Tr}(\gamma_\beta \gamma_\mu \gamma_\alpha \gamma_\nu \gamma_5) = ?$$

$$\hookrightarrow \operatorname{Tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\eta) = \operatorname{Tr}(\gamma_5 \cdot (g^{\mu\nu} - i\sigma^{\mu\nu})\gamma^\rho\gamma^\eta)$$

$$= \underbrace{\operatorname{Tr}(\gamma_5 \gamma^\rho \gamma^\eta)}_{=0 \text{ por ejercicio 11j)}} \cdot g^{\mu\nu} - i\operatorname{Tr}(\gamma_5 \sigma^{\mu\nu}\gamma^\rho\gamma^\eta) \overset{\text{ver 9a)}}{=}$$

$$= \frac{\epsilon^{\mu\nu\alpha\beta}}{2}\operatorname{Tr}(\sigma_{\alpha\beta}\gamma^\rho\gamma^\eta) = \frac{\epsilon^{\mu\nu\alpha\beta}}{2} g_{\alpha\delta}g_{\beta\epsilon}\operatorname{Tr}(\sigma^{\delta\epsilon}\gamma^\rho\gamma^\eta)$$

$$= \frac{i\epsilon^{\mu\nu\alpha\beta}}{4} g_{\alpha\delta}g_{\beta\epsilon}[\operatorname{Tr}(\gamma^\delta\gamma^\epsilon\gamma^\rho\gamma^\eta) - \operatorname{Tr}(\gamma^\epsilon\gamma^\delta\gamma^\rho\gamma^\eta)] \overset{\text{ver 11d)}}{=}$$

g simétrico ↑

$$= i\epsilon^{\mu\nu\alpha\beta} g_{\alpha\delta}g_{\beta\epsilon}[g^{\delta\epsilon}g^{\rho\eta} - g^{\delta\rho}g^{\epsilon\eta} + g^{\delta\eta}g^{\epsilon\rho} - g^{\epsilon\delta}g^{\rho\eta} + g^{\epsilon\rho}g^{\delta\eta} - g^{\epsilon\eta}g^{\delta\rho}] =$$

$$= 2i\epsilon^{\mu\nu\alpha\beta} g_{\alpha\delta}g_{\beta\epsilon} \cdot [g^{\delta\eta}g^{\epsilon\rho} - g^{\delta\rho}g^{\epsilon\eta}] = 2i\epsilon^{\mu\nu\alpha\beta} \cdot [g_\alpha^{\ \eta}g_\beta^{\ \rho} - g_\alpha^{\ \rho}g_\beta^{\ \eta}]$$

$$= 2i\epsilon^{\mu\nu\alpha\beta}g_\alpha^{\ \eta}g_\beta^{\ \rho} - 2i\epsilon^{\mu\nu\alpha\beta}g_\alpha^{\ \rho}g_\beta^{\ \eta} \overset{g^\alpha{}_\beta \equiv \delta^\alpha{}_\beta \ldots \text{Kronecker}}{=} 2i\epsilon^{\mu\nu\eta\rho} - 2i\epsilon^{\mu\nu\rho\eta} \overset{\epsilon \text{ antisimét.}}{=}$$

$$= -4i\epsilon^{\mu\nu\rho\eta} //$$

$$\operatorname{Tr}(\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_5) = \operatorname{Tr}(\gamma_5\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\nu) = g_{a\beta}g_{b\mu}g_{c\alpha}g_{d\nu}\operatorname{Tr}(\gamma_5\gamma^a\gamma^b\gamma^c\gamma^d)$$

$$= -4i\epsilon^{abcd}g_{a\beta}g_{b\mu}g_{c\alpha}g_{d\nu} = -4i\epsilon_{\beta\mu\alpha\nu} //$$

→ Se podría haber obtenido del ejercicio 11K)

$$\rightarrow 2p_1^{\mu}\, p_A^{\nu}\left[\mathrm{Tr}\left(\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\nu\right)+\mathrm{Tr}\left(\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_5\right)\right]$$

$$= 2p_1^{\mu}p_A^{\nu}\cdot\left[4(g_{\beta\mu}g_{\alpha\nu}-g_{\beta\alpha}g_{\mu\nu}+g_{\beta\nu}g_{\mu\alpha})+(-4i)\,\epsilon_{\beta\mu\alpha\nu}\right]$$

$$= 8\cdot\left[p_{1\beta}\,p_{A\alpha}-g_{\beta\alpha}(p_1p_A)+p_{1\alpha}\,p_{A\beta}-i\,\epsilon_{\beta\mu\alpha\nu}\,p_1^{\mu}p_A^{\nu}\right]$$

Idem para la otra traza pero con los índices arriba:

$$\mathrm{Tr}\left[\underline{1}(1+\gamma_5)\gamma^\beta\, \not{p}_2\, \gamma^\alpha(1-\gamma_5)\not{p}_3\right]=2p_{2\mu}p_{3\nu}\cdot\mathrm{Tr}\left[(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu)+(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu\gamma_5)\right]$$

$$= 8\left[p_2^{\beta}p_3^{\alpha}-g^{\beta\alpha}(p_2p_3)+p_2^{\alpha}p_3^{\beta}-i\,\epsilon^{\beta\mu\alpha\nu}\,p_{2\mu}p_{3\nu}\right]$$

Multiplicamos (cambiar índices de μ y ν de un producto para no repetir): ya que están cerrados. α y β sí son los mismos

$$\sum|\mathcal{M}|^2=\frac{G_F^2}{4}\cdot 64\cdot\Big[(p_1p_2)(p_Ap_3)-(p_1p_A)(p_2p_3)+(p_1p_3)(p_Ap_2)$$

$$-i\,\epsilon^{\beta\mu'\alpha\nu'}p_{1\beta}\,p_{A\alpha}\,p_{2\mu'}\,p_{3\nu'}-(p_1p_A)(p_2p_3)+\overset{=g^{\alpha\beta}g_{\alpha\beta}}{4}(p_1p_A)(p_2p_3)-(p_1p_A)(p_2p_3)$$

→ 0 porque g es diagonal → β = α → ε se anula

$$+\,i\,\epsilon^{\beta\mu'\alpha\nu'}(p_1p_A)\cancel{p_{2\mu'}}\,p_{3\nu'}\,g_{\beta\alpha}+(p_1p_3)(p_Ap_2)-(p_1p_A)(p_2p_3)+(p_1p_2)(p_Ap_3)$$

$$-\,i\,\epsilon^{\beta\mu'\alpha\nu'}p_{1\alpha}\,p_{A\beta}\,p_{2\mu'}\,p_{3\nu'}-i\,\epsilon_{\beta\mu\alpha\nu}\,p_1^{\mu}p_A^{\nu}p_2^{\beta}p_3^{\alpha}+i\,\epsilon_{\beta\mu\alpha\nu}\underset{0}{\cancel{p_1^{\mu}p_A^{\nu}}}(p_2p_3)g^{\beta\alpha}$$

$$-\,i\,\epsilon_{\beta\mu\alpha\nu}\,p_1^{\mu}p_A^{\nu}p_2^{\alpha}p_3^{\beta}-\underbrace{\epsilon_{\beta\mu\alpha\nu}\,\epsilon^{\beta\mu'\alpha\nu'}}_{=\epsilon_{\beta\alpha\mu\nu}\epsilon^{\beta\alpha\mu'\nu'}}p_1^{\mu}p_A^{\nu}p_{2\mu'}p_{3\nu'}\Big]$$

$$=16G_F^2\cdot\Big[2(p_1p_2)(p_Ap_3)+2(p_1p_3)(p_Ap_2)-(p_1p_A)(p_2p_3)\cdot(4-1\cdot1-1-1)$$

→ permuto α↔β, cambio signo

$$-\,i\,\epsilon^{\beta\mu'\alpha\nu'}p_{1\beta}\,p_{A\alpha}\,p_{2\mu'}\,p_{3\nu'}-i\,\epsilon^{\beta\mu'\alpha\nu'}(-p_{1\beta}\,p_{A\alpha})\,p_{2\mu'}\,p_{3\nu'}$$

→ permuto

$$-\,i\,\epsilon_{\beta\mu\alpha\nu}\,p_1^{\mu}p_A^{\nu}p_2^{\beta}p_3^{\alpha}-i\,\epsilon_{\beta\mu\alpha\nu}\,p_1^{\mu}p_A^{\nu}(-p_2^{\beta}p_3^{\alpha})\quad = 0$$

→ ver 1a)

$$-(-2)\cdot\left(g^{\mu'}_{\mu}g^{\nu'}_{\nu}-g^{\mu'}_{\nu}g^{\nu'}_{\mu}\right)\cdot p_1^{\mu}p_A^{\nu}p_{2\mu'}p_{3\nu'}\Big]$$

$$=32G_F^2\left[(p_1p_2)(p_Ap_3)+(p_1p_3)(p_Ap_2)+(p_1p_2)(p_Ap_3)-(p_Ap_2)(p_1p_3)\right]$$

$$=64G_F^2\,(p_1p_2)(p_Ap_3)$$

Anchura de desintegración: $\Gamma=\frac{1}{2E_i}\sum_{\lambda_f}\int dQ_N\,|M|^2$ → promediamos polarizaciones

↳ muón en reposo: $E_A=m_A$

$$\Gamma=\frac{1}{2m_\mu}\cdot\int\cancel{\tfrac{1}{2}}\,\overline{\sum}|\mathcal{M}|^2\,dQ_3=\frac{32G_F^2}{m_\mu}\int(p_1p_2)(p_Ap_3)\,dQ_3$$

Calculemos dQ_3

$$dQ_3\equiv\frac{1}{(2\pi)^5}\,\frac{d^3\vec{p}_1}{2E_1}\,\frac{d^3\vec{p}_2}{2E_2}\,\frac{d^3\vec{p}_3}{2E_3}\cdot\delta^{(4)}(p_A^{\mu}-p_1^{\mu}-p_2^{\mu}-p_3^{\mu})$$

→ $\int dp\cdot\delta(p-a)=1$

↳ elimino el d^3p_3

$$=\frac{1}{8(2\pi)^5}\cdot\frac{1}{E_1E_2E_3}\,d^3\vec{p}_1\,d^3\vec{p}_2\cdot\delta(E_A-E_1-E_2-E_3)$$

Cambio de variable a esféricas:

$$d^3\vec{p_1}\, d^3\vec{p_2} = |\vec{p_1}|^2 \cdot |\vec{p_2}|^2\, d|\vec{p_1}|\, d|\vec{p_2}| \cdot d\Omega_1\, d\Omega_2 \quad ; \quad d\Omega_i = d\cos(\theta_i)\, d\varphi_i$$

Como $E_i^2 = m_i^2 + |\vec{p_i}|^2 \rightarrow 2E_i\, dE_i = 2|\vec{p_i}|\, d|\vec{p_i}|$ (masa i fija)

$$\hookrightarrow d^3\vec{p_1}\, d^3\vec{p_2} = |\vec{p_1}|\,|\vec{p_2}| \cdot E_1 E_2\, dE_1\, dE_2\, d\Omega_1\, d\Omega_2$$

En sistema CM (A en reposo): $\vec{p_3} = -\vec{p_1} - \vec{p_2}$

$\hookrightarrow E_3^2 = m_3^2 + |\vec{p_3}|^2 = m_3^2 + |\vec{p_1}|^2 + |\vec{p_2}|^2 + 2|\vec{p_1}|\,|\vec{p_2}| \cdot \cos\theta_{12}$; $m_3, \underbrace{|\vec{p_1}|, |\vec{p_2}|}_{\text{independientes}}$ fijo

$\hookrightarrow E_3\, dE_3 = |\vec{p_1}|\,|\vec{p_2}|\, d(\cos\theta_{12})$

$\rightarrow d\Omega_2 \equiv d(\cos\theta_{12})\, d\varphi_2$

$$\Rightarrow dQ_3 = \frac{1}{8(2\pi)^5} \cdot \frac{1}{E_1 E_2 E_3} \cdot E_1 E_2\, dE_1\, dE_2\, d\Omega_1\, |\vec{p_1}|\,|\vec{p_2}|\, d(\cos\theta_{12})\, d\varphi_2 \cdot \delta(E_A - E_1 - E_2 - E_3)$$

$$= \frac{1}{8(2\pi)^5} \cdot dE_1\, dE_2\, d\Omega_1\, d\varphi_2 \cdot \underbrace{dE_3 \cdot \delta(E_A - E_1 - E_2 - E_3)}_{=1 \text{ al integrar}}$$

$$= \frac{1}{8(2\pi)^5}\, dE_1\, dE_2\, d\Omega_1\, d\varphi_2$$

En SCM ángulos 1 aleatorios $\rightarrow \int d\Omega_1\, d\varphi_2 = 4\pi \cdot 2\pi$; poner explícitamente $\cos^2\theta_{12} \leq 1$

$$dQ_3 = \Theta(1 - \cos^2\theta_{12}) \cdot \frac{1}{(2\pi)^5} \cdot \frac{1}{8}\, dE_1\, dE_2 \cdot 8\pi^3 = \frac{1}{32\pi^3}\, \Theta(1 - \cos^2\theta_{12})\, dE_1\, dE_2$$

Empleando las masas invariantes:

$$s_{13} = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2p_1 p_3 = (p_A - p_2)^2 = (\sqrt{s} - p_2)^2 \overset{\text{en SCM}}{=} s + m_2^2 - 2\sqrt{s}\, E_2$$

$$\hookrightarrow E_2 = \frac{s + m_2^2 - s_{13}}{2\sqrt{s}}$$

$$s_{23} = (p_2 + p_3)^2 = (p_A - p_1)^2 = s + m_1^2 - 2\sqrt{s}\, E_1 \rightarrow E_1 = \frac{s + m_1^2 - s_{23}}{2\sqrt{s}}$$

Como m_i, s son fijos

$$dE_1 = -\frac{ds_{23}}{2\sqrt{s}} \quad ; \quad dE_2 = -\frac{ds_{13}}{2\sqrt{s}}$$

$$\hookrightarrow dQ_3 = \frac{1}{32\pi^3}\, \Theta(1 - \cos^2\theta_{12})\, \frac{ds_{13}\, ds_{23}}{4s} = \frac{1}{s \cdot 128\pi^3}\, ds_{13}\, ds_{23} \cdot \Theta(1 - \cos^2\theta_{12})$$

Como vimos en ejercicio 5c)

$(m_1 + m_3)^2 \leq s_{13} \leq (\sqrt{s} - m_2)^2$. $(m_2 + m_3)^2 \leq s_{23} \leq (\sqrt{s} - m_1)^2$

Hay que integrar entre $s_{13\,\min}^{\ \max}$ y $s_{23\,\min}^{\ \max}$ dado en 5c) $\rightarrow$ ya no hace falta $\Theta(\ldots)$

En nuestro caso, $m_1, m_3 \simeq 0$; muón en reposo

$0 \leq s_{13} \leq (m_A - m_2)^2$ $\hookrightarrow$ ver 5d).2 5c) intercambiando $\underline{\underline{1 \leftrightarrow 2}}$, $m_2 \equiv m$

$m_2^2 \leq s_{23} \leq m_A^2$ $\hookrightarrow$ $s_{13}^{\pm} = \left\langle \begin{array}{l} 0 \\ (m_A^2 - \ldots) + m_2^2\left(1 - \frac{m_A^2}{\ldots}\right) \end{array}\right.$

$$\Gamma = \frac{32G_F^2}{m_\mu}\int (p_1p_2)(p_Ap_3)\,d\Phi_3 = \frac{32G_F^2}{m_\mu}\int_{m_2^2}^{m_A^2} ds_{23}\int_0^{s_{13}^-} ds_{13}\ \frac{1}{128\pi^3 \underset{\searrow s=m_A=m_\mu}{s}}(p_1p_2)(p_Ap_3)$$

$$= \frac{G_F^2}{4m_\mu^3\pi^3}\int_{m_2^2}^{m_A^2} ds_{23}\int_0^{s_{13}^-}(p_1p_2)(p_Ap_3)\,ds_{13}$$

→ Escribamos p_ip_j en función de s_{13}, s_{23}

$$s_{12} = m_1^2 + m_2^2 + 2p_1p_2 \rightarrow p_1p_2 = \frac{s_{12} - m_1^2 - m_2^2}{2} \overset{m_1\to 0}{=} \frac{s_{12} - m_2^2}{2}$$

Aparte: $s_{13} + s_{23} + s_{12} = M_A^2 + m_1^2 + m_2^2 + m_3^2 \overset{m_1, m_3 \to 0}{=} M_A^2 + m_2^2$

↳ $$p_1p_2 = \frac{M_A^2 - s_{13} - s_{23}}{2}$$

$$p_Ap_3 = (p_1+p_2+p_3)p_3 = p_1p_3 + p_2p_3 + \overset{0 \text{ ya que } m_3\to 0}{p_3^2}$$

$$s_{13} = \underset{0}{m_1^2} + \underset{0}{m_3^2} + 2p_1p_3 \rightarrow p_1p_3 = s_{13}/2$$

$$s_{23} = m_2^2 + \underset{0}{\cancel{m_3^2}} + 2p_2p_3 \rightarrow p_2p_3 = \frac{s_{23} - m_2^2}{2}$$

$$\therefore (p_1p_2)\cdot(p_Ap_3) = \frac{1}{4}\cdot\left(M_A^2 - (s_{13}+s_{23})\right)\cdot\left((s_{13}+s_{23}) - m_2^2\right)$$

$$= \frac{1}{4}\left(M_A^2(s_{13}+s_{23}) - M_A^2m_2^2 - (s_{13}+s_{23})^2 + m_2^2(s_{13}+s_{23})\right)$$

$$= \frac{1}{4}\left((M_A^2+m_2^2)(s_{13}+s_{23}) - (s_{13}+s_{23})^2 - M_A^2m_2^2\right)$$

$x \equiv s_{23}$; $y \equiv s_{13}$ $\Rightarrow \Gamma = \dfrac{G_F^2}{16m_\mu^3\pi^3}\cdot I$

$$I = \int_{m_2^2}^{m_A^2} dx\int_0^{y^-} dy\left[(M_A^2+m_2^2)(x+y) - (x+y)^2 - M_A^2m_2^2\right]$$

$$= \int^{m_A^2} dx\cdot\left[(M_A^2+m_2^2)\frac{(x+y)^2}{2} - \frac{(x+y)^3}{3} - M_A^2m_2^2\,y\right]_0^{M_A^2 - x + M_2^2\left(1-\frac{m_A^2}{x}\right)}$$

$$= \int_{m_2^2}^{m_A^2} dx\left[\frac{(M_A^2+m_2^2)}{2}\left(M_A^2+m_2^2\left(1-\frac{m_A^2}{x}\right)\right)^2 - \frac{1}{3}\left(M_A^2+m_2^2\left(1-\frac{m_A^2}{x}\right)\right)^3 - M_A^2m_2^2\cdot\left(M_A^2 - x + M_2^2\left(1-\frac{m_A^2}{x}\right)\right)\right.$$
$$\left. - (M_A^2+m_2^2)\frac{x^2}{2} + \frac{x^3}{3}\right]$$

$$= \int_{m_2^2}^{m_A^2} dx\left\{\left(M_A^2+m_2^2\left(1-\frac{m_A^2}{x}\right)\right)^2\cdot\left[\frac{M_A^2+m_2^2}{2} - \cancel{\frac{M_A^2+m_2^2\left(1-\frac{m_A^2}{x}\right)}{3}}\right] - M_A^4m_2^2\left(M_A^2 - x\right.\right.$$
$$\left.\left. + M_2^2\left(1-\frac{m_A^2}{x}\right)\right) - (M_A^2+m_2^2)\frac{x^2}{2} + \frac{x^3}{3}\right]\Big\}$$

→ Ver siguiente página

~~$= \frac{(m_A^2+m_2^2)}{2\cdot 3}\left(m_A^2 x + m_2^2(x - m_A^2 \ln x)\right)^3 - \frac{1}{3\cdot 4}\left(m_A^2 x + m_2^2(x - m_A^2 \ln x)\right)^4$~~

Para simplificar notación: $M \equiv m_A^2$, $m \equiv m_2^2$

$$\int_m^M dx \int_0^{M-x+m(1-M/x)} dy \left[(M+m)x + (M+m)y - x^2 - 2xy - y^2 - M m\right]$$

$$= \int_m^M dx \cdot \left[((M+m)x - x^2 - Mm)y + (M+m-2x)\frac{y^2}{2} - y^3/3\right]_0^{M-x+m(1-M/x)}$$

$$= \int_m^M dx \left[((M+m)x - x^2 - Mm)\left(M - x + m - \frac{Mm}{x}\right) + \frac{1}{2}(M+m-2x)\left(M-x+m-\frac{Mm}{x}\right)^2 - \frac{1}{3}\left(M-x+m-\frac{Mm}{x}\right)^3\right]$$

$$= \int_m^M dx \left[x\left((M+m) - x - \frac{Mm}{x}\right)^2 + \frac{1}{2}\left((M+m)-x-\frac{Mm}{x}\right)^2 (M+m-2x) - \frac{1}{3}\left((M+m) - x - \frac{Mm}{x}\right)^3\right]$$

$$= \int_m^M dx \left((M+m) - x - \frac{Mm}{x}\right)^2 \cdot \left(x + \frac{(M+m)}{2} - x - \frac{(M+m)}{3} + \frac{x}{3} + \frac{Mm}{3x}\right)$$

$$= \int_m^M dx \left((M+m) - x - \frac{Mm}{x}\right)^2 \cdot \left(\frac{x}{3} + \frac{(M+m)}{6} + \frac{Mm}{3x}\right)$$

$$= \int_m^M dx \cdot \frac{1}{3}\left((M+m) - x - \frac{Mm}{x}\right)^2 \cdot \left(x + \frac{(M+m)}{2} + \frac{Mm}{x}\right)$$

$$= \int_m^M \frac{dx}{3}\left((M+m)^2 + x^2 + \frac{M^2m^2}{x^2} - 2x(M+m) - 2\frac{Mm}{x}(M+m) + 2Mm\right)\left(x + \frac{Mm}{x} + \frac{(M+m)}{2}\right)$$

$$= \int_m^M \frac{dx}{3}\Big\{\left[(M+m)^2 + 2mM\right]\cdot x + Mm\left((M+m)^2+2mM\right)/x + \frac{(M+m)}{2}\left((M+m)^2+2Mm\right)$$
$$+ x^3 + Mmx + (M+m)x^2/2 + M^2m^2/x + \frac{M^3m^3}{x^3} + \frac{M^2m^2(M+m)}{2x^2}$$
$$- 2x^2(M+m) - 2(M+m)Mm - x(M+m)^2 - 2Mm(M+m) - 2M^2\frac{m^2}{x^2}(M+m) - \frac{Mm(M+m)}{x}\Big\}$$

$$= \int_m^M \frac{dx}{3}\Big\{\frac{M+m}{2}(M^2+m^2+4Mm) - 4(M+m)Mm$$
$$+ x\left[M^2+m^2+4mM + Mm - (M+m)^2\right] + x^2(M+m)\left(\frac{1}{2}-2\right) + x^3$$
$$+ \frac{1}{x}\left[Mm(M^2+m^2+4mM) + M^2m^2 - Mm(M^2+m^2+2mM)\right]$$
$$+ \frac{1}{x^2}M^2m^2(M+m)\cdot\left(\frac{1}{2}-2\right) + M^3m^3/x^3\Big\}$$

$$= \int_m^M \frac{dx}{3} \left\{ \frac{M+m}{2}(M^2+m^2-4mM) + x\cdot(3Mm) + x^2\left(-\frac{3}{2}\right)\cdot(M+m) + x^3 \right.$$

$$\left. + \frac{1}{x}(3M^2m^2) + \frac{1}{x^2}M^2m^2(M+m)\cdot\left(-\frac{3}{2}\right) + \frac{M^3m^3}{x^3} \right\}$$

$$= \frac{1}{3}\left\{ \frac{M+m}{2}(M^2+m^2-4mM)x + \frac{3}{2}Mmx^2 - \frac{1}{2}(M+m)x^3 + \frac{x^4}{4} \right.$$

$$\left. + 3M^2m^2 \ln x + \frac{3}{2}M^2m^2(M+m)\cdot\frac{1}{x} - \frac{M^3m^3}{2x^2} \right\}_m^M$$

$$= \frac{1}{3}\left\{ \frac{M+m}{2}(M^2+m^2-4mM)\cdot(M-m) + \frac{3}{2}Mm(M^2-m^2) - \frac{1}{2}(M+m)(M^3-m^3) \right.$$

$$+ \frac{1}{4}(M^4-m^4) + 3M^2m^2 \ln\frac{M}{m} + \frac{3}{2}(M+m)Mm^2 - \frac{3}{2}(M+m)M^2m$$

$$\left. - \frac{M\,m^3}{2} + \frac{M^3m}{2} \right\}$$

$$= \frac{1}{3}\left\{ \frac{M+m}{2}(\cancel{M^3} + Mm^2 - 4M^2m - M^2m - \cancel{m^3} + 4m^2M) + \frac{3}{2}(M+m)(M-m)Mm \right.$$

$$- \frac{M+m}{2}(\cancel{M^3} - \cancel{m^3}) + \frac{1}{4}(M^2+m^2)(M^2-m^2) + 3M^2m^2 \ln\frac{M}{m}$$

$$\left. + \frac{3}{2}(M+m)(Mm^2-M^2m) - \frac{1}{2}Mm(\underbrace{m^2-M^2}_{(m+M)(m-M)}) \right\}$$

$$= \frac{1}{3}\left\{ \frac{M+m}{2}\Big(5Mm^2 - 5M^2m + 3M^2m - 3Mm^2 + \frac{1}{2}(M^2+m^2)(M-m) \right.$$

$$\left. + 3Mm^2 - 3M^2m - Mm(m-M)\Big) + 3M^2m^2 \ln M/m \right.$$

$$= \frac{1}{6}\left\{ (M+m)\cdot\left\{ 4Mm^2 - 4M^2m + \frac{1}{2}(M^3 - M^2m + m^2M - m^3)\right)\right\} + \frac{6}{6}M^2m^2 \ln\frac{M}{m}$$

$$= \frac{1}{12}\{M+m)\cdot\left\{M^3 + 9Mm^2 - 9M^2m - m^3\right\} + \frac{12}{12}M^2m^2 \ln\frac{M}{m}$$

$$= \frac{1}{12}\left\{ M^4 + 9M^2m^2 - 9M^3m - m^3M + mM^3 + 9Mm^3 - 9M^2m^2 - m^4 + 12M^2m^2 \ln\frac{M}{m}\right\}$$

$$= \frac{1}{12}\left\{ M^4 - 8M^3m + 8Mm^3 - m^4 + 12M^2m^2 \ln\frac{M}{m} \right\}$$

$$= \frac{1\,M^4}{12}\left\{ 1 - 8\frac{m}{M} + 8\left(\frac{m}{M}\right)^3 - \left(\frac{m}{M}\right)^4 - 12\left(\frac{m}{M}\right)^2 \ln\frac{m}{M} \right\}$$

$$= \frac{M^4}{12}\cdot f\left(\frac{m}{M}\right), \quad \text{con} \quad f(x) = 1 - 8x + 8x^3 - 12x^2 \ln x \,.$$

$$\Downarrow$$

$$\Gamma = \frac{G_F^2}{16 M_A^3 \pi^3}\cdot I = \frac{G_F^2}{16\cdot 12\pi^3}\cdot\frac{M_A^8}{M_A^3}\cdot f\left(\frac{m_2^2}{M_A^2}\right) = \frac{G_F^2 m_A^5}{192\pi^3}\cdot f\left(\frac{m_2^2}{M_A^2}\right)$$

Vida media del muón:

$$\tau_\mu = \frac{\hbar}{\Gamma_\mu} = \frac{192\pi^3\hbar}{G_F^2 m_\mu^5\cdot f\left(\frac{m_e^2}{m_\mu^2}\right)}$$

G_F se puede determinar a partir de τ_μ experimental:

$(\tau_\mu)_{exp} = (2,1969803 \pm 0,0000022)\ \mu s$ (según las notas de teoría)

El G_F obtenido es:

valores de PDG — valor aceptado (notas teoría)

$$G_F = \left(\frac{192 \pi^3 \hbar}{m_\mu^5 \tau_\mu f\left(\frac{m_e^2}{m_\mu^2}\right)} \right)^{1/2} = 1,18608528 \cdot 10^{-5}\,\text{GeV}^{-2} \approx (1,1663788 \pm 0,0000007) \times 10^{-5}\,\text{GeV}^{-2}$$

La fórmula deducida es válida también para la desintegración del tauón, teniendo en cuenta el branching ratio $B \simeq 17,85\%$ (notas teoría)

$m_\tau \simeq 1,777$ GeV, $G_F = 1,166 \ldots$ GeV^{-2}

$$\tau_\tau = \frac{192 \pi^3 \hbar \cdot B}{G_F^2 m_\tau^5 f\left(\frac{m_e^2}{m_\tau^2}\right)} = 2,91 \cdot 10^{-13}\,\text{s} = 0,291\,\text{ps}$$ → coincide con experimental

La fórmula se podría haber deducido, salvo constantes, por argumentos dimensionales:

$\mathcal{M} \sim G_F$ (1 vértice)

$|\mathcal{M}|^2 \sim G_F^2$

Como $\mathcal{L}$ tiene 4 campos fermiónicos, cada uno $\propto E^{3/2}$

$\mathcal{L} \sim G_F E^6$

acción unidimensional

Como $[S] = [d^4x \mathcal{L}] \equiv 1 \propto E^{-4} \cdot G_F \cdot E^6 \rightarrow [G_F] = E^{-2}$

$[\Gamma] = E$

$\Gamma \propto |M|^2 = G_F^2$ → La única escala de masas (si $m_e \to 0$) es m_μ

$\Gamma \propto G_F^2 \cdot m_\mu^5$ necesariamente / $[\Gamma] = E$

La integral de dQ_3 implica un factor $\frac{1}{128\pi^3}$

$\Gamma \sim \frac{G_F^2 m_\mu^5}{128\pi^3}$ → El cálculo exacto sólo añade un factor $\frac{3}{2}$ y la $f(x)$, pero no cambia el orden de magnitud de Γ.

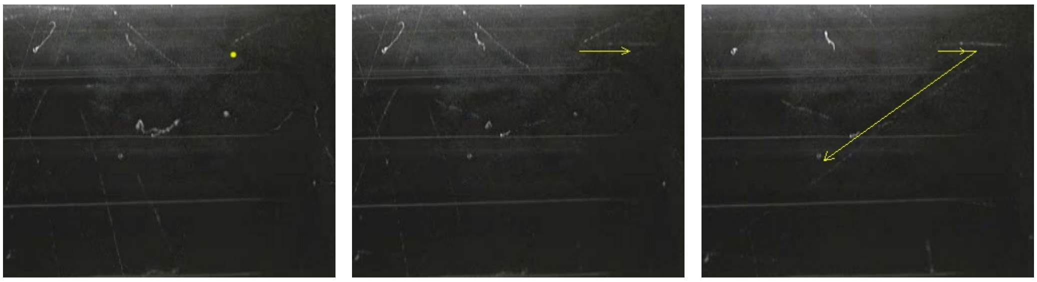
El parámetro efectivo G_F esconde la masa del bosón W^-:

$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} \rightarrow G_F \sim \frac{1}{(290\,\text{GeV})^2}$ → Sólo verás el W a una escala de energías de ~ 250 GeV.

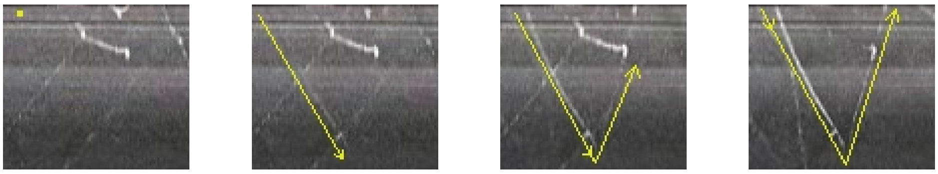
$$W \propto \frac{1}{q^2 - M_W^2} \left(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{M_W^2} \right) \overset{q^2 \ll M_W^2}{\propto} \frac{g^{\alpha\beta}}{M_W^2} \equiv \lambda \sim G_F$$ (teoría efectiva tipo ϕ^3)

Desintegración en vuelo de un muón en la cámara de niebla del IFIC

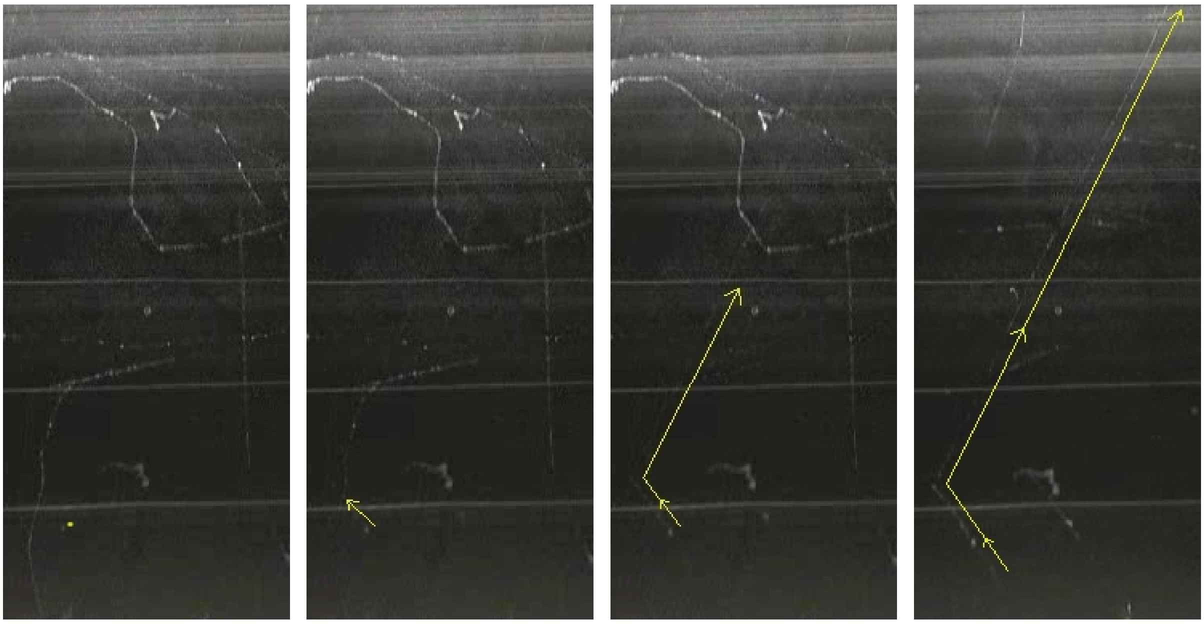
La identificacion la realicé en una práctica del laboratorio de Física Nuclear (en noviembre de 2010). Se hace una selección de los tres candidatos más visibles, que se presentan en las figuras a continuación. Las imágenes no tienen buena definición, por lo que se ha marcado con una línea amarilla paralela a las trazas del suceso. Las trazas corresponden al muón (mas gruesa el ser más pesado) y el electrón (más fina). Ambas son bastante rectas y se distinguen de otros sucesos por la forma de L, la secuencia gruesa-fina y la poca desviación de la trayectoria. Por otro lado, los neutrinos no dejan traza visible.



(a) BG-03:16,71 (b) BG-03:16,87 (c) BG-03:17,11



(a) BL_1-00:02,16 (b) BL_1-00:02,24 (c) BL_1-00:02,32 (d) BL_1-00:02,64



(a) BG-00:54,88 (b) BG 00:54,96 (c) BG 00:55,04 (d) BG 00:55,36

Fernando Hueso González

Link a Memoria de la Cámara de Niebla de difusión: (se ve más en detalle la medida de la desintegración del muón)
http://mural.uv.es/ferhue/4o/fnp/labfnp_p3.pdf

(32)

↳ Ver 30 d)

$$\mathcal{M} \underset{\swarrow E_i \ll M}{=} -\frac{g^2}{M^2}\{-(\bar{u}_3\gamma_5 V_4)(\bar{V}_2\gamma_5 u_1) + (\bar{V}_2\gamma_5 V_4)(\bar{u}_3\gamma_5 u_1)\}.$$

Teoría efectiva tipo $\phi^4 \rightarrow \mathcal{L}_I = -\frac{\lambda}{2!}(\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_5\psi)$

$$A = -i\frac{\lambda}{2!}\int d^4x \, \langle 0 | b_4 a_3 : (\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) : (x) \; b_2^+ a_1^+ | 0 \rangle \cdot 2 \quad \text{... fijo 1}$$

} 2 top. → 1 contr, signo −

→ 4 contr, signo +

$$= -i\lambda(2\pi)^4 \delta^{(4)}(\dots) \cdot (-(\bar{u}_3\gamma_5 V_4)(\bar{V}_2\gamma_5 u_1) + (\bar{V}_2\gamma_5 V_4)(\bar{u}_3\gamma_5 u_1))$$

Coincide! $\lambda = -g^2/M^2$

1ª :

2ª

(33) → ver hoja aparte

(34)

$$\mathcal{L}_W = \frac{g}{2\sqrt{2}}\{W_\alpha^+ \sum_{l=e,\mu,\tau}[\bar{\nu}_e\gamma^\alpha(1-\gamma_5)l] + W_\alpha \sum_{l=e,\mu,\tau}\bar{l}\gamma^\alpha(1-\gamma_5)\nu_e\}$$

a)

$$A = \frac{ig}{2\sqrt{2}}\int d^4x \, \langle 0 | a_e b_{\nu_e} : W_\alpha^+ \bar{\nu}_e \gamma^\alpha (1-\gamma_5) l : a_W^+ | 0 \rangle \quad \rightarrow \text{no acopla}$$

$$+ \frac{ig}{2\sqrt{2}}\int d^4x \, \langle 0 | a_e b_{\nu_e} : W_\alpha \bar{l} \gamma^\alpha (1-\gamma_5) \nu_e : a_W^+ | 0 \rangle$$

$$= -\frac{ig}{2\sqrt{2}}(2\pi)^4 \delta^{(4)}(\dots) \cdot 1 \; \bar{u}_e \gamma^\alpha (1-\gamma_5) V_{\nu_e}$$

$$\mathcal{M} = \frac{g}{2\sqrt{2}} \bar{u}_e \gamma^\alpha (1-\gamma_5) V_{\nu_e} \; ; \; \mathcal{M}^+ = \frac{g}{2\sqrt{2}} V_{\nu_e}^+ \cdot \gamma^0\gamma^0 (1-\gamma_5)\gamma^0\gamma^\alpha\gamma^0\gamma^0 u_e$$

$$= \frac{g}{2\sqrt{2}} \bar{V}_{\nu_e} (1+\gamma_5)\gamma^\alpha u_e$$

$\mathcal{M}^+\mathcal{M} = \rceil$