TQC
ejc a entregar 1 - a^2+b^2 conservación probabilidad
ejc a entregar 2 - lagrangiano Klein Gordon
ejc a entregar 3 - función de ondas simétrica Klein Gordon, microcausalidad
ejc a entregar 4 - función de ondas asimétrica Dirac, momento P_mu y relaciones de conmutación
ejc a entregar 5 - lagrangiano Electromagnético, Gauge, Yukawa, Dyson I_2 ++ consultar
http://www.physics.rutgers.edu/~shapiro/504/lects/beam17_6.pdf
ejc a entregar 6 - Matriz de dispersión S unitaria, condición de Fermi, sección eficaz de Rutherford
ejc a entregar 7 - Teorema de Wick, Compton, identidad de Gordon, abgamma
ejc a entregar 8 - paridad e inversión temporal
archivo mathetmatica feyncalc dispersión compton

③

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- Sea $\rho_0 = \psi^+(x)\psi(x)$ la densidad de probabilidad del campo de Dirac. Probar que el paquete está normalizado si $\sum_K (|a_r|^2 + |b_r|^2) = 1$, con $\sum_K \equiv \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0} = \int d^4k\, \Theta(k^0)\cdot\delta^{(4)}(k^2 - m^2)$

y siendo $\psi(x) = \sum_{K,r} \left(a_r(\vec{k})\, u^r(\vec{k})\, e^{-ikx} + b_r^+(\vec{k})\, v^r(\vec{k})\, e^{ikx}\right)$

Dem: $\int \rho_0 \cdot dV = 1 \quad \forall x^0$ (Tomo $k^0 \equiv E > 0$)

$$\hookrightarrow \int d^3x \sum_{\substack{K,K'\\ r,r'}} \{ a_r^+(\vec{k}) a_{r'}(\vec{k}')\, u^r(\vec{k})^+ u^{r'}(\vec{k}')\, e^{i(k-k')x}$$
$$+ a_r^+(\vec{k})\, b_{r'}(\vec{k}')^+ u^r(\vec{k})^+ v^{r'}(\vec{k}')\, e^{i(k+k')x}$$
$$+ b_r(\vec{k}) a_{r'}(\vec{k}')\, v^r(\vec{k})^+ u^{r'}(\vec{k}')\, e^{-i(k+k')x}$$
$$+ b_r(\vec{k})\, b_{r'}(\vec{k}')^+ v^r(\vec{k})^+ v^{r'}(\vec{k}')\, e^{-i(k-k')x} \}$$

$\rightarrow$ Integramos, salen deltas, se comen exponenciales

$$= \sum_{\substack{K,K'\\ r,r'}} \{ [a_r^+(\vec{k}') a_{r'}(\vec{k}')\, u^r(\vec{k}')^+ u^{r'}(\vec{k}')$$
$$+ b_r(\vec{k}')\, b_{r'}(\vec{k}')^+ v^r(\vec{k}')^+ v^{r'}(\vec{k}')] \cdot \overset{(2\pi)^3}{}\delta^{(3)}(\vec{k} - \vec{k}') \cdot 1 = e^{i(k^0-k^{0\prime})x^0}$$
$$+ [a_r^+(-\vec{k}')\, b_{r'}(\vec{k}')^+ u^r(-\vec{k}')^+ v^{r'}(\vec{k}') \cdot e^{2ik^{0\prime}x^0} + c.c.] \}$$

como $\overset{E}{\overset{\parallel}{k^0}} > 0$, y $k^2 = m^2$ $\therefore$ $k^0 = k^{0\prime}$

$(2\pi)^3\delta^{(3)}(\vec{k} + \vec{k}')$

Por propiedades de los espinores elementales:

$$\begin{cases} u^r(\vec{k})^+ u^{r'}(\vec{k}) = v^r(\vec{k})^+ v^{r'}(\vec{k}) = 2k^0 \delta^{rr'} \\ u^r(\vec{k})^+ v^{r'}(-\vec{k}) = 0 \end{cases}$$

(con $N = \sqrt{k^0 + m}$)

$\Downarrow$

$$\int \rho_0\, d^3x = \sum_{K,r} \sum_{K',r'} [a_r^+(\vec{k}') a_{r'}(\vec{k}') + b_r(\vec{k}')\, b_{r'}(\vec{k}')^+] \cdot \overset{(2\pi)^3 \cdot \delta^{(3)}(\vec{k} - \vec{k}')}{} 2k'^0 \delta^{rr'} + 0$$

$$= \sum_{K,r} \int \frac{d^3k'}{\cancel{2k'^0}} \cdot \frac{1}{(2\pi)^3} \cdot \cancel{(2\pi)^3} \cdot 2k'^0 \cdot \delta^{(3)}(\vec{k} - \vec{k}') \cdot [a_r^+(\vec{k}') a_r(\vec{k}') + b_r(\vec{k}')\, b_r^+(\vec{k}')]$$

$$= \sum_{K,r} [a_r^+(\vec{k})\, a_r(\vec{k}) + b_r(\vec{k})\, b_r^+(\vec{k})] =$$ $\nearrow$ si a, b no son operadores

$$= \sum_{K,r} [|a_r(\vec{k})|^2 + |b_r(\vec{k})|^2] = 1$$

//

Con las convenciones anteriores, el desarrollo de Fourier de la solución de la ecuación de Dirac se escribe (cfr. § 2)

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2E}\left[a_r(k)u^r(k)e^{-ikx} + b_r^+(k)v^r(k)e^{ikx}\right] \tag{11.8}$$

que frecuentemente se escribirá

$$\psi(x) = \sum_k \left[a_r u^r e^{-ikx} + b_r^+ v^r e^{ikx}\right] \tag{11.9}$$

en donde, simbólicamente,

$$\sum_k \equiv \frac{1}{(2\pi)^3}\int \frac{d^3k}{2E} \tag{11.10}$$

El paquete (11.8) estará normalizado si

$$\sum_k \left(|a_r|^2 + |b_r|^2\right) = 1 \tag{11.11}$$

ya que

$$(\psi,\psi) = \int d^3x \sum_k \sum_{k'} \Big\{ a_r^+(k)a_s(k')u^{+r}(k)u^s(k')e^{-i(k'-k)x} +$$

$$b_r(k)b_s^+(k')v^r(k)^+v^s(k')e^{-i(k-k')x} + a_r^+(k)b_s^+(k')u^r(k)^+v^s(k')e^{i(k+k')x}$$

$$b_r(k)a_s(k')v^r(k)^+u^s(k')e^{-i(k+k')x} \Big\}$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3k'}{2E'} \sum_k \Big\{ a_r^+(k)a_s(k')u^r(k)^+u^s(k')e^{-i(E'-E)x^0}\delta(\vec{k}-\vec{k}')$$

$$+ b_r(k)b_s^+(k')v^r(k)^+v^s(k')e^{i(E'-E)x^0}\delta(\vec{k}-\vec{k}') +$$

$$+ a_r^+(k)b_s^+(k')u^r(k)^+v^s(k')\delta(\vec{k}+\vec{k}')e^{i(E+E')x^0} +$$

$$+ b_r(k)a_s(k')v^r(k)^+u^s(k')\delta(\vec{k}-\vec{k}')e^{-i(E+E')x^0} =$$

$$= \sum_k \frac{1}{2E}\Big\{ a_r^+a_s \underbrace{u^r(k)u^s(k)}_{2E\delta^{rs}} + b_r(k)b_s^+(k)\underbrace{v^r(k)^+v^s(k)}_{2E\delta^{rs}}$$

$$+ a_r^+(\vec{k})b_s^+(-\vec{k})\underbrace{u^r(\vec{k})^+v^s(-\vec{k})}_{u_+^{r+}u_-^s=0}e^{i2Ex^0} + b_r(\vec{k})a_s(-\vec{k})\underbrace{v^r(\vec{k})^+u^s(-\vec{k})}_{u_-^r(-\vec{k})u_+^s(-\vec{k})=0}e^{-2iEx^0}\Big\}$$

$$= \sum_k \left\{ a_r^+(k)a^r(k) + b_r(k)b^{+r}(k) \right\} \Rightarrow (11.11) \quad .$$

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- Encontrar Lagrangiano de Klein-Gordon complejo.

Suponemos campo de Klein-Gordon ϕ complejo, con $\phi \neq \phi^+$

Imponemos que el lagrangiano $\mathcal{L}$ sea hermítico: $\mathcal{L} = \mathcal{L}^+$

Aplicando las ecuaciones de Euler-Lagrange $\begin{cases} \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \dfrac{\partial \mathcal{L}}{\partial \phi} = 0 \\ \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)} - \dfrac{\partial \mathcal{L}}{\partial \phi^+} = 0 \end{cases}$ debemos obtener la ecuación de Klein Gordon:

$$(\Box + m^2)\phi^+ = 0 \quad ; \quad (\Box + m^2)\phi = 0$$

Como $\mathcal{L}$ es hermítico, se intuye que debe ser de la forma:
$\mathcal{L} \propto \phi^+ \phi$ o bien $\mathcal{L} \propto \phi + \phi^+$, con $\mathcal{L}(\phi, \phi^+, \partial_\mu \phi, \partial_\mu \phi^+)$

- ↓ Aplicamos E-L y comparamos con K-G:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi^+ + m^2 \phi^+ \Rightarrow \mathcal{L} = \partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi + f(\phi^+, \partial_\mu \phi^+)$$

- Integro asociando términos separadamente

↳ Lo igualo con K-G para ϕ^+ porque al derivar respecto a ϕ nos queda la parte en ϕ^+ (si no, $\mathcal{L}$ tendría términos del tipo ϕ^{2+} que no sería hermítico).
Y viceversa:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^+} = \partial_\mu \partial^\mu \phi + m^2 \phi \quad \rightarrow f = 0$$

$$\hookrightarrow \boxed{\mathcal{L}_{KG} = \partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi} //$$

Es hermítico ~~y cumple la ecuación de Klein-Gordon~~ da lugar a (que no es lo mismo)

- Encontrar la corriente conservada bajo la transformación

$$\begin{cases} \phi'(x) = e^{-iq\alpha} \phi(x) \\ \phi^{*\prime}(x) = e^{iq\alpha} \phi^*(x) \end{cases} \rightarrow \text{Deja invariante el Lagrangiano por ser hermítico (invariante de fase)}$$

Estudiemos la transformación como función de α:
∴ parámetro

$$\phi'(\alpha) = e^{-iq\alpha} \phi \qquad \rightarrow \text{Identidad si } \alpha = 0$$

$$\frac{d\mathcal{L}}{d\alpha} = 0 \rightarrow \text{Invariante por } \alpha, \text{ con } \mathcal{L} = \partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi = \mathcal{L}' = \partial^\mu \phi'^+ \partial_\mu \phi' - m^2 \phi'^+ \phi'$$

$$L(\phi, \partial_\mu \phi, \phi^+, \partial_\mu \phi^{+\prime})$$

$$\overset{\parallel}{\sum_{i=1}^{2}} \left[\frac{\partial L}{\partial \phi'_i} \cdot \frac{\partial \phi'_i}{\partial \alpha} + \frac{\partial L}{\partial(\partial_\mu \phi'_i)} \cdot \frac{d(\partial_\mu \phi'_i)}{d\alpha} \right] = 0$$

con $\phi'_1 \equiv \phi'$; $\phi'_2 \equiv \phi'^+$

$$\sum_{i=1}^{2}\Big[\overbrace{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i')}}^{\text{sustituyo E-Lag.}}\cdot\frac{\partial\phi_i'}{\partial\alpha}+\frac{\partial L}{\partial(\partial_\mu\phi_i')}\cdot\frac{d}{d\alpha}(\partial_\mu\phi_i')\Big]=0 \rightarrow \text{Derivada del Producto}$$

$$\Rightarrow \partial_\mu\sum_{i=1}^{2}\Big(\frac{\partial L}{\partial(\partial_\mu\phi_i)}\cdot\frac{d}{d\alpha}(\phi_i)\Big]=0$$

$$\Rightarrow \partial_\mu j^\mu = 0 \;// \quad \Rightarrow \boxed{j^\mu=\sum_{i=1}^{2}\frac{\partial L}{\partial(\partial_\mu\phi_i')}\cdot\frac{d}{d\alpha}(\phi_i')}$$

Corriente conservada

(Tma. de Noether)

$$\Downarrow$$

$$j^\mu=\partial^\mu\phi'^+\cdot(-iq)\cdot e^{-iq\alpha}\phi+(\partial^\mu\phi')\cdot(iq)\,e^{-iq\alpha}\phi^+$$

$$=iq\big((\partial^\mu\phi)\phi^+-(\partial^\mu\phi^+)\phi\big)\;//$$

⇒ Corriente conservada asociada a la invariancia de fase en el lagrangiano

La carga conservada (será) ~~puede interpretarse como:~~

$$Q=\int d^3x\;j^0=\int d^3x\,(iq)\,(\dot\phi\,\phi^+-\dot\phi^+\phi)\;//$$

Alternativamente:

$$\mathcal{L}=\mathcal{L}(\phi_i,\partial_\mu\phi_i)=\partial_\mu\phi^+\partial^\mu\phi-m^2\phi^+\phi$$

o por E-L

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i)=\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i+\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i')}\delta\phi_i\Big)-\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big)\cdot\delta\phi_i$$

por ser inv. Lagr. (simetría

$$=0=\partial_\mu j^\mu \Rightarrow j^\mu=\sum_{i=1}^{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\cdot\delta\phi_i$$

$$\underset{\phi_1}{\phi'}=\phi\cdot e^{-iq\alpha}\simeq\phi(1-iq\alpha)\rightarrow\delta\phi_1\equiv\phi'-\phi=-iq\alpha\phi$$

$$\phi'^+=\underset{\phi_2}{\phi^+}e^{+iq\alpha}\simeq\phi(1+iq\alpha)\rightarrow\delta\phi_2=\phi'^+-\phi^+=iq\alpha\phi^+$$

$$\rightarrow j^\mu=\partial^\mu\phi^+\delta\phi+\partial^\mu\phi\,\delta\phi^+=(\partial^\mu\phi^+)(-iq\alpha)\phi+(\partial^\mu\phi)(iq\alpha)\phi^+$$

$$\overset{\alpha=1}{=}iq\big((\partial^\mu\phi)\phi^+-(\partial^\mu\phi^+)\phi\big)$$

⇒ Coincide ✓

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- Comprobar que la "función de ondas" $\langle x_1 x_2 | k_1 k_2 \rangle$ asociada a dos bosones$^{(1,2)}$ es simétrica respecto al intercambio de las partículas 1-2.

③

$$|x\rangle \equiv \phi^{(+)*}(x)|0\rangle \quad \rightarrow \quad \langle x| = \langle 0|\phi^{(+)}(x)$$

$$\phi^{(+)}(x) = \sum_k a(k)e^{-ikx}$$

$$|k\rangle \equiv a^+(\vec{k})|0\rangle$$

$$\Downarrow$$

$$\langle x_1 x_2 | k_1 k_2 \rangle = \langle 0|\phi^{(+)}(x_2)\,\phi^{(+)}(x_1)\,a^+(\vec{k_1})\,a^+(\vec{k_2})|0\rangle$$

$$= \sum_{k,k'} e^{-i(k x_1 + k' x_2)} \langle 0| a(\vec{k'})\,a(\vec{k})\,a^+(\vec{k_1})\,a^+(\vec{k_2})|0\rangle$$

Relas de conmuta para bosones

$$\boxed{\begin{aligned} &[a(\vec{k}), a^+(\vec{k}')] = \Delta_{\vec{k}\vec{k}'} \\ &[a, a] = 0 = [a^+, a^+] \\ &a(\vec{k})a^+(\vec{k}') = \Delta_{\vec{k}\vec{k}'} + a^+(\vec{k}')a(\vec{k}) \end{aligned}}$$

$$\Rightarrow P \equiv \langle 0| a(\vec{k}')a(\vec{k})\,a^+(\vec{k_1})a^+(\vec{k_2})|0\rangle =$$

$$= \langle 0| a(\vec{k}')\left([a(\vec{k}), a^+(\vec{k_1})] + a^+(\vec{k_1})a(\vec{k})\right)a^+(\vec{k_2})|0\rangle$$

$$= \langle 0| \Big\{ \Delta_{\vec{k}\vec{k_1}}\left(\Delta_{\vec{k}'\vec{k_2}} + a^+(\vec{k_2})a(\vec{k}')\right) + \left(\Delta_{\vec{k}'\vec{k_1}} + a^+(\vec{k_1})\cdot a(\vec{k}')\right)\cdot$$

$$\cdot\left(\Delta_{\vec{k}\vec{k_2}} + a^+(\vec{k_2})a(\vec{k})\right)\Big\}|0\rangle$$

$$= \langle 0| \Big\{ \Delta_{\vec{k}\vec{k_1}}\Delta_{\vec{k}'\vec{k_2}} + \Delta_{\vec{k}\vec{k_1}}a^+(\vec{k_2})a(\vec{k}') + \Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}} +$$

$$+ \Delta_{\vec{k}'\vec{k_1}}a^+(\vec{k_2})a(\vec{k}) + a^+(\vec{k_1})a(\vec{k}')\Delta_{\vec{k}\vec{k_2}} + a^+(\vec{k_1})a(\vec{k}')\cdot$$

$$\cdot a^+(\vec{k_2})a(\vec{k})\Big\}|0\rangle$$

Sólo sobreviven los términos sin operadores, pues $a|0\rangle = 0$, $\langle 0|a^+ = 0$

$$P = \left(\Delta_{\vec{k}\vec{k_1}}\Delta_{\vec{k}'\vec{k_2}} + \Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}}\right)\cdot\underbrace{\langle 0|0\rangle}_{=1}$$

Por tanto, aplicando que $\boxed{\sum_{k'} f(k')\cdot\Delta_{\vec{k}\vec{k}'} = f(k)}$

$$\langle x_1, x_2 | k_1 k_2 \rangle = \sum_{k,k'} e^{-i(kx_1 + k'x_2)}\left\{\Delta_{\vec{k}\vec{k_1}}\Delta_{\vec{k}'\vec{k_2}} + \Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}}\right\}$$

$$= e^{-i(k_1 x_1 + k_2 x_2)} + e^{-i(k_2 x_1 + k_1 x_2)}$$

Es obvio que al intercambiar los índices 1 y 2 de k, no de la x (o viceversa), la función de ondas no cambia, signo + entre ambos términos, por lo que concluimos que es simétrica respecto al intercambio de partículas. (estadística de Bose): $\phi(x_1, x_2) = \phi(x_2, x_1)$

- Si por error cuantizamos el campo de Klein - Gordon real (escalar neutro, por simplificar) con anticonmutadores $\{a(k), a^+(k')\} = \Delta_{\vec{k}\vec{k}'}$, etc.; probar que se viola el principio de microcausalidad: $\{\phi(x), \phi(y)\} \neq 0$ para $(x-y)^2 < 0$

$\{a,a\} = \{a^+,a^+\} = 0$

$$\phi(x) = \sum_k (a(\vec{k})\, e^{-ikx} + a^+(\vec{k}) e^{ikx})$$

$$\{\phi(x), \phi(y)\} = \sum_{k,k'} \Big(\overset{0}{\{a(\vec{k}), a(\vec{k}')\}} \cdot e^{-i(kx+k'y)} + \overset{\Delta_{\vec{k}\vec{k}'}}{\{a(\vec{k}), a^+(\vec{k}')\}} \cdot e^{-i(kx-k'y)}$$

$$+ \underset{\Delta_{\vec{k}\vec{k}'}}{\underset{\shortparallel}{\{a^+(\vec{k}), a(\vec{k}')\}}} e^{i(kx-k'y)} + \underset{0}{\underset{\shortparallel}{\{a^+(\vec{k}), a^+(\vec{k}')\}}} \cdot e^{i(kx+k'y)} \Big)$$

$$= \sum_{k,k'} \left(\Delta_{\vec{k}\vec{k}'}\, e^{-ik(x-y)} + \Delta_{\vec{k}\vec{k}'} \cdot e^{ik(x-y)} \right) = \sum_k \left(e^{-ik(x-y)} + e^{+ik(x-y)} \right)$$

$$\Rightarrow \Delta^\pm(x) \equiv \mp \frac{i}{(2\pi)^3} \int \frac{d^3\vec{k}}{2k^0}\, e^{\mp ikx} = \mp i \sum_k e^{\mp ikx}$$

↳ $\{\phi(x), \phi(y)\} = i(\Delta^+(x-y) - \Delta^-(x-y)) \equiv \Delta_1(x-y)$ //

↳ No se obtiene $i\Delta(x-y) = i(\Delta^+(x-y) + \Delta^-(x-y))$ debido al cambio de signo entre los 2 términos.

Se puede probar ⊛ que, a diferencia de la función $\Delta(x-y)$, la función $\Delta_1(x-y)$ viola el principio de microcausalidad, pues $\Delta_1(x-y) \neq 0$ para $(x-y)^2 < 0$. Si cuantizamos con conmutadores, sí se obtendría $\Delta(x-y)$, lo que será (una vez lo comparemos con el campo de Dirac) la motivación para formular el teorema de conexión espín - estadística.

⊛ $\Delta_1(x^0 = 0, \vec{x}) = \sum_{\vec{k}} (e^{-i\vec{k}\vec{x}} + e^{i\vec{k}\vec{x}}) \neq 0$ y con $x^2 \overset{=-\vec{x}^2}{<} 0$:

Como Δ_1 es invariante bajo T. $\mathcal{L}^\uparrow$, $\Delta_1(x) \neq 0$ para $\forall x^2 < 0$

incluso con $x^0 \neq 0$

$\textcircled{3}^{\downarrow}$

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T. Cuántica de Campos I

- Comprobar que la "función de ondas" asociada a dos fermiones$^{(1,2)}$ $\langle x_1 x_2 | k_1, r_1; k_2 r_2\rangle$ es antisimétrica respecto al intercambio de las partículas 1-2, lo que conduce que no puede haber una doble ocupación de un estado fermiónico.

$|k, r\rangle \equiv a_r^+(\vec{k})|0\rangle$; $|\underset{r_1, r_2}{k_1 k_2}\rangle \equiv a_{r_1}^+(\vec{k_1})a_{r_2}^+(\vec{k_2})|0\rangle$; $f_{\vec{k}}^{\,r}(x) \ominus u^r(\vec{k})\cdot e^{-ikx}$; $\equiv$

$a_r(\vec{k})|0\rangle = 0$; $g_{\vec{k}}^{\,r}(x) = v^r(\vec{k})\, e^{ikx}$

$$\psi = \sum_{k,r}\left(a_r(\vec{k})\, f_{\vec{k}}^{\,r}(x) + b_r^+(\vec{k})\, g^r_{\vec{k}}(x)\right) \underset{\equiv}{\ominus} \psi^{(+)}(x) + \psi^{(-)}(x)$$

en KG no

$$\langle x| \equiv \langle 0|\psi^{(+)}(x) \rightarrow \langle x_1 x_2| \equiv \langle 0|\psi^{(+)}(x_1)\,\psi^{(+)}(x_2)$$

orden es convención pero IMPORTA

lo llamo

$$P \equiv \langle x_1 x_2 | k_1, r_1; k_2, r_2\rangle = \langle 0|\psi^{(+)}(x_1)\,\psi^{(+)}(x_2)\, a_{r_1}^+(\vec{k_1})\, a_{r_2}^+(\vec{k_2})|0\rangle$$

$$= \langle 0| \sum_{k,r} a_r(\vec{k})\, f_{\vec{k}}^{\,r}(x_1)\cdot \sum_{k',r'} f_{\vec{k}'}^{\,r'}(x_2)\, a_{r'}(\vec{k}')\cdot a_{r_1}^+(\vec{k_1})\, a_{r_2}^+(\vec{k_2})|0\rangle$$

$$= \langle 0| \boxed{\sum_{\substack{k,k'\\ r,r'}} f_{\vec{k}}^{\,r}(x_1)\, f_{\vec{k}'}^{\,r'}(x_2)}\cdot \underbrace{a_r(\vec{k})\, a_{r'}(\vec{k}')\, a_{r_1}^+(\vec{k_1})\, a_{r_2}^+(\vec{k_2})}_{\equiv A}|0\rangle$$

puede salir fuera

$\equiv A \rightarrow$ N-ordenamos

Relaciones canónicas de anticonmutación:

Utilizamos que: $\{a_r^+(\vec{k}), a_s^+(\vec{k}')\} = 0 = \{a_r(\vec{k}), a_s(\vec{k}')\}$

y $\langle 0|a_r(\vec{k})\, a_{r'}(\vec{k}')\, a_{r_1}^+(\vec{k}_1)\, a_{r_2}^+(\vec{k}')|0\rangle$ $\{a_r(\vec{k}), a_s^+(\vec{k}')\} = \delta_{rs}\,\Delta_{\vec{k}\vec{k}'}$

$= \Delta$

$$A = a_r(\vec{k})\cdot\left[\delta_{r'r_1}\Delta_{\vec{k}'\vec{k_1}} - a_{r_1}^+(\vec{k_1})\, a_{r'}(\vec{k}')\right]\cdot a_{r_2}^+(\vec{k_2}) \qquad a_r(\vec{k})$$

$$= \delta_{r'r_1}\Delta_{\vec{k}'\vec{k_1}}\cdot\left[\delta_{r r_2}\Delta_{\vec{k}\vec{k_2}} - a_{r_2}^+(\vec{k_2})\, a_r(\vec{k})\right] - \left[\delta_{r,r_1}\Delta_{\vec{k}\vec{k_1}} - a_{r_1}^+(\vec{k_1})\right]\cdot$$

$$\cdot\left[\delta_{r'r_2}\Delta_{\vec{k}'\vec{k_2}} - a_{r_2}^+(\vec{k_2})\, a_{r'}(\vec{k}')\right]$$

$$= \delta_{r'r_1}\delta_{r,r_2}\Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}} - \delta_{r'r_1}\Delta_{\vec{k}'\vec{k_1}}\, a_{r_2}^+(\vec{k_2})\, a_r(\vec{k})$$

$$- \delta_{r,r_1}\delta_{r',r_2}\Delta_{\vec{k}\vec{k_1}}\Delta_{\vec{k}'\vec{k_2}} + \delta_{r,r_1}\Delta_{\vec{k}\vec{k_1}}\, a_{r_2}^+(\vec{k_2})\, a_{r'}(\vec{k}') + \delta_{r'r_2}\Delta_{\vec{k}'\vec{k_2}}\cdot$$

$$\cdot\, a_{r_1}^+(\vec{k_1})\, a_r(\vec{k}) - a_{r_1}^+(\vec{k_1})\, a_r(\vec{k})\, a_{r_2}^+(\vec{k_2})\, a_{r'}(\vec{k}')$$

Los dos términos con operadores tienen un op. destrucción a la derecha. Al aplicar sobre el estado vacío, se cancelan:

$$A|0\rangle = \delta_{r'r_1}\delta_{r,r_2}\Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}}|0\rangle - \delta_{r,r_1}\delta_{r'r_2}\Delta_{\vec{k}\vec{k_1}}\Delta_{\vec{k}'\vec{k_2}}|0\rangle$$

↳ Nótese el cambio de signo respecto a K-G, que conducirá a la antisimetría

Y queda que:

$$P = \sum_{\substack{k,k'\\ r,r'}} f_{\vec{k}}^{\,r}(x_1)\, f_{\vec{k}'}^{\,r'}(x_2)\cdot\left(\delta_{r'r_1}\delta_{r r_2}\Delta_{\vec{k}'\vec{k_1}}\Delta_{\vec{k}\vec{k_2}} - \delta_{r,r_1}\delta_{r'r_2}\Delta_{\vec{k}'\vec{k_2}}\Delta_{\vec{k}\vec{k_1}}\right)\cdot$$

$1 = \langle 0|0\rangle$

sumatorio y deltas fuerzan el valor de $f_{\vec{k}_i}^{r_i}(x_i)$

$$\Rightarrow P = f_{\vec{k}_2}^{r_2}(x_1)\cdot f_{\vec{k}_1}^{r_1}(x_2) - f_{\vec{k}_1}^{r_1}(x_1)\, f_{\vec{k}_2}^{r_2}(x_2) \;//$$

Si intercambiamos las partículas 1 y 2, es decir, los subíndices que describen a la partícula (k,r), pero no el de la posición, que es "externo" a la partícula, queda que: ↳(o bien viceversa)

$$\tilde{P} \equiv P\Big(\underset{\substack{\text{en } k,r\\ \text{NO en } x}}{1\leftrightarrow 2}\Big)^{\substack{\text{ó en } x\\ \text{NO en } k,r}} = f_{\vec{k}_1}^{r_1}(x_1)\, f_{\vec{k}_2}^{r_2}(x_2) - f_{\vec{k}_2}^{r_2}(x_1)\, f_{\vec{k}_1}^{r_1}(x_2) = -P \;//$$

es decir, la "función de ondas" es antisimétrica para fermiones, lo que constituye el principio de exclusión de Pauli! $\psi(x_1,x_2) \overset{\shortparallel}{\ } -\psi(x_2,x_1)$

- Calcular P^μ (cuadrimomento) en términos de los operadores de creación y destrucción y las relaciones de conmutación con el campo.

$$\mathcal{L}_{DIRAC} = -\bar{\psi}(-i\gamma^\mu \partial_\mu + m)\psi$$

Invariancia bajo traslaciones - - - - - → conservación, tensor energía impulso $\Theta^{\mu\nu}$

$\Downarrow$ $\qquad \Theta^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi$

$$P^\mu = \int d^3x\; \Theta^{0\mu} \overset{\llcorner}{=} i\int d^3x\; \psi^+_{(x)} \partial^\mu \psi_{(x)} \equiv i(\psi, \partial^\mu \psi)$$

$$\psi(x) = \sum_{\vec{k},r}\Big(a_r(\vec{k})\, \underbrace{u^r(\vec{k})\, e^{-ikx}}_{f_{\vec{k}}^r(x)} + b_r^+(\vec{k})\, \underbrace{v^r(\vec{k})\, e^{ikx}}_{g_{\vec{k}}^r(x)}\Big)$$

$$\partial^\mu \psi(x) = \sum_{\vec{k},r}\Big(a_r(\vec{k})(-ik^\mu) f_{\vec{k}}^r(x) + b_r^+(\vec{k})(ik^\mu) g_{\vec{k}}^r(x)\Big)$$

$$= (-i)\sum_{\vec{k},r} k^\mu \Big(a_r(\vec{k}) f_{\vec{k}}^r(x) - b_r^+(\vec{k}) g_{\vec{k}}^r(x)\Big)$$

$\Downarrow$

$$P^\mu = i(-i)\cdot \sum_{\substack{\vec{k}\vec{k}'\\ r r'}} k'^\mu \Big(a_r(\vec{k}) f_{\vec{k}}^r(x) + b_r^+(\vec{k}) g_{\vec{k}}^r(x),\; a_{r'}(\vec{k}') f_{\vec{k}'}^{r'}(x) - b_{r'}^+(\vec{k}') g_{\vec{k}'}^{r'}(x)\Big)$$

↙ Sacamos operadores del producto escalar, los de la izquierda salen adjuntos. Hay que respetar el orden

$$= \sum_{\substack{\vec{k},\vec{k}'\\ r r'}} k'^\mu \cdot \Big\{ a_r^+(\vec{k})\, a_{r'}(\vec{k}')\cdot\big(f_{\vec{k}}^r(x), f_{\vec{k}'}^{r'}(x)\big) - a_r^+(\vec{k})\, b_{r'}^+(\vec{k}')\cdot\big(f_{\vec{k}}^r(x), g_{\vec{k}'}^{r'}(x)\big)$$
$$+ b_r(\vec{k})\, a_{r'}(\vec{k}')\cdot\big(g_{\vec{k}}^r(x), f_{\vec{k}'}^{r'}(x)\big) - b_r(\vec{k})\, b_{r'}^+(\vec{k}')\big(g_{\vec{k}}^r(x), g_{\vec{k}'}^{r'}(x)\big)\Big\}$$

Propiedades de f y g:

- $$\boxed{\big(f_{\vec{k}}^r(x), f_{\vec{k}'}^{r'}(x)\big)} = \int d^3x\, f_{\vec{k}}^r(x)^+ f_{\vec{k}'}^{r'}(x) = \overbrace{\int d^3x\cdot e^{i(k-k')x}}^{(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') = \frac{\Delta_{\vec{k}\vec{k}'}}{2E}}\; \overbrace{u_r^+(\vec{k})\, u_{r'}(\vec{k}')}^{2E\cdot\delta_{rr'} \text{ (propiedades de espinores)}} = \Delta_{\vec{k}\vec{k}'}\,\delta_{rr'}$$

$$e^{i(k-k')x} = e^{i(E-E')x^0}\, e^{-i(\vec{k}-\vec{k}')\vec{x}}\,!$$

$$\big(g_{\vec{k}}^r(x), g_{\vec{k}'}^{r'}(x)\big) = \Delta_{\vec{k}\vec{k}'}\,\delta^{rr'}$$

Análogamente

$$\big(f_{\vec{k}}^r(x), g_{\vec{k}'}^{r'}(x)\big) = 0 = (g,f)^+$$

$u_r^+ v_{r'} = 0 \qquad ↳ (g,f) = 0$

→ Por tanto, de los 4 productos escalares que hay en P^μ, sólo sobreviven dos (ver siguiente página)

$$P^\mu = \sum_{\substack{\vec{k},\vec{k}' \\ r,r'}} K'^\mu \left\{ a_r^+(\vec{k})\, a_{r'}(\vec{k}')\, \Delta_{\vec{k}\vec{k}'}\, \delta^{rr'} - b_r(\vec{k})\, b_{r'}^+(\vec{k}')\, \Delta_{\vec{k}\vec{k}'}\, \delta^{rr'} \right\}$$

$$\boxed{= \sum_{\vec{k},r} K^\mu \left\{ a_r^+(\vec{k})\, a_r(\vec{k}) - b_r(\vec{k})\, b_r^+(\vec{k}) \right\}} \;/\!/$$

cambias origen

$:P^\mu: = P^\mu - \langle 0 | P^\mu | 0 \rangle$ … restas valor esperado en el vacío

$$\hookrightarrow P^\mu = \sum_{\vec{k},r} K^\mu \Big\{ a_r^+(\vec{k})\, a_r(\vec{k}) - \underbrace{\{b_r(\vec{k}), b_r^+(\vec{k})\}}_{=\Delta_{\vec{k}\vec{k}}\delta_{rr}} + b_r^+(\vec{k})\, b_r(\vec{k}) \Big)\Big)$$

$\langle 0) P^\mu | 0 \rangle = - \sum_{\vec{k},r} K^\mu \Delta_{\vec{k}\vec{k}} \delta_{rr}$ → tetravector, NO contiene operadores

$$\hookrightarrow \boxed{:P^\mu: = \sum_{\vec{k},r} K^\mu \left(a_r^+(\vec{k})\, a_r(\vec{k}) + b_r^+(\vec{k})\, b_r(\vec{k}) \right)\Big)} \;/\!/$$

$$= \sum_{\vec{k},r} K^\mu \left(N_r(\vec{k}) + \bar{N}_r(\vec{k}) \right)\Big)$$

→ Nótese que las relaciones de conmutación de P^μ son las mismas que las de $:P^\mu:$

ya que $[P^\mu, A] = [:P^\mu:, A] + [\langle 0 | P^\mu | 0 \rangle, A] = [:P^\mu:, A] + 0$

$\langle 0 | P^\mu | 0 \rangle$ no contiene operadores

Relaciones de conmutación de P^μ (ó $:P^\mu:$)

$$[AB, C] = ABC - CAB = ABC + ACB - ACB - CAB = A\{B, C\} - \{A, C\}B$$

$$[P^\mu, \psi(x)] = \sum_{\substack{\vec{k},r \\ \vec{k}',r'}} K^\mu \Big[\underbrace{a_r^+(\vec{k})\, a_r(\vec{k})}_{N_r(\vec{k})} - b_r(\vec{k})\, b_r^+(\vec{k}),\; a_{r'}(\vec{k}')\, f_{\vec{k}'}^{r'}(x) + b_{r'}^+(\vec{k}')\, g_{\vec{k}'}^{r'}(x) \Big]$$

↳ 4 conmutadores: calculo por separado

① $[N_r(\vec{k}), a_{r'}(\vec{k}')] = [a_r^+(\vec{k})\, a_r(\vec{k}), a_{r'}(\vec{k}')] = a_r^+(\vec{k}) \overset{0}{\{a_r(\vec{k}), a_{r'}(\vec{k}')\}} - \{a_r^+(\vec{k}), a_{r'}(\vec{k}')\}\, a_r(\vec{k}) = -\Delta_{\vec{k}\vec{k}'} \delta_{rr'}\, a_r(\vec{k})$

② ↳ $[-b_r(\vec{k})\, b_r^+(\vec{k}), a_{r'}(\vec{k}')] = b_r(\vec{k}) \underset{0}{\{b_r^+(\vec{k}), a_{r'}(\vec{k}')\}} - \overset{0}{\{b_r(\vec{k}), a_{r'}(\vec{k}')\}}\, b_r^+(\vec{k})$

$= 0$

$$\Rightarrow \boxed{[P^\mu, a_{r'}(\vec{k}')] = -\sum_{\vec{k},r} K^\mu \overset{①}{\Delta_{\vec{k}\vec{k}'} \delta_{rr'}\, a_r(\vec{k})} + \overset{②}{0} = -a_{r'}(\vec{k}') \cdot K'^\mu} \;/\!/$$

③ $[a_r^+(\vec{k})\, a_r(\vec{k}), b_{r'}^+(\vec{k}')] = a_r^+(\vec{k}) \underset{0}{\{a_r(\vec{k}), b_{r'}^+(\vec{k}')\}} - \underset{0}{\{a_r^+(\vec{k}), b_{r'}^+(\vec{k}')\}} \cdot a_r(\vec{k})$

$= 0$

④ $[b_r(\vec{k})\, b_r^+(\vec{k}), b_{r'}^+(\vec{k}')] = b_r(\vec{k}) \cdot \overset{0}{\{b_r^+(\vec{k}), b_{r'}^+(\vec{k}')\}} - \{b_r(\vec{k}), b_{r'}^+(\vec{k}')\} \cdot b_r^+(\vec{k})$

$= -\delta_{rr'} \Delta_{\vec{k}\vec{k}'}\, b_r^+(\vec{k})$

$$\boxed{[P^\mu, b_{r'}^+(\vec{k}')] = \sum_{\vec{k},r} K^\mu (-)(-)\delta_{rr'} \Delta_{\vec{k}\vec{k}'}\, b_r^+(\vec{k}) = K'^\mu\, b_{r'}(\vec{k}')}$$

$$[P^\mu, \psi(x)] = \sum_{\substack{\vec{k},r \\ \vec{k}',r'}} K^\mu \left(-\Delta_{\vec{k}\vec{k}'} \delta_{rr'}\, f_{\vec{k}'}^{r'}(x)\, a_r(\vec{k}) + 0 - 0 - \left(-\delta_{rr'} \Delta_{\vec{k}\vec{k}'}\, g_{\vec{k}'}^{r'}(x)\, b_r^+(\vec{k})\right)\right)$$

$$= -\sum_{\vec{k},r} K^\mu \left(f_{\vec{k}}^r(x)\, a_r(\vec{k}) - g_{\vec{k}}^r(x)\, b_r^+(\vec{k}) \right) \;/\!/$$

Como $\partial^\mu f_{\vec{k}}^{\,r}(x) \equiv \frac{\partial}{\partial x_\mu}\left(u_r(\vec{k})e^{-ik^\mu x_\mu}\right) = -ik^\mu f_{\vec{k}}^{\,r}(x)$

$$\partial^\mu g_{\vec{k}}^{\,r}(x) \equiv \frac{\partial}{\partial x_\mu}\left(v_r(\vec{k})e^{ik^\mu x_\mu}\right) = ik^\mu g_{\vec{k}}^{\,r}(x)$$

Entonces $i\partial^\mu \psi = \sum_{k,r} k^\mu \left(f_{\vec{k}}^{\,r}(x)\, a_r(\vec{k}) - g_{\vec{k}}^{\,r}(x)\, b_r^+(\vec{k})\right)$

y queda:

$$\boxed{[P^\mu, \psi(x)] = -i\partial^\mu \psi(x)}$$

Análogamente:

$$[P^\mu, \psi^+_{(x)}] = \sum_{\substack{k,r\\k,r'}} k^\mu \left[a_r^+(\vec{k})a_r(\vec{k}) - b_r(\vec{k})b_{r'}^+(\vec{k}),\ a_{r'}^+(\vec{k}')f_{\vec{k}'}^{\,r'}(x)^+ + b_{r'}(\vec{k}')g_{\vec{k}'}^{\,r'}(x)^+\right]$$

① $[a_r^+(\vec{k})a_r(\vec{k}), a_{r'}^+(\vec{k}')] = a_r^+(\vec{k})\{a_r(\vec{k}), a_{r'}^+(\vec{k}')\} - 0 = a_r^+(\vec{k})\delta_{rr'}\Delta_{\vec{k}\vec{k}'}$

② $[a_r^+(\vec{k})a_r(\vec{k}), b_{r'}(\vec{k}')] = 0$; ③ $[b_r(\vec{k})b_{r'}^+(\vec{k}), a_{r'}^+(\vec{k}')] = 0$ (a y b conmutan)

$$\boxed{[P^\mu, a_{r'}^+(\vec{k}')]} \overset{\text{uso 1 y 3}}{=} \sum_{\vec{k},r} k^\mu \overset{①}{a_r^+(\vec{k})\delta_{rr'}\Delta_{\vec{k}\vec{k}'}} - \overset{③}{0} = \boxed{a_{r'}^+(\vec{k}')\, k'^\mu}$$

④ $[b_r(\vec{k})b_{r'}^+(\vec{k}), b_{r'}(\vec{k}')] = b_r(\vec{k})\{b_{r'}^+(\vec{k}), b_{r'}(\vec{k}')\} - 0 = b_r(\vec{k})\delta_{rr'}\Delta_{\vec{k}\vec{k}'}$

$\hookrightarrow \boxed{[P^\mu, b_{r'}(\vec{k}')]} = \overset{②}{0} - \sum_{\vec{k},r} k^\mu \cdot b_r(\vec{k})\delta_{rr'}\Delta_{\vec{k}\vec{k}'} = \boxed{-b_{r'}(\vec{k}')\, k'^\mu}$

$$\boxed{[P^\mu, \psi^+(x)]} = \sum_{\substack{k,r\\k',r'}} k^\mu \left(f_{\vec{k}'}^{\,r'}(x)^+ a_r^+(\vec{k}) - g_{\vec{k}'}^{\,r'}(x)^+ b_r(\vec{k})\right)\cdot \Delta_{\vec{k}\vec{k}'}\delta_{rr'}$$

$$= \sum_{k,r} k^\mu \left(f_{\vec{k}}^{\,r}(x)^+ a_r^+(\vec{k}) - g_{\vec{k}}^{\,r}(x)^+ b_r(\vec{k})\right)$$

Como $\partial^\mu f_{\vec{k}}^{\,r}(x)^+ = +ik^\mu f_{\vec{k}}^{\,r}(x)^+$

$$\partial^\mu g_{\vec{k}}^{\,r}(x)^+ = -ik^\mu g_{\vec{k}}^{\,r}(x)^+$$

Entonces: $-i\partial^\mu \psi^+ = \sum_{k,r} k^\mu \left(a_r^+(\vec{k})f_{\vec{k}}^{\,r}(x)^+ - b_r(\vec{k})g_{\vec{k}}^{\,r}(x)^+\right)$

y queda:

$$\boxed{[P^\mu, \psi^+(x)] = -i\partial^\mu \psi^+(x)}$$

$P^\mu \equiv :P^\mu: = \sum_{\vec{k},r} k^\mu (N_{k,r} + \bar{N}_{k,r})$

→ omitimos $:P:$ por simplicidad, pero recordamos que está N-ordenado

- $\boxed{P^2}(a_r^+(\vec{k})|0\rangle) = P^\mu g_{\mu\nu}P^\nu a_r^+(\vec{k})|0\rangle = g_{\mu\nu}P^\mu(P^\nu a_r^+(\vec{k}))|0\rangle$

$= g_{\mu\nu}P^\mu\cdot([P^\nu, a_r^+(\vec{k})] + a_r^+(\vec{k})P^\nu)|0\rangle = g_{\mu\nu}P^\mu\cdot a_r^+(\vec{k})\cdot k^\nu|0\rangle$

$+ \overset{:P^\mu:|0\rangle = 0}{0} = k_\mu([P^\mu, a_r^+(\vec{k})] + a_r^+(\vec{k})\underbrace{P^\mu)|0\rangle}_{0} = k_\mu\cdot a_r^+(\vec{k})k^\mu|0\rangle$

$= k^2 a_r^+(\vec{k})|0\rangle = \boxed{m^2(a_r^+(\vec{k})|0\rangle)}$

hiperboloide de masas $k^2 = m^2$

Por tanto $a_r^+(\vec{k})|0\rangle \equiv |k,r\rangle$ es propio de P^2 con valor propio m^2.

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$\boxed{3}$

- Sea la densidad lagrangiana del campo EM libre:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;, \quad \text{con} \left\{ \begin{array}{l} F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \\ A^\mu = (\Phi, \vec{A}) \\ \vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}} \;;\; \vec{B} = \vec{\nabla} \wedge \vec{A} \Rightarrow B_i = \epsilon_{ijk} \partial_j A_k = +\epsilon_{ijk} \vec{\nabla}_j A_k \\ \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \dot{\vec{A}} = -\dot{\vec{B}} \;;\; \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{E} = 0 \end{array} \right.$$

Tomo unidades naturales: $c = 1$

$\Downarrow$

Encontrar $\mathcal{L}$ en términos de $\vec{E}$ y $\vec{B}$

μ, ν: índices Lorentz: 1, 2, 3, 4 (letras griegas)

$\underbrace{F^{\mu\mu} = F_{\mu\mu} = 0}$; $\underbrace{F^{\mu\nu} = -F^{\nu\mu}}$; $\underbrace{F_{\mu\nu} = -F_{\nu\mu}}$ (tensor antisimétrico).

j, i: índices espaciales = 1, 2, 3 (índices latinos)

porque $\partial^i = \frac{\partial}{\partial x_i}$ y $(\vec{\nabla})^i = \frac{\partial}{\partial x^i}$

$$\underbrace{F^{0i}} = -F^{i0} = \partial^0 A^i - \partial^i A^0 = (\dot{\vec{A}})^i - (-\vec{\nabla})^i A^0 = (\dot{\vec{A}} + \vec{\nabla}\phi)^i = \underbrace{-(\vec{E})^i}$$

$$\underbrace{F^{ij}} = -F^{ji} = \partial^i A^j - \partial^j A^i = -((\vec{\nabla})^i A^j - (\vec{\nabla})^j A^i)$$

→ signo − porque $(\vec{\nabla})^i = \frac{\partial}{\partial x^i} = -\partial^i$

con $j > i$:

$$= -(\delta^i_m \delta^j_n \vec{\nabla}^m A^n - \delta^j_m \delta^i_n \vec{\nabla}^m A^n)$$

$$= -(\delta^i_m \delta^j_n - \delta^j_m \delta^i_n) \vec{\nabla}^m A^n = -\epsilon^{ijk} \epsilon_{kmn} \vec{\nabla}^m A^n$$

$$= \underbrace{-\epsilon^{ijk} B_K}$$

$\epsilon^{ijk} = \epsilon_{ijk}$; $(\vec{A})^i = (\vec{A})_i$ (No me refiero a si el índice es covariante o contravariante) supongo vector contravariante usual

Por tanto, F queda:
$$\underbrace{F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & -\vec{E} & \\ \hline & 0 & -B_z & B_y \\ \vec{E} & B_z & 0 & -B_x \\ & -B_y & B_x & 0 \end{array} \right)}$$

$$F_{0i} = g_{0\alpha} g_{i\beta} F^{\alpha\beta} = g_{00} g_{ii} F^{0i} = 1 \cdot (-1) F^{0i} = -F^{0i}$$

$$F_{ij} = g_{i\alpha} g_{j\beta} F^{\alpha\beta} = g_{ii} g_{jj} F^{ij} = (-1) \cdot (-1) F^{ij} = F^{ij}$$

Por tanto,
$$\underbrace{F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & \vec{E} & \\ \hline & 0 & -B_z & B_y \\ -\vec{E} & B_z & 0 & -B_x \\ & -B_y & B_x & 0 \end{array} \right)}$$

Por tanto, descartando los términos diagonales $F^{\mu\mu}$ (nulos):

$$\mathcal{L} = -\frac{1}{4}(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij})$$

entendiendo que hay índices sumados al estar al cuadrado

$$= -\frac{1}{4}(2 \cdot F_{0i} F^{0i} + F_{ij} F^{ij}) = -\frac{1}{4}(2 \cdot (-)(F^{0i})^2 + (F^{ij})^2)$$

$$= -\frac{1}{4}((-2) \cdot (-(\vec{E})_i)^2 + (-)^2 (\epsilon_{ijk})^2 (B_K)^2) = -\frac{1}{4}(-2(E_x^2 + E_y^2 + E_z^2)$$
$$+ ((\epsilon_{123})^2 + (\epsilon_{213})^2) B_z^2 + ((\epsilon_{312})^2 + (\epsilon_{321})^2) B_x^2 + ((\epsilon_{132})^2 + (\epsilon_{312})^2) B_y^2)$$

$$\mathcal{L} = -\frac{1}{4}\left(-2\cdot\vec{E}^2 + 2\vec{B}^2\right) = \underbrace{\frac{\vec{E}^2 - \vec{B}^2}{2}}$$ que es invariante Lorentz (viene del Lagrangiano)

- Escribir $\textcircled{H}^{\mu\nu}$, H, $\vec{P}$

$$\textcircled{H}^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)}\cdot\partial^\nu A^\lambda$$ (tensor energía-momento canónico)

Desarrollemos previamente el Lagrangiano:

$$\hookrightarrow \mathcal{L} = -\frac{1}{4}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\cdot(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

$$= -\frac{1}{4}\left[(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) + (\partial_\beta A_\alpha)(\partial^\beta A^\alpha) - (\partial_\beta A_\alpha)(\partial^\alpha A^\beta) - (\partial_\alpha A_\beta)(\partial^\beta A^\alpha)\right]$$

intercambio α y β en término segundo y tercero

$$= -\frac{1}{2}\left[(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - (\partial_\alpha A_\beta)(\partial^\beta A^\alpha)\right]$$

$$= -\frac{1}{2}(\partial_\alpha A_\beta)\left[\underbrace{(\partial^\alpha A^\beta) - (\partial^\beta A^\alpha)}_{=F^{\alpha\beta}}\right] = -\frac{1}{2}(\partial_\alpha A^\gamma)g_{\gamma\beta}\cdot$$

$$\cdot\left[(\partial_\delta A^\beta)g^{\delta\alpha} - (\partial_\delta A^\alpha)g^{\delta\beta}\right]$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)} = -\frac{1}{2}\left[g_\alpha{}^\mu g^\gamma{}_\lambda g_{\gamma\beta}\cdot F^{\alpha\beta} + \partial_\alpha A_\beta\cdot g^{\delta\alpha}\cdot g_\delta{}^\mu g^\beta{}_\lambda - \partial_\alpha A_\beta\cdot g^{\delta\beta}g_\delta{}^\mu g^\alpha{}_\lambda\right]$$

$$= -\frac{1}{2}\left[F^\mu{}_\lambda + \underbrace{\partial^\mu A_\lambda - \partial_\lambda A^\mu}_{=F^\mu{}_\lambda}\right] = -F^\mu{}_\lambda \,/\!/$$

$$\Rightarrow \textcircled{H}^{\mu\nu} = -g^{\mu\nu}\cdot\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\right) - F^\mu{}_\lambda\cdot\partial^\nu A^\lambda \,/\!/ \quad [1]$$

$$= \frac{g^{\mu\nu}}{4}\cdot F^{\alpha\beta}F_{\alpha\beta} - F^\mu{}_\lambda\cdot(F^{\nu\lambda} + \partial^\lambda A^\nu)$$

$$= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^\mu{}_\lambda F^{\nu\lambda} - \partial^\lambda(F^\mu{}_\lambda A^\nu) + \underbrace{(\partial^\lambda F^\mu{}_\lambda)}_{\overset{\parallel}{\partial_\lambda F^{\mu\lambda}} = j^\mu = 0}A^\nu$$

$$= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - \underbrace{F^\mu{}_\lambda F^{\nu\lambda}}_{=T^{\mu\nu}=T^{\nu\mu}} - \partial^\lambda(F^\mu{}_\lambda A^\nu) \,/\!/ \quad [2]$$

$$\mathcal{P}^\nu = \textcircled{H}^{0\nu} \overset{[1]}{=} \frac{1}{4}g^{0\nu}\cdot 2(\vec{B}^2 - \vec{E}^2) - F^0{}_i\,\partial^\nu A^i = \frac{1}{2}g^{0\nu}(\vec{B}^2 - \vec{E}^2) - (\vec{E})_i\,\partial^\nu A^i$$

$$\underset{\sim}{\mathcal{H}} = \textcircled{H}^{00} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) - \vec{E}\cdot\dot{\vec{A}} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) + \vec{E}\cdot(\vec{E} + \vec{\nabla}\phi)$$

$$= \underbrace{\frac{1}{2}(\vec{B}^2 + \vec{E}^2) + \vec{E}\cdot\vec{\nabla}\phi} \,/\!/$$ (densidad hamiltoniana)

$$\mathcal{P}^j = -(\vec{E})_i \cdot \partial^j A^i = +(\vec{E})_i \cdot \frac{\partial}{\partial x^j} A^i = \vec{E} \cdot \frac{\partial \vec{A}}{\partial x^j}$$

$F^0{}_i = -F^{0i}$

$F^0{}_0 = 0$

Mejor usar expresión [2]:

$$\mathcal{P}^j = \Theta^{0j} = -F^0{}_\lambda F^{j\lambda} - \partial^\lambda (F^0{}_\lambda A^j) = -F^0{}_i F^{ji} - \partial^i (F^0{}_i \vec{A}^j)$$

$$= -(\vec{E})_i (-)\epsilon^{jik} B_k - \partial^i ((\vec{E})_i (\vec{A})^j) = (\vec{E} \times \vec{B})^j - (\partial^i E_i)\vec{A}^j - \vec{E}_i (\partial^i \vec{A}^j)$$

$$= (\vec{E} \times \vec{B})^j + \underset{\overset{\parallel}{\rho = 0}}{(\vec{\nabla} \cdot \vec{E})} \cdot \vec{A}^j + (\vec{E} \cdot \vec{\nabla}) \cdot \vec{A}^j$$

$\rho = 0$ → campo EM libre, sin fuentes

$$= [\vec{E} \times \vec{B} + (\vec{E} \cdot \vec{\nabla} \cdot \vec{A}]^j \Rightarrow \underbrace{\vec{\mathcal{P}} = \vec{E} \times \vec{B} + (\vec{E} \cdot \vec{\nabla}) \cdot \vec{A}}$$

Al integrar en todo el volumen, el 2º término, que es una derivada total, no afectará, por el teorema de Stokes (o de la divergencia), al momento total:

→ ¿porqué es una deriv. total?

$$\mathbb{P} = \int d^3x\, \mathcal{P}$$

2º término: $\int d^3x (-)\partial^\lambda (F^0{}_\lambda A^j) = \int dV \cdot (-)\partial^i (\vec{E}_i \vec{A}^j)$

$$= \int_V dV\, \vec{\nabla} (\vec{E} A^j) = \int_{S_\infty} dS (\vec{E} A^j) = 0$$

en S_∞, $\vec{E}$ se anula (bien comportado)

Por tanto,

vector de Poynting

$$\underbrace{\vec{\mathbb{P}} = \int d^3x\, (\vec{E} \times \vec{B}) = \int dV \cdot \vec{S}}$$

Análogamente para el hamiltoniano:

$$\underbrace{H = \int d^3x\, \mathcal{H} = \int d^3x\, \frac{(\vec{B}^2 + \vec{E}^2)}{2}}$$

ya que el término $\int d^3x\, \vec{E} (\vec{\nabla} \phi)$ se anula:

$$\int dV\, \vec{E} (\vec{\nabla} \phi) = \int_V dV\, \vec{\nabla} (\vec{E} \phi) - \int dV \underset{\overset{\parallel}{\rho = 0}}{(\vec{\nabla} \cdot \vec{E})} \cdot \phi = \int_{S_\infty} dS\, (\vec{E} \cdot \phi) = 0$$

Ppo. fundamental de Gauge (QED):

- $(D_\mu \varphi)' = (\partial_\mu + iq A'_\mu) \varphi' = \partial_\mu \varphi' + iq A'_\mu \varphi' =$

$\varphi' = e^{-iq\chi(x)} \varphi$

Derivada covariante

$$= e^{-iq\chi(x)} \cdot \partial_\mu \varphi + (-iq)(\partial_\mu \chi(x)) \cdot e^{-iq\chi(x)} \varphi + iq (A_\mu + \partial_\mu \chi(x)) e^{-iq\chi(x)} \varphi$$

$= 0$

$$= e^{-iq\chi(x)} (\partial_\mu \varphi + iq A_\mu) \cdot \varphi = e^{-iq\chi(x)} (D_\mu \varphi)$$

$$\Rightarrow \underbrace{(D_\mu \varphi)' = e^{-iq\chi(x)} \cdot (D_\mu \varphi)}$$

- Escribir el lagrangiano de interacción de Yukawa para un protón, descrito por un campo de Dirac, interaccionando con un pión pseudoescalar. Respetar el lagrangiano efectivo de la interacción fuerte y la simetría de la interacción.
  Lo no viola paridad

El lagrangiano completo sería:

$\mathcal{L} = \mathcal{L}_{pión} + \mathcal{L}_{protón} + \mathcal{L}_{int}$

pión → bosón → K-G → $\mathcal{L}_{mes} = (\partial^\mu \phi)^* (\partial_\mu \phi) - m_\pi^2 \phi^* \phi$

protón → fermión → Dirac → $\mathcal{L}_{prot} = \overline{\psi}(i\gamma^\mu \partial_\mu - m_p)\psi$

$\mathcal{L}_{int}$: debe ser un escalar de Lorentz, por tanto debe contener un bilineal de Dirac

$\mathcal{L}_{int} \sim \overline{\Psi}\,\Gamma\,\Psi$ (invar. de fase)

Como el pión es un mesón pseudoescalar, la matriz γ_5 estará involucrada: $\Gamma = \gamma_5$ (paridad pión impar)

Aparte, aparecerá el campo del pión ϕ, y una constante de acoplamiento adimensional para que $\mathcal{L}$ tenga las adecuadas $[M^4]$.

$[M^4] = [L^{-4}]$

$[\phi] = [M]$
$[\psi] = [M^{3/2}]$

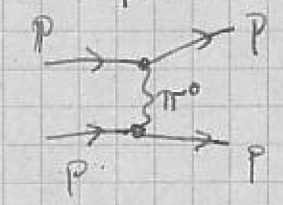
signo - por ser una interacción atractiva

$$\mathcal{L}_{int} = \overset{\text{signo}}{-}\,\overline{\psi}\,\gamma_5\,\phi\,\psi \cdot g$$

$[g] = 1$

Vértice de interacción: que involucra tres campos

Interacción nuclear:



Si ϕ tiene paridad impar ($P_\pi = -1$ intrínseca), entonces $\mathcal{L}_{int}$ es invariante bajo paridad, como se da en la interacción fuerte.

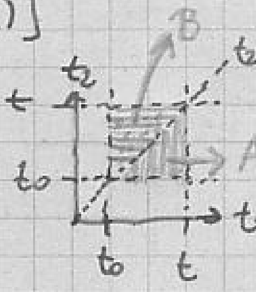
$\phi(t,\vec{x}) \to -\phi(t,-\vec{x})$; $\psi(t,\vec{x}) \to \gamma^0 \psi(t,-\vec{x})$ $\quad \gamma^0\gamma_5 = -\gamma_5\gamma_0$, $\gamma_0^2 = 1$

$$P(\mathcal{L}_{int}) = -g\,\psi^+(t,-\vec{x})\underset{\gamma^0}{(\gamma^0)^+}\gamma^0\gamma_5 \cdot (-)\phi(t,-\vec{x})\,\gamma^0\,\psi(t,-\vec{x}) = -\overline{\psi}\gamma_5\phi\psi g = \mathcal{L}_{int}(t,\vec{x})$$

producto cronológico

$$\bullet\ I_n = \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n\ P\left[H_I(t_1)\ldots H_I(t_n)\right]$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) H_I(t_2) \ldots H_I(t_n)$$



Comprobarlo explícitamente para $n = 2$

$$I_2 = \frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdot \left\{ \theta(t_1 - t_2) H_I(t_1) H_I(t_2) + \theta(t_2 - t_1) H_I(t_2) H_I(t_1) \right\}$$

Divido en 2 tramos las integrales:

$$= \frac{1}{2}\left\{ \int_{t_0}^{t} dt_1 \left[\int_{t_0}^{t_1} dt_2\ \underbrace{\theta(t_1 - t_2)}_{=1 \text{ porque } t_2 < t_1} H_I(t_1) H_I(t_2) + \overbrace{\int_{t_1}^{t} dt_2\, \theta(t_1 - t_2) H_I(t_1) H_I(t_2)}^{=0 \text{ porque } t_2 > t_1} \right] \right.$$

A B

(dibujo zona A)

$$\left. + \int_{t_0}^{t} dt_2 \left[\int_{t_0}^{t_2} dt_1\, \underbrace{\theta(t_2 - t_1)}_{=1 \text{ porque } t_1 < t_2} H_I(t_2) H_I(t_1) + \int_{t_2}^{t} dt_1\, \underbrace{\theta(t_2 - t_1)}_{=0 \text{ porque } t_1 > t_2} H_I(t_2) H_I(t_1) \right] \right\}$$

B (ver dibujo zona B) A

$$= \frac{1}{2}\left\{ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, H_I(t_2) H_I(t_1) \right\}$$

Dado que los índices 1 y 2 son mudos, intercambio 1 y 2 en el segundo término:

$$\underset{\sim\sim}{I_2} = \frac{1}{2}\left\{ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) \right\}$$

$$= \cancel{2} \cdot \frac{\cancel{1}}{\cancel{2}} \cdot \underbrace{\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2)}_{} \qquad \text{q.e.d.}$$

$$I_n \equiv \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; H_I(t_1)\cdots H_I(t_n) \qquad (1)$$

$$(t_0 = -\infty)$$

$$P\left(H_I(t_1)\cdots H_I(t_n)\right) \equiv \sum_{\text{perm.}} \theta(t_{\sigma(1)} - t_{\sigma(2)})\cdots\theta(t_{\sigma(n-1)} - t_{\sigma(n)})\, H_I(t_{\sigma(1)})\cdots H_I(t_{\sigma(n)})$$

y se trata de ver que I_n en (1) coincide con

$$I_n = \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \; P\left(H(t_1)\cdots H(t_n)\right) \qquad (2)$$

para el caso ultrasencillo de $n=2$, donde (1) da $I_2 = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2)$

$$I_2 \overset{\substack{(2)\\ \downarrow}}{=} \frac{1}{2!}\left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; \underset{\substack{\downarrow \\ t_2 < t_1}}{\theta(t_1 - t_2)}\, H(t_1) H_I(t_2) + \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; \underset{\substack{\downarrow \\ t_1 < t_2}}{\theta(t_2 - t_1)}\, H(t_2) H(t_1)\right)$$

$$= \frac{1}{2}\left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; \overset{\Downarrow}{H_I}(t_1) H_I(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \; H(t_2) H_I(t_1)\right) \quad \updownarrow\; t_1 \leftrightarrow t_2$$

$$\overset{\substack{(1)\\ \downarrow}}{=} \frac{1}{2}\left(I_2 + \underset{\substack{\| \\ I_2}}{\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1) H_I(t_2)}\right) = I_2 \text{ según } (1)$$

Ejercicios para entregar el lunes 21 Nov

① Mostrar, utilizando el teorema de Wick, cómo aparecen los diagramas de segundo orden de la dispersión de Compton $e^- + \gamma \to e^- + \gamma$, i.e. cómo proceden de los términos de $S^{(2)}$

$$\frac{e^2}{2!}\left\{\int d^4x_1 d^4x_2 \; : \underbrace{\bar{\Psi}(x_1)\gamma_\mu \Psi(x_1)\bar{\Psi}(x_2)\gamma_\nu \Psi(x_2)}_{} A^\mu(x_1) A^\nu(x_2) : + \right.$$

$$\left. + \int d^4x_1 d^4x_2 \; : \bar{\Psi}(x_1)\gamma_\mu \underbrace{\Psi(x_1)\bar{\Psi}(x_2)}_{} \gamma_\nu \Psi(x_2) A^\mu(x_1) A^\nu(x_2) : \right\}$$

② Probar la identidad de Gordon para dos espinores $u_r(\vec{p})$, $u_s(\vec{p}')$ de Dirac,

$$\bar{u}_r(\vec{p}')\gamma^\mu u_s(\vec{p}) = \frac{1}{2m}\bar{u}_r(p')\left[(p+p')^\mu + i\sigma^{\mu\nu}(p'-p)_\nu\right] u_s(\vec{p})$$

③ Probar que, por ejemplo ($\not{a} = \gamma^\mu a_\mu$, como siempre)

$$\gamma^\mu \not{a} \not{b} \gamma_\mu = 4a^\nu b_\nu \equiv 4ab$$

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- Probar que $\ddot{P}_{i\to \text{todo } f} = 1$ está garantizado por $S^+S = \mathbb{1}$, siendo S^+ la matriz de dispersión (unitaria).

$\boxed{2\frac{1}{5}}$

$$P_{i\to f} = \frac{|\langle f|S|i\rangle|^2}{\langle i|i\rangle} = \frac{\langle i|S^+|f\rangle\langle f|S|i\rangle}{\langle i|i\rangle}$$

$$\underbrace{P_{i\to\text{todo } f}} = \sum_f P_{i\to f} = \sum_f \frac{\langle i|S^+|f\rangle\langle f|S|i\rangle}{\langle i|i\rangle}$$

$$= \frac{\langle i|S^+\overset{\text{identidad de Parseval} = 1 \text{ (base completa)}}{\left(\sum_f |f\rangle\langle f|\right)} S|i\rangle}{\langle i|i\rangle} = \frac{\langle i|S^+\cdot\mathbb{1}\cdot S|i\rangle}{\langle i|i\rangle}$$

$$\overset{S^+S=\mathbb{1}}{=} \frac{\langle i|i\rangle}{\langle i|i\rangle} = \underset{\sim}{1} \quad \text{q.e.d.} \qquad P \text{ sólo } = 1 \text{ si } S^+S = 1$$

La conservación de la probabilidad está garantizada por $S^+S = 1$

- Probar que la condición $\partial_\mu A^\mu|\psi\rangle = 0$ no es compatible con el operador A.

$\forall\, |\psi\rangle$ físico

Lo que escribe (ahí) corresponde a probar que $\partial_\mu A^\mu = 0$ es inconsistente con $[A_\mu(x'), A_\nu(x)] = -ig_{\mu\nu} D(x-x')$

Dem:

El operador A cumple (rel. canónica cuantizada)

$$[A^\mu(x), A^\nu(x')] = -ig^{\mu\nu} D(x-x') \quad ; \; A^\mu = (A^\mu)^*$$

Si derivamos respecto a X^μ (sin prima) en ambos lados de la igualdad:

$$\partial_\mu [A^\mu(x), A^\nu(x')] = -i\,\partial^\nu D(x-x')$$

derivada sólo afecta a x, no a x'

no; el miembro de la izquierda es un operador, el de la derecha una función. Faltan las a's

$$[\partial_\mu A^\mu(x), A^\nu(x')] = -i\partial^\nu D(x-x')$$

$$(\partial_\mu A^\mu(x))A^\nu(x') - A^\nu(x')\partial_\mu A^\mu(x) = -i\partial^\nu\left\{\sum_K (-i)(e^{-ik(x-x')} - e^{ik(x-x')}\right\}$$

Si calculamos el valor esperado en el estado físico $|\psi\rangle$ normalizado e imponemos la condición $\partial_\mu A^\mu|\psi\rangle = 0$, equivalente a $\langle\psi|\partial_\mu A^\mu = 0$ ya que $A^\mu = (A^\mu)^*$, entonces la igualdad queda:

se ha de imitado al escribir, aunque OK $\langle\psi|(\ldots)|\psi\rangle$

$$\langle\psi|\underbrace{(\partial_\mu A^\mu(x))}_{=0}A^\nu(x')|\psi\rangle - \langle\psi|A^\nu(x')\underbrace{\partial_\mu A^\mu(x)|\psi\rangle}_{=0} = i\sum_k k^\nu(e^{-ik(x-x')} + e^{ik(x-x')})\cdot\underset{1}{\langle\psi|\psi\rangle}$$

casi. La definición lleva un $-i$ que no sobrevive con a i de la derivada

$$\Delta(x'-x') = -i\sum_{\vec{k}}\left\{e^{-ik(x'-x)} - e^{ik(x'-x)}\right\}$$

$$\Downarrow$$

$$\underset{\sim}{0} = \cancel{i}\sum_k k^\nu(e^{-ik(x-x')} + e^{ik(x-x')}) \neq 0 \text{ da en general}$$

Como llegamos a una contradicción ($0 \neq 0$), queda probado que la condición $\partial_\mu A^\mu|\psi\rangle = 0$ no es compatible con el operador A^μ.

$\Delta(x'-x) = -i\sum_{\vec{k}}\left\{e^{-ik(x'-x)} - e^{ik(x'-x)}\right\}$

$k_\mu \text{ "=" } i\partial_\mu$

- Calcular $\left(\frac{d\sigma}{d\Omega}\right)_{CM}$ del proceso $1+2 \to 1'+2'$, donde las partículas son fermiones de spin ½, con $m_1 = m_1' \neq m_2 = m_2'$; distinguibles 1-2 y la interacción es electromagnética.

Aplicar el límite relativista para obtener la fórmula de Rutherford.

$\longrightarrow t$

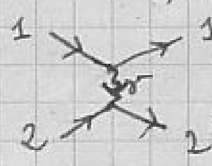


diagrama único, $1 \neq 2$, $1' \neq 2'$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\overline{\sum_s}|A|^2 = \frac{1}{64\pi^2 s}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\sum_{\substack{r_1',r_2'\\ r_1,r_2}}|A|^2$$

promedio spin inicial / suma spines finales

el. de matriz reducido (conexo)

A se calcula mediante las reglas de Feynman:

- Fermión 1/2 incoming : u_{r_1} / u_{r_2}
- Vértice fotón : $e\gamma^\mu$
- Línea fotónica interna : $-\frac{g_{\mu\nu}}{q^2+i\epsilon}$; $q = p_1 - p_1'$

- Fermión 1'/2' outgoing : $\overline{u}_{r_1'}$, $\overline{u}_{r_2'}$, con $\overline{u} = u^+\gamma^0$

Siguiendo las líneas de carga a la inversa:

$$A = \left(\overline{u}_{r_1'}(\vec{p_1}')\,e\gamma^\mu\,u_{r_1}(\vec{p_1})\right)\cdot\frac{-g_{\mu\nu}}{q^2+i\epsilon}\cdot\left(\overline{u}_{r_2'}(\vec{p_2}')\,e\gamma^\nu\,u_{r_2}(\vec{p_2})\right)$$

$$|A|^2 = e^4\cdot\left(u_{r_2}^+(\vec{p_2})\overset{=\gamma^0\gamma^\nu\gamma^0}{(\gamma^\nu)^+}\overset{=\gamma^0}{(\gamma^0)^+}u_{r_2'}(\vec{p_2}')\cdot\frac{g_{\mu\nu}}{q^2+i\epsilon}\cdot u_{r_1}^+(\vec{p_1})(\gamma^\mu)^+(\gamma^0)^+\cdot u_{r_1'}(\vec{p_1}')\right)\cdot$$

$$\cdot\left(\overline{u}_{r_1'}(\vec{p_1}')\gamma^{\mu'}u_{r_1}(\vec{p_1})\cdot\frac{g_{\mu'\nu'}}{q^2+i\epsilon}\cdot\overline{u}_{r_2'}(\vec{p_2}')\gamma^{\nu'}u_{r_2}(\vec{p_2})\right)$$

$(\gamma^0)^2 = 1$

$$= \frac{e^4}{(q^2+i\epsilon)^2}\cdot\left(\overline{u}_{r_2}(\vec{p_2})\gamma^\nu u_{r_2'}(\vec{p_2}')\cdot\overline{u}_{r_1}(\vec{p_1})\gamma_\nu u_{r_1'}(\vec{p_1}')\right)\cdot\left(\overline{u}_{r_1'}(\vec{p_1}')\gamma_{\nu'}u_{r_1}(\vec{p_1})\cdot\overline{u}_{r_2'}(\vec{p_2}')\gamma^{\nu'}u_{r_2}(\vec{p_2})\right)$$

Introduzco notación de índices espinoriales y reordeno buscando factores $u\overline{u}$:

$$= \frac{e^4}{(q^2+i\epsilon)}\left\{\left(u_{r_2}(\vec{p_2})_\delta\,\overline{u}_{r_2}(\vec{p_2})_\alpha(\gamma^\nu)_{\alpha\beta}\,u_{r_2'}(\vec{p_2}')_\beta\,\overline{u}_{r_2'}(\vec{p_2}')_\gamma(\gamma^{\nu'})_{\gamma\delta}\right)\cdot\left(u_{r_1}(\vec{p_1})_\eta\,\overline{u}_{r_1}(\vec{p_1})_\rho(\gamma_\nu)_{\rho\sigma}u_{r_1'}(\vec{p_1}')_\sigma\cdot\right.\right.$$

$=0$

$$\left.\left.\cdot\,\overline{u}_{r_1'}(\vec{p_1}')_\epsilon(\gamma_{\nu'})_{\epsilon\eta}\right)\right\} \to \text{Son trazas si sumas.}$$

Aparte $\sum_{r_1} u_{r_1}(\vec{p_1})\,\overline{u}_{r_1}(\vec{p_1}) = \not{p}_1 + m_1$ (propiedades espinores)

$$\sum_{\substack{r_1,r_2\\ r_1',r_2'}}|A|^2 = \frac{e^4}{q^4}\cdot\underbrace{\mathrm{Tr}\left[(\not{p}_2+m_2)\gamma^\nu(\not{p}_2'+m_2')\gamma^{\nu'}\right]}_{T_2}\cdot\underbrace{\mathrm{Tr}\left[(\not{p}_1+m_1)\cdot\gamma_\nu\cdot(\not{p}_1'+m_1')\gamma_{\nu'}\right]}_{T_1}$$

↳ Ambas trazas son equivalentes, cambiando 1 por 2 y bajando ambos índices de las γ. Calculo una solamente.

$$T_2 \equiv \mathrm{Tr}[(\not{p}_2+m_2)\gamma^{\nu}(\not{p}_2'+m_2')\gamma^{\nu'}] \overset{m_2=m_2'}{=} \mathrm{Tr}[\not{p}_2\gamma^{\nu}\not{p}_2'\gamma^{\nu'}] + \mathrm{Tr}[\not{p}_2\gamma^{\nu}m_2\gamma^{\nu'}]$$

$$+\mathrm{Tr}[m_2\gamma^{\nu}\not{p}_2'\gamma^{\nu'}] + \mathrm{Tr}[m_2\gamma^{\nu}m_2\gamma^{\nu'}] = p_{2\mu}p'_{2\mu'}\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\mu'}\gamma^{\nu'}]$$

$$+ m_2 p_{2\mu}\underset{\overset{\shortparallel}{0}}{\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\nu'}]} + m_2 p'_{2\mu'}\underset{\overset{\shortparallel}{0}}{\mathrm{Tr}[\gamma^{\nu}\gamma^{\mu'}\gamma^{\nu'}]} + m_2^2\,\mathrm{Tr}[\gamma^{\nu}\gamma^{\nu'}]$$

$\Rightarrow$ Tr (# impar γ) = 0

$$T_2 = p_{2\mu}p'_{2\mu'}\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\mu'}\gamma^{\nu'}] + m_2^2\,\mathrm{Tr}[\gamma^{\nu}\gamma^{\nu'}]$$

Propiedades γ: → $\mathrm{Tr}(\gamma^{\alpha}\gamma^{\beta}) = 4g^{\alpha\beta}$

↳ $\mathrm{Tr}(\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{\epsilon}) = 4(g^{\alpha\beta}g^{\delta\epsilon} - g^{\alpha\delta}g^{\beta\epsilon} + g^{\alpha\epsilon}g^{\beta\delta})$

$$T_2 = 4\cdot\left[p_{2\mu}p'_{2\mu'}\cdot(g^{\mu\nu}g^{\mu'\nu'} - g^{\mu\mu'}g^{\nu\nu'} + g^{\mu\nu'}g^{\mu'\nu}) + m_2^2\cdot g^{\nu\nu'}\right]$$

$$= 4\left[p_2^{\nu}p_2'^{\nu'} - (p_2\cdot p_2')g^{\nu\nu'} + p_2^{\nu'}p_2'^{\nu} + m_2^2 g^{\nu\nu'}\right]$$

$$= 4\cdot[p_2^{\nu}p_2'^{\nu'} + p_2^{\nu'}p_2'^{\nu} + (m_2^2 - (p_2p_2'))g^{\nu\nu'}]$$

Introducimos variable de Mandelstam:

$$t = q^2 = (p_1 - p_1')^2 = (p_2' - p_2)^2 = m_1^2 + m_1^2 - 2(p_1p_1') = m_2^2 + m_2^2 - 2(p_2'p_2)$$

↳ $p_2'p_2 = p_2p_2' \Rightarrow m_2^2 - (p_2p_2') = \frac{t}{2} = m_1^2 - (p_1p_1')$

$$T_2 = 4\cdot[p_2^{\nu}p_2'^{\nu'} + p_2^{\nu'}p_1'^{\nu} + \frac{t}{2}g^{\nu\nu'}]$$ → análogo para T_1 cambiando 2 por 1 y bajando índices

$$\overline{\sum_s}|A|^2 = \sum_{\substack{r_1,r_2\\ r_1',r_2'}}\frac{|A|^2}{4} = \frac{e^4}{t^2}\cdot\frac{4\cdot4}{4}\cdot(p_2^{\nu}p_2'^{\nu'} + p_2^{\nu'}p_1'^{\nu} + \frac{t}{2}g^{\nu\nu'})\cdot(p_{1\nu}p'_{1\nu'} + p_{1\nu'}p'_{1\nu} + \frac{t}{2}g_{\nu\nu'})$$

$$= \frac{e^4}{t^2}\cdot 4\,\Big[(p_2p_1)(p_2'p_1') + (p_2p_1')(p_2'p_1) + \frac{t}{2}(p_2p_2')$$
$$+(p_2p_1')\cdot(p_2'p_1) + (p_2p_1)(p_2'p_1') + \frac{t}{2}(p_2p_2')$$
$$+\frac{t}{2}(p_1p_1' + p_1p_1') + \frac{t^2}{4}\cdot\underset{\overset{\shortparallel}{4}}{g^{\nu}_{\nu}}\Big]$$

$$= \frac{e^4\cdot 4}{t^2}\circ[2(p_2p_1)(p_2'p_1') + 2(p_2p_1')(p_2'p_1) + t(p_2p_2' + p_1p_1') + t^2]$$

↳ invariante, pues son todo productos escalares → lo estudiaremos en el SCM.
↳(que se pueden medir experimentalmente)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\cdot\frac{1}{4}\sum_{\substack{r_1,r_2\\ r_1',r_2'}}|A|^2 = \frac{e^4}{16\pi^2 s t^2}\Big\{2[(p_2p_1)(p_2'p_1') + (p_2p_1')(p_2'p_1)] +$$
$$+ t(p_2p_2' + p_1p_1') + t^2\Big\}$$

se puede sustituir en función de s,t,u

$$\frac{e^4}{16\pi^2} = \left(\frac{e^2}{4\pi}\right)^2 = \alpha^2$$; α: cte. estructura fina en unid. naturales

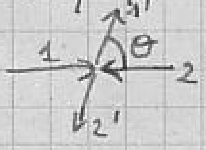
Apliquemos ahora el límite no relativista: $(m \gg |\vec{p}|)$
y estudiemos el problema en el S.C.M: $\quad \hookrightarrow E \sim m$

$$p_a p_b = E_a E_b - \vec{p}_a \vec{p}_b \simeq E_a E_b \overset{\sim}{\rightarrow} m_a m_b$$

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2 p_2 p_1 \simeq m_1^2 + m_2^2 + 2 m_2 m_1 = (m_1 + m_2)^2$$

$$t = (p_1 - p_1')^2 = 2m_1^2 - 2 p_1 p_1' = 2m_1^2 - 2E_1 E_1' + 2\vec{p_1}\vec{p_1}'$$

$$= -2 (E_1 E_1' - m_1^2 - 2 |\vec{p}_1| \cdot |\vec{p}_1'| \cdot \cos\Theta)$$



Por otro lado,

$$E_i = \gamma m_i \overset{\text{Taylor}}{\simeq} \left(1 + \frac{1}{2} v_i^2\right) m_i = m_i + \frac{m_i v_i^2}{2} = m_i + \frac{\vec{p_i}^2}{2m_i}$$

Cons. energía:

$$E_1 + E_2 = E_1' + E_2' \Rightarrow m_1 + \frac{\vec{p_1}^2}{2m_1} + m_2 + \frac{\vec{p_2}^2}{2m_2} = m_1 + \frac{\vec{p_1}'^2}{2m_1} + m_2 + \frac{\vec{p_2}'^2}{2m_2}$$

Como estamos en S.C.M: $|\vec{p_1}| = |\vec{p_2}|$; $|\vec{p_1}'| = |\vec{p_2}'|$ y queda que:

$$\vec{p_1}^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = \vec{p_1}'^2 \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$

$$\Rightarrow |\vec{p_1}| = |\vec{p_1}'| \equiv K \Rightarrow E_1 = E_1'$$

En el lím NR, $p_1 \cdot p_2$, por ej., está dominado por las E, que van como la masa: $p_1 \cdot p_2 \rightarrow m_1 m_2$

Por tanto, t queda como:

$$t \simeq -2 (\underbrace{E_1^2 - m_1^2}_{= |\vec{p_1}|^2} - 2 K^2 \cos\Theta) = -2 (K^2 - K^2 \cos\Theta) = -2K^2 (1 - \cos\Theta)$$

$$\rightarrow \sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta) \quad \text{(prop. trigonométrica)}$$

$$t = -4K^2 \sin^2\frac{\theta}{2} \ .$$

Como $t \propto K$ (momento), y es límite no relativista, $K \ll m_i$, se cumple $t \ll m_i^2$ (despreciable frente a términos con sólo masas.)

↓ Sustituyendo todas las aproximaciones:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \simeq \frac{\alpha^2}{s t^2} \cdot \left\{ 2 (m_2^2 m_1^2 + m_2^2 m_1^2) + t^{\nearrow 0} \cdot (m_2^2 + m_1^2) + t^{2 \nearrow 0} \right\}$$

$$= \frac{4\alpha^2}{s t^2} \cdot (m_1^2 m_2^2) = 4\alpha^2 \cdot \frac{m_1^2 m_2^2}{(m_1 + m_2)^2} \cdot \frac{1}{t^2}$$

$$= 4\alpha^2 \left(\overset{\mu}{\overset{|||}{\frac{m_1 m_2}{m_1 + m_2}}}\right)^2 \cdot \frac{1}{16 K^4 \sin^4\frac{\theta}{2}} = \frac{(\alpha\mu)^2}{4K^4 \sin^4\frac{\Theta}{2}} = \left(\frac{\alpha\mu}{2K^2 \sin^2\frac{\Theta}{2}}\right)^2$$

$\hat{=}$ Sección eficaz de Rutherford

Gupta-Bleuler: cómo imponer la condición de Lorentz $\partial_\mu A^\mu(x) = 0$ cuando el campo está cuantizado

$$[A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} D(x-y)$$

Es evidente que no se puede exigir que el operador A_μ satisfaga $\partial^\mu A_\mu = 0$ pues eso implicaría la contradicción

$$0 = [\partial^\mu A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} \partial^\mu D(x-y) \neq 0 .$$

La condición $\partial^\mu A_\mu |\psi\rangle = 0$ sobre cualquier estado físico (que se debe a Fermi) tampoco es aceptable. Si se toma $|\psi\rangle = |0\rangle$

$$\partial^\mu A_\mu |0\rangle = \partial^\mu A^{(-)}_\mu |0\rangle = 0$$ (pues, para la parte de destrucción, $\partial^\sigma A^{(+)}_\sigma |0\rangle = 0$)

Por tanto,

$$\Downarrow$$

$$A^{(-)}_\nu(y)^* \partial^\sigma_x A^{(-)}(x)|0\rangle = 0 = \partial^\sigma_x (A^{(-)*}_\nu(y) A^{(-)}_\mu(x))|0\rangle$$

$$\equiv \partial^\sigma_x \left(A^{(-)}_\sigma(x) A^{(-)*}_\nu(y) + [A^{(-)*}_\nu(y), A^{(-)}_\sigma(x)]\right)|0\rangle$$

$A^{(-)*}_\nu$ es de destrucción, aniquila el vacío,

$$\overset{\downarrow}{=} \partial^\sigma_x [A^{(-)*}_\nu(y), A^{(-)}_\sigma(x)]|0\rangle = 0$$

$$\Downarrow$$

$$\partial_\sigma D^+(y-x)|0\rangle = 0 \qquad (*)$$

que es, de nuevo, una contradicción. **Más brevemente**: $\partial^\mu A_\mu|0\rangle$ no puede ser cero, pues es $(\partial_\mu A^{\mu(+)} + \partial_\mu A^{(-)}_\mu)|0\rangle$ y $A^{(-)}_\mu$ contiene operadores de creación.

Así pues, la condición correcta es la de G-B (condición de Lorentz para parte de destrucción $A^{(+)}_\mu$), $\boxed{\partial^\mu A^{(+)}_\mu |\psi\rangle = 0}$ $(\Rightarrow \langle\psi|\partial^\mu A^{(-)}_\mu = 0)$ que implica

$$\Downarrow$$

$$\langle\psi|\partial^\mu A_\mu(x)|\psi\rangle = 0$$

(*) Como en el caso de K-G, donde $\Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y)$, $D(x-y)$ tiene la misma descomposición; de hecho, si se usa para la Δ de K-G la notación $\Delta(x) \equiv \Delta(x;m)$, $D(x) = \Delta(x; m=0)$. Obsérvese que la masa entra en la definición a través de $\int \frac{d^3p}{2E}$ pues $E = \sqrt{m^2+\vec{p}^{\,2}}$.

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①

Dispersión Compton: $e^- + \gamma \rightarrow e^- + \gamma$ ③

Matriz de dispersión $S = \sum_n S_n = \frac{\sum (i)^n}{n!} \int d^4x_1 \ldots \int d^4x_n T(\mathcal{L}_I(x_1) \ldots \ldots \mathcal{L}_I(x_n))$

$\mathcal{L}_I(x) = -e\,\overline{\psi}(x)\gamma_\mu \psi(x) A^\mu(x)$

A 2º orden: $(n=2)$

Usaré que:
$[A^\mu(x), \psi(y)] = 0$
$[A^\mu(x), \overline{\psi}(y)] = 0$

$$S_2 = \frac{i^2}{2!} \int d^4x_1 \int d^4x_2 \cdot T\left((-)e^2\overline{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1) \overline{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2)\right)$$

$$= -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2 \cdot T\left(\overline{\psi}(x_1)\gamma_\mu \psi(x_1)\overline{\psi}(x_2)\gamma_\nu \psi(x_2) A^\mu(x_1) A^\nu(x_2)\right)$$

Por tanto, tenemos el producto de 6 campos (4 fermiónicos y 2 electromagnéticos). Si nos centramos únicamente en el proceso de dispersión Compton, con 2 fotones y 2 fermiones involucrados, entonces los campos A^μ y A^ν se tendrán que contraer con 2 fotones (saliente y entrante). Otros 2 campos fermiónicos se contraerán con los 2 fermiones, con lo que sólo quedarán 2 de los 6 campos por contraer.
Al aplicar el tma. de Wick, nos centraremos sólo en el término que contrae 2 campos fermiónicos del producto de 6, pues hemos de dejar 4 libres (2 em. y 2 ferm.) para las partículas entrantes y salientes.

$$T\left(\overline{\psi}(x_1)\gamma_\mu \psi(x_1)\overline{\psi}(x_2)\gamma_\nu \psi(x_2) A^\mu(x_1) A^\nu(x_2)\right)$$

$= :\ \overline{\psi}_1\ \gamma_\mu\ \psi_1\ \overline{\psi}_2\ \gamma_\nu\ \psi_2\ \underbrace{A^\mu_1\ A^\nu_2}\ :$ → Descartamos, no es disp. Compton! ← propagador fotónico

$+ :\ \underbrace{\overline{\psi}_1\ \gamma_\mu\ \psi_1}\ \overline{\psi}_2\ \gamma_\nu\ \psi_2\ A_1{}^\mu A^\nu{}_2 :$ ⟹ $= 0$ porque al aplicarse el tma. de Wick hay que omitir contracciones a tiempos iguales

$+ :\ \underbrace{\quad '' \quad}\quad '' \quad :$ ⟹ $= 0$ porque $\underbrace{\overline{\psi}(x)\ \overline{\psi}(y)} = 0$

$+ :\ \underbrace{\qquad\qquad\qquad} :$ ✓

$+ :\quad \underbrace{\quad} \quad :$ ✓

$+ :\quad \underbrace{\qquad\qquad} :$ ⟹ $= 0$ porque $\overbrace{\psi(x)\ \psi(y)} = 0$

$+ :\qquad \underbrace{\qquad} :$ ⟹ $= 0$ (NO contraer a tiempos iguales)

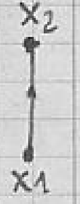
Con lo que sólo nos quedan dos términos:

$$S_2^{\text{COMPTON}} = -\frac{e^2}{2!}\int\Big[:\overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 A_1^\mu A_2^\nu :$$

$$+ :\overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 A_1^\mu A_2^\nu :\Big] \cdot d^4x_1\, d^4x_2$$

fermiónico

Cada término representa un propagador entre x_1 y x_2.



En el primer término, necesariamente, ψ_1 deberá contraerse con el fermión entrante y $\overline{\psi}_2$ con el saliente. En el 2º, ψ_2 con el entrante y $\overline{\psi}_1$ con el saliente. El campo A_1 tiene dos posibilidades en cada caso de ir a x_1 o a x_2.

$\Downarrow$

$$\langle f | S_2^{\text{compton}} | i \rangle = \langle 0 | a_{r'}(\vec{p}\,') a^{\lambda'}(\vec{k}') \cdot S_2^{\text{com}} | a^{+\lambda}(\vec{k}) a_r^+(\vec{p}) | 0 \rangle$$

posibilidad 2

$$= -\frac{e^2}{2!}\int d^4x_1\, d^4x_2 \cdot \Big\{ \langle 0 | a_{r'}(\vec{p}\,') a^{\lambda'}(\vec{k}') : \overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 A_1^\mu A_2^\nu : a^{+\lambda}(\vec{k}) a_r^+(\vec{p}) | 0 \rangle$$

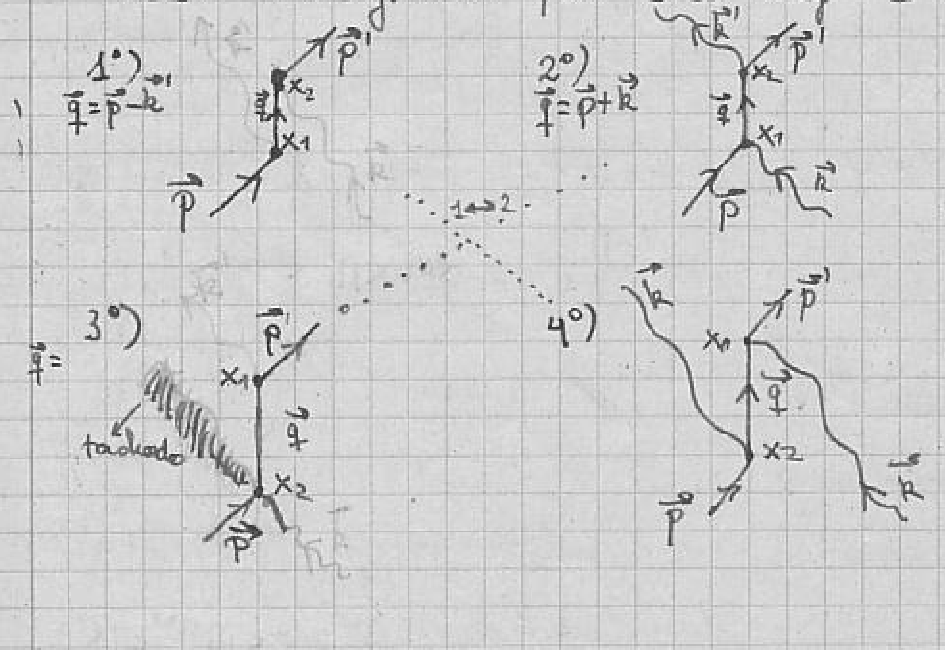
posibilidad 1

3 cruces líneas fermiónicas : signo -

$$(-\overline{\psi}_2 \psi_1)$$

$$+ \langle 0 | a_{r'}(\vec{p}\,') a^{\lambda'}(\vec{k}') : \overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 A_1^\mu A_2^\nu : a^{+\lambda}(\vec{k}) a_r^+(\vec{p}) | 0 \rangle \Big\}$$

→ Salen 4 diagramas pero sólo hay 2 topologías distintas:



Si intercambiamos las variables mudas 1 y 2 (vértices) en los diagramas 3 y 4, vemos que estos son equivalentes a los 2 y 1 respectivamente.

Por tanto, tenemos una contribución doble de 2 diagramas, que cancelarán el $\frac{1}{2!}$.

② $u_s(\vec{p})$ → Probar la identidad de Gordon

$u_{\bar{r}}(\vec{p}')$; $\underset{r \equiv r'}{s \equiv r}$ (cambio notación)

La demostración parte de una "idea feliz". Comenzamos con la ec. de Dirac en espacio de momentos para un espinor elemental (nos centramos en la parte con u):

$$(\gamma^\mu p_\mu - m)\, u_r(\vec{p}) = (\not{p} - m)\, u_r(\vec{p}) = 0 \quad ①$$

La ecuación adjunta (con otro momento, polariz.)

$$u^+_{r'(\vec{p}')}((\gamma^\mu)^+ p_\mu - m) = 0 \qquad | \cdot \gamma^0$$

$$\overline{u}_{r'(\vec{p}')}(\not{p}' - m) = 0 \quad ②$$

idea feliz

Como buscamos una γ^μ entre $\overline{u}$ y u, calculamos:

$$\overset{= 0}{②} \cdot \gamma^\mu \cdot u_r(\vec{p}) + \overline{u}_{r'}(\vec{p}')\gamma^\mu \cdot \overset{= 0}{①} = 0$$

$$\overline{u}_{r'}(\vec{p}') \cdot [(\not{p}' - m)\gamma^\mu + \gamma^\mu(\not{p} - m)]\, u_r(\vec{p}) = 0$$

$$\overline{u}_{r'}(\vec{p}') \cdot [-2m\gamma^\mu + \not{p}'\gamma^\mu + \gamma^\mu \not{p}]\, u_r(\vec{p}) = 0$$, paso masa a la dcha

$$\overline{u}_{r'}(\vec{p}') \cdot 2m\gamma^\mu\, u_r(\vec{p}) = \overline{u}_{r'}(\vec{p}')[\not{p}'\gamma^\mu + \gamma^\mu \not{p}]\, u_r(\vec{p})$$

$$\Rightarrow \not{p}'\gamma^\mu + \gamma^\mu \not{p} = p'_\nu \gamma^\nu \gamma^\mu + p_\nu \gamma^\mu \gamma^\nu$$

$$\Rightarrow \gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} \cdot \frac{2}{i}\sigma^{\mu\nu} + \frac{1}{2} \cdot 2g^{\mu\nu}$$

$$= g^{\mu\nu} - i\sigma^{\mu\nu} \quad ; \quad \gamma^\nu\gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu}$$

antisimétrico: $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$
… conmutador opuesto implica cambio signo

$$\hookrightarrow \not{p}'\gamma^\mu + \gamma^\mu \not{p} = p'_\nu(g^{\mu\nu} + i\sigma^{\mu\nu}) + p_\nu(g^{\mu\nu} - i\sigma^{\mu\nu})$$

$$= p'^\mu + p^\mu + i\sigma^{\mu\nu}(p'_\nu - p_\nu)$$

$$\underbrace{\overline{u}_{r'}(\vec{p}')\gamma^\mu u_r(\vec{p}) = \frac{1}{2m}\overline{u}_r(\vec{p}') \cdot [(p + p')^\mu + i\sigma^{\mu\nu} \cdot (p' - p)_\nu]\, u_r(\vec{p})}$$

$$\gamma^\mu \gamma^\nu = \{\underbrace{\gamma^\mu, \gamma^\nu}_{}\}^{2g^{\mu\nu}} - \gamma^\nu \gamma^\mu$$

③ $\gamma^\mu \not{a} \not{b} \gamma_\mu = a_\nu b_\rho \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = a_\nu b_\rho (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) \gamma^\rho \gamma_\mu$

$= 2 a_\nu b_\rho \gamma^\rho \gamma^\nu - a_\nu b_\rho \gamma^\nu (2 g^{\mu\rho} - \gamma^\rho \gamma^\mu) \gamma_\mu$

$= 2\not{b}\not{a} - 2\not{a}\not{b} + \not{a}\not{b} \gamma^\mu \gamma_\mu$

Por otro lado:

$\not{a}\not{b} = a_\mu b_\nu \gamma^\mu \gamma^\nu = a_\mu b_\nu (g^{\mu\nu} - i\sigma^{\mu\nu}) = a \cdot b - i a_\mu b_\nu \sigma^{\mu\nu}$

$\not{b}\not{a} = a_\nu b_\mu \gamma^\mu \gamma^\nu = a_\nu b_\mu (g^{\mu\nu} - i\sigma^{\mu\nu}) = \underset{a\cdot b}{\underset{\parallel}{\underbrace{b \cdot a}}} - i a_\nu b_\mu \sigma^{\mu\nu}$

$\hookrightarrow \not{a}\not{b} + \not{b}\not{a} = -i (a_\mu b_\nu \sigma^{\mu\nu} + a_\mu b_\nu \sigma^{\nu\mu}) + \cancel{2ab}\ 2(ab)$

cambio μ por ν (mudos)

$= 2ab - i a_\mu b_\nu (\sigma^{\mu\nu} + \overset{-\sigma^{\mu\nu}}{\overset{\parallel}{\sigma^{\nu\mu}}}) = 2ab$

Aparte,

$\gamma^\mu \gamma_\mu = g_{\mu\nu} \gamma^\mu \gamma^\nu = g_{\mu\nu} (g^{\mu\nu} - i\sigma^{\mu\nu}) = g^\mu_\mu - i g_{\mu\nu} \sigma^{\mu\nu}$

Sim. x Antis. = 0

$= g^\mu{}_\mu = 4$

Juntando todo:

$\gamma^\mu \not{a}\not{b} \gamma_\mu = 2\not{b}\not{a} - 2\not{a}\not{b} + 4\not{a}\not{b} = 2(\not{b}\not{a} + \not{a}\not{b}) = 2 \cdot 2ab = 4ab$

Las dos posibles contracciones que dan lugar al propagador del e son

$$: \bar{\Psi}(x_2)\gamma_\nu \underbrace{\Psi(x_2)\bar{\Psi}(x_1)}\gamma_\mu \Psi(x_1) \quad A^\mu(x_1)A^\nu(x_2) :$$

$$: \bar{\Psi}(x_1)\gamma_\mu \underbrace{\Psi(x_1)\bar{\Psi}(x_2)}\gamma_\nu \Psi(x_2) \quad A^\mu(x_1)A^\nu(x_2) :$$

La contribución al elemento de matriz S en segundo orden procede de las contracciones externas con los oper. (omitiendo indicación de posibles polarizaciones y con $p \Rightarrow$ electrón, $q \Rightarrow$ fotón) de los estados $|i\rangle$ y $|f\rangle$, i.e.,

$$\underbrace{\langle 0|\, a(p')a(q')}_{\langle f|} \; - - - - \; \underbrace{a^+(p)a^+(q)|0\rangle}_{|i\rangle}$$

$\left[\begin{array}{l} A^{(-)}, A^{(+)} \text{ contienen } a, a^+ \\ \psi^{(+)}, \bar{\psi}^{(-)} \text{ contienen } a, \underline{a^+} \end{array}\right.$

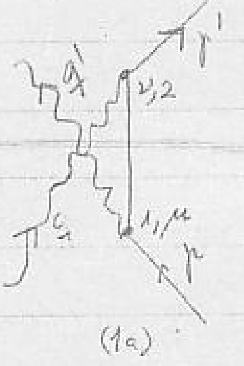
Las contribuciones vienen de

$$(1) \quad \left(\bar{\Psi}^{(-)}(x_2)\gamma_\nu \overbrace{\Psi(x_2)\bar{\Psi}(x_1)}\gamma_\mu \Psi^{(+)}(x_1)\right)\left(\overset{(1a)}{A^\mu(x_1)^{(-)}A^{\nu(+)}(x_2)} + \overset{(1b)}{A^{\nu(-)}(x_2)A^{\mu(+)}(x_1)}\right)$$

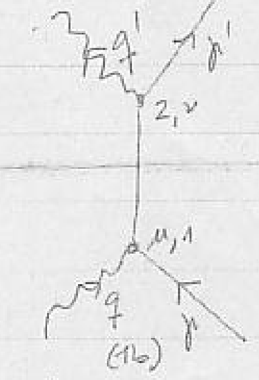
$$(2) \quad \left(\bar{\Psi}^{(-)}(x_1)\gamma_\mu \overbrace{\Psi(x_1)\bar{\Psi}(x_2)}\gamma_\nu \Psi^{(+)}(x_2)\right)\left(\underset{(2a)}{A^\mu(x_1)^{(-)}A^\nu(x_2)^{(+)}} + \underset{(2b)}{A^\nu(x_2)^{(-)}A^{\mu(+)}(x_1)}\right)$$

(por ej., $A^{(+)}(x_1)A^{(+)}(x_2)$ no puede contribuir ; $\overbrace{a(p')\bar{\Psi}} = \bar{u}(p')e^{ipx}$, etc. y las exponenciales darán lugar a δ's de conservación de momentos). Preocupándonos sólo de los diagramas,

(1) de lugar a



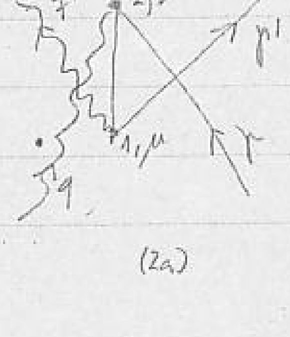
(1a)



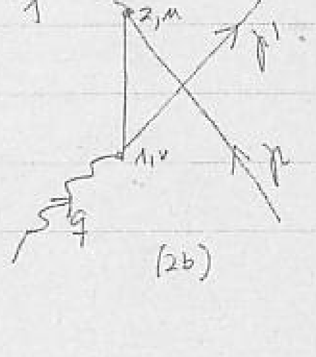
(1b)

Usar $\frac{1}{2!}$ del desarrollo es equivalente a quedarse con los topológicamente distintos e.g., con (1a) y (1b)

y (2) a



(2a)



(2b)

(−) falta

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1

- Probar, a partir de la ecuación de evolución de Heisenberg para un campo cuantizado

$-i\partial_0 \varphi(x) = [H, \varphi(x)]$ ②

que el paso de x a $T\cdot x$ (cambio de signo en t) requiere que la operación inversión temporal sea antiunitaria.

2'5

Antiunitaria implica $\mathcal{T} i \mathcal{T}^{-1} = -i$ (conjuga nºs complejos)
Si fuese unitaria no cambiaría el signo

Partimos, por analogía a lo que demostramos en el 2º ejercicio → página siguiente
con paridad, de que (para KG) → por simplificar

$\mathcal{T}\varphi(x)\mathcal{T}^{-1} = \varphi(Tx)$ (ignorando fases)

$\mathcal{T}P^0\mathcal{T}^{-1} = \mathcal{T}H\mathcal{T}^{-1} = H$

$\mathcal{T}(a(\vec{k}))\mathcal{T}^{-1} = a(-\vec{k})$

falta el − de arriba, que no afecta al razonamiento, pero falta

$\mathcal{T}(i\partial_0\varphi(x))\mathcal{T}^{-1} = \mathcal{T}[H,\varphi(x)]\mathcal{T}^{-1}$ } Inserto identidad $\mathcal{T}^{-1}\mathcal{T}$

$\mathcal{T}(i\partial_0\varphi(x))\mathcal{T}^{-1} = \mathcal{T}H\mathcal{T}^{-1}\mathcal{T}\varphi(x)\mathcal{T}^{-1} - \mathcal{T}\varphi(x)\mathcal{T}^{-1}\mathcal{T}H\mathcal{T}^{-1}$

$\mathcal{T}(i\partial_0\varphi(x))\mathcal{T}^{-1} = H\varphi(Tx) - \varphi(Tx)H$

$\mathcal{T}(i\partial_0\varphi(x))\mathcal{T}^{-1} = [H,\varphi(Tx)]$ ①

$i\partial_0\varphi(x) = \sum_{\vec{k}} k_0 \left(a(\vec{k})e^{-ikx} - a^{+}(\vec{k})e^{ikx}\right)$

$k^0 \in \mathbb{R}$, $\mathcal{T}k^0\mathcal{T}^{-1} = k^0$

$\mathcal{T}(i\partial_0\varphi(x))\mathcal{T}^{-1} = \sum_{\vec{k}} k_0\left(\mathcal{T}a(\vec{k})\mathcal{T}^{-1}\mathcal{T}e^{-ikx}\mathcal{T}^{-1} - \mathcal{T}a^{+}(\vec{k})\mathcal{T}^{-1}\mathcal{T}e^{ikx}\mathcal{T}^{-1}\right)$ ver pág siguiente

$\sum_{\vec{k}} = \sum_{\vec{k}'}$

$= \sum_{\vec{k}} k_0\left(a(-\vec{k})\mathcal{T}e^{-ikx}\mathcal{T}^{-1} - a^{+}(-\vec{k})\mathcal{T}e^{ikx}\mathcal{T}^{-1}\right)$ cambio $\vec{k}$ por $-\vec{k}'$: $\boxed{\begin{matrix}\vec{k}' = -\vec{k}\\ k_0' = k_0\end{matrix}}$

$= \sum_{\vec{k}'} k_0'\left(a(\vec{k}')\mathcal{T}e^{-i(k_0'x^0+\vec{k}'\vec{x})}\mathcal{T}^{-1} - a^{+}(\vec{k}')\mathcal{T}e^{i(k_0'x^0+\vec{k}'\vec{x})}\mathcal{T}^{-1}\right)$ ③

$\mathcal{T}k^\mu\mathcal{T}^{-1} = k^\mu$
k^μ es un c-número real
(no así la exp. ver detrás)

Por ① y ②, si $x' = Tx$

↳ $i\partial_0'\varphi(x') = [H,\varphi(Tx)]$, entonces $i\partial_0'\varphi(x') =$ ③

↳ $= \sum_{\vec{k}'} k_0'\left(a(\vec{k}')e^{-ik'x'} - a^{+}(\vec{k}')e^{ik'x'}\right)$

$x' = Tx$
$\begin{cases} x_0' = -x_0 \\ \vec{x}' = \vec{x}\end{cases}$

La igualdad con ③ sólo se cumple si:

$\mathcal{T}e^{-i(k_0'x^0+\vec{k}'\vec{x})}\mathcal{T}^{-1} = e^{-ik'x'} \Rightarrow \alpha = -ik'x' = -i(k_0'x^{0\prime} - \vec{k}'\cdot\vec{x}') = +i(k_0'x^0 + \vec{k}'\cdot\vec{x})$

↳ Lo que obliga a que $\mathcal{T}e^{-i\alpha}\mathcal{T}^{-1} = e^{i\alpha}$ → conjuga los

Análogo con a^+

operador momento de K-G

operador paridad

P en función de operadores de creación y destrucción.

Dem:

$$P_\mu = \sum_{\vec{k}} k^\mu \left(a^+(\vec{k})\, a(\vec{k}) + b^+(\vec{k})\, b(\vec{k}) \right) \Rightarrow \text{ya N-ordenado}$$

Partimos de que: $(\mathcal{P} = \mathcal{P}^+)$; $\mathcal{P}\mathcal{P}^{-1} = I$; $\mathcal{P}$ unitario

(ignoramos fases) $\mathcal{P} a^+(\vec{p}) \mathcal{P}^{-1} = a^+(-\vec{p})$; $\mathcal{P} a(\vec{p}) \mathcal{P}^{-1} = a(-\vec{p})$

Idem con b. introduzco $I = \mathcal{P}^{-1}\mathcal{P}$ entre operadores

$$\mathcal{P} P_\mu \mathcal{P}^{-1} = \mathcal{P} \sum_{\vec{k}} k^\mu \left(\mathcal{P}^{-1}\mathcal{P} a^+(\vec{k}) \mathcal{P}^{-1}\mathcal{P} a(\vec{k}) + \mathcal{P}^{-1}\mathcal{P} b^+(\vec{k}) \mathcal{P}^{-1}\mathcal{P} b(\vec{k}) \right) \mathcal{P}^{-1}$$

$(\mathcal{P} k^\mu \mathcal{P}^{-1} = k^\mu) \Rightarrow$ la componente μ es escalar (número real), no contiene operadores

$$= \sum_{\vec{k}} k^\mu \left(a^+(-\vec{k})\, a(-\vec{k}) + b^+(-\vec{k})\, b(-\vec{k}) \right)$$

Cambio de variable
$\vec{k}' = -\vec{k}$

$k^{0\prime} = +\sqrt{\vec{k}'^2 + m^2} = +\sqrt{\vec{k}^2 + m^2} = k^0$

$\sum_{\vec{k}} = \iiint_{-\infty}^{+\infty} d^3\vec{k} = \iiint_{+\infty}^{-\infty} (-)d^3\vec{k}' = \iiint_{-\infty}^{+\infty} d^3\vec{k}' = \sum_{\vec{k}'}$

$$= \sum_{\vec{k}'} (k^{0\prime}, -\vec{k}') \left(a^+(\vec{k}')\, a(\vec{k}') + b^+(\vec{k})\, b(\vec{k}') \right)$$

$$= \sum_{\vec{k}'} (k^{0\prime}, -\vec{k}') \left(a^+(\vec{k}')\, a(\vec{k}') + b^+(\vec{k}')\, b(\vec{k}') \right)$$

$$= (P^0, -\vec{P})$$ $\Rightarrow$ paridad cambia el signo de las componentes espaciales del cuadrivector P_μ.

$\times k^\mu$ contiene / no contiene operadores

c.q.d.

La cuestión es si en (S) $\quad -i\partial_t \varphi(t,\vec{x}) = [H, \varphi(t,x)]$

implica en (S') $\quad -i\partial_{t'} \varphi(t',\vec{x}) = [H, \varphi(t',\vec{x})]$ con $t' = -t$

Puesto que $\tau [H, \varphi(t,x)] \tau^{-1} = \underline{\qquad}$

para el miembro de la izqda, $(\tau \partial_t \tau^{-1} = \partial_t)$

$$\tau(-i\partial_t \varphi(t,\vec{x}))\tau^{-1} = \tau(-i)\tau^{-1}\, \partial_t\, \varphi(t',\vec{x}) =$$

$$= -\tau(-i)\tau^{-1}\, \partial_{t'}\, \varphi(t',\vec{x})$$ ¡prima!

que la ec. de ev. de Heisenberg sea la misma en S y S' $\Rightarrow$ $\tau(-i)\tau^{-1} = i$ i.e. τ es antilineal,

$\tau(c)\tau^{-1} = c^*$

$\forall c \in \mathbb{C}$

#1 • $j^\mu = \frac{i}{2m}(\phi^* \overleftrightarrow{\partial^\mu} \phi)$

$$\partial_\mu j^\mu = \frac{i}{2m}(\partial_\mu(\phi^*\partial^\mu\phi) - \partial_\mu((\partial^\mu\phi^*)\phi))$$

$$= \frac{i}{2m}(\underset{\downarrow}{\partial_\mu}\phi^*\partial^\mu\phi + \phi^*\partial_\mu\partial^\mu\phi - (\partial_\mu\partial^\mu\phi^*)\phi - \partial^\mu\phi^*\partial_\mu\phi)$$

$$= \frac{i}{2m}(\partial^\nu\phi^* \overbrace{g_{\mu\nu}\partial^\mu}^{=\partial_\nu}\phi \underbrace{-\partial^\mu}_{=0}\phi^*\partial_\mu\phi + \phi^*\Box\phi - \Box\phi^*\phi)$$

$$= \frac{i}{2m}(\phi^* \overleftrightarrow{\Box} \phi)$$

Si ϕ es campo de K-G, cumple $(\Box + m^2)\phi = 0$

$(\Box + m^2)\phi^* = 0$

$\downarrow$

$$\partial_\mu j^\mu = \frac{i}{2m}(\phi^*(-m^2\phi) - (-m^2\phi^*)\phi) = 0$$

↳ Ecuac de conservac de la corriente de probabilidad

• Para el campo de Schrödinger ψ:

$j^0 = \psi^+\psi$ ···> no relativista, componente temporal aparte

$j^i = \frac{i}{2m}(\phi^* \overleftrightarrow{\partial^i} \phi)$

$$\partial_0 j^0 = \dot\psi^+\psi + \psi^+\dot\psi$$

$$\partial_i j^i = \frac{i}{2m}(\partial_i\psi^*\partial^i\psi + \overbrace{\psi^*\partial_i\partial^i\psi - \partial_i\partial^i\psi^*\psi - \partial^i\psi^*\partial_i\psi}^{=0})$$

$$= \frac{i}{2m}(\psi^* \overleftrightarrow{(\partial_i\partial^i)} \psi)$$ ← Si ψ es sol. ec. Schröd, cumple:

$(2mi\partial_0 - \partial^i\partial_i)\psi = 0$

$(-2im\partial_0 - \partial^i\partial_i)\psi^+ = 0$

✓

$$= \frac{i}{2m}(\psi^*(2mi\partial_0\psi) - (-2mi\partial_0\psi^+)\psi)$$

$$= -(\psi^*\dot\psi + \dot\psi^*\psi)$$

↳ $\partial_\mu j^\mu = \dot\psi^*\psi + \psi^*\dot\psi - (\psi^*\dot\psi + \dot\psi^*\psi) = 0$ ✓

#2 . $\gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu$

En la realización de DP: $\swarrow (\gamma^0)^3 = \gamma^0$ ✓

$\mu=0$: $(\gamma^0)^+ = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}^+ = \gamma^0$ ✓ $\quad (\gamma^0)^2 = \mathbb{1}_4$

$\mu = i$:

$$(\gamma^i)^+ = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -\sigma^{i*} \\ \sigma^{i+} & 0 \end{pmatrix}$$

$$(\gamma^0)(\gamma^i)^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\sigma^{i+} \\ \sigma^{i+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^{i+} \\ -\sigma^{i+} & 0 \end{pmatrix}$$

$$(\gamma^0)(\gamma^i)^+(\gamma^0) = \begin{pmatrix} 0 & -\sigma^{i+} \\ -\sigma^{i+} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^{i+} \\ -\sigma^{i+} & 0 \end{pmatrix}$$

Probemos que $\sigma^{i+} = \sigma^i$

$i = 1 \to (\sigma^1)^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1$

$i = 2 \to (\sigma^2)^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2$

$i = 3 \to (\sigma^3)^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3$ ✓

$\Rightarrow \gamma^0 (\gamma^i)^+ \gamma^0 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma^i$ ✓

$\Rightarrow \gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu$ // ✓

#3 • $(i\gamma^\mu(\partial_\mu - ieA_\mu(x)) - m)\psi^c(x) = 0$

con $\psi^c = C\psi^*$; $C = i\gamma^2$; $C(\gamma^\mu)^* C^{-1} = -\gamma^\mu$

$\downarrow$

$(-\cancel{\gamma^*\partial_\mu}\gamma^2 \cancel{+ie\gamma^\mu A_\mu}\gamma^2 - im\cancel{\gamma^2})\psi^* = 0$

$(i(-C\gamma^{\mu *}C^{-1})(\partial_\mu - ieA_\mu(x)) - m)C\psi^* = 0 \qquad C^{-1}\cdot|$

$\downarrow$

$\{-i(\gamma^\mu)^*(\partial_\mu - ie \cdot A_\mu \cdot) - m\}\psi^* = 0$

lo que es la ecua complejo conjugado ($\neq$ adjunta) de la ec. de Dirac en presencia de un campo E.M.

#4 • $u_r\bar{u}^r = \not{k} + m$; $v_r\bar{v}^r = \not{k} - m$

$$u^1 = N\begin{pmatrix} 1 \\ 0 \\ k_z/E+m \\ \frac{k_x+ik_y}{E+m} \end{pmatrix} \qquad \overline{u^1} = N^*\begin{pmatrix} 1 & 0 & \frac{k_z}{E+m} & \frac{k_x-ik_y}{E+m} \end{pmatrix}\overset{\gamma^0}{\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}}$$

$$= N^*\begin{pmatrix} 1 & 0 & -\frac{k_z}{E+m} & \frac{-k_x+ik_y}{E+m} \end{pmatrix}$$

$$u^1\overline{u_1} = |N|^2\begin{pmatrix} 1 & 0 & -\frac{k_z}{E+m} & -\frac{k_x+ik_y}{E+m} \\ 0 & 0 & 0 & 0 \\ \frac{k_z}{E+m} & 0 & \frac{-k_z^2}{(E+m)^2} & \frac{(-k_x+ik_y)k_z}{(E+m)^2} \\ \frac{k_x+ik_y}{E+m} & 0 & \frac{-k_z(k_x+ik_y)}{(E+m)^2} & \frac{-k_y^2-k_x^2}{(E+m)^2} \end{pmatrix}$$

Dems:

$\gamma^\mu = -C(\gamma^\mu)^* C^{-1}$, con $C = i\gamma^2$; $C^{-1} = C^* = i\gamma^2 = C^+$

$(\gamma^0)^* = \gamma^0 \longrightarrow -i\gamma^2(\gamma^0)^* i\gamma^2 = \gamma^2\gamma^0\gamma^2 = -\gamma^0(\gamma^2)^2 = \gamma^0$ ✓

$(\gamma^1)^* = \gamma^1 \longrightarrow -i\gamma^2\gamma^1 i\gamma^2 = -\gamma^1(\gamma^2)^2 = \gamma^1$ ✓

$(\gamma^3)^* = \gamma^3 \longrightarrow -i\gamma^2\gamma^3 i\gamma^2 = -\gamma^3(\gamma^2)^2 = \gamma^3$ ✓

$(\gamma^2)^* = -\gamma^2 \longrightarrow -i\gamma^2(-\gamma_2)i\gamma^2 = -(\gamma^2)^2(\gamma^2) = \gamma^2$ ✓

$$u^2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{k_x - ik_y}{E+m} \\ \frac{-k_z}{E+m} \end{pmatrix} \qquad \overline{u_2} = N^* \begin{pmatrix} 0 & 1 & \frac{k_x+ik_y}{E+m} & \frac{-k_z}{E+m} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= N^* \begin{pmatrix} 0 & 1 & \frac{-(kx+iky)}{E+m} & \frac{Kz}{E+m} \end{pmatrix}$$

$$u^2 \overline{u_2} = |N|^2 \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-(kx+iky)}{E+m} & \frac{Kz}{E+m} \\ 0 & \frac{kx-iky}{E+m} & -\frac{(Kx^2+ky^2)}{(E+m)^2} & \frac{kz\,(Kx-iKy)}{(E+m)^2} \\ 0 & -\frac{kz}{E+m} & \frac{(Kx+iKy)Kz}{(E+m)^2} & -\frac{Kz^2}{(E+m)^2} \end{pmatrix}$$

$$u^1\overline{u_1} + u^2\overline{u_2} = |N|^2 \cdot \begin{pmatrix} 1 & 0 & \frac{-kz}{E+m} & -\frac{(kx-iky)}{E+m} \\ 0 & 1 & -\frac{kx+iky}{E+m} & \frac{Kz}{E+m} \\ \frac{kz}{E+m} & \frac{kx-iky}{E+m} & -\frac{|\vec{k}|^2}{(E+m)^2} & 0 \\ \frac{kx+iky}{E+m} & -\frac{kz}{E+m} & 0 & -\frac{|\vec{k}|^2}{(E+m)^2} \end{pmatrix}$$

$$|N|^2 = (E+m)$$

$$= \begin{pmatrix} E+m & 0 & -Kz & -(k_x-iky) \\ 0 & E+m & -(kx+iky) & Kz \\ kz & kx-iky & -|\vec{k}|^2/E+m & 0 \\ kx+iky & -kz & 0 & -\frac{|\vec{k}|^2}{E+m} \end{pmatrix}$$

$E = k^0$; $k^2 = m^2 = (k^0)^2 - |\vec{k}|^2$; $|\vec{k}|^2 = (k^0)^2 - m^2 = (k^0+m)(k^0-m)$; $-\frac{|\vec{k}|^2}{E+m} = -(k^0-m)$

$$= \begin{pmatrix} (k^0+m)\cdot I_2 & 0 \\ 0 & (-k^0+m)I_2 \end{pmatrix} + kz \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

$$+ kx \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} + ky \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = m \cdot I_4 + k^0\gamma^0 + k^i\gamma^i$$

$$= m + k^\mu \gamma_\mu = \not{k} + m$$ ✓

Hay una notación más abreviada para hacerlo:

$$v_1 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\overline{v_1} = v_1^+ \gamma^0 = N^* \left((0\ 1) \frac{(\vec{\sigma}\cdot\vec{k})^+}{E+m} \quad (0\ 1) \right) \gamma^0 = N^* \left((0\ 1) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \quad -(0\ 1) \right)$$

$$(\vec{\sigma}\cdot\vec{k})^+ = \begin{pmatrix} k_z & k_x - ik_y \\ +k_x + ik_y & -k_z \end{pmatrix}^{T*} = (\vec{\sigma}\cdot\vec{k})$$

$$v_1 \overline{v_1} = |N|^2 \cdot \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1\ 0) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) \end{pmatrix}$$

$$v_2 = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\overline{v_2} = v_2^+ \gamma^0 = \left((1\ 0) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \quad (1\ 0) \right) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \left((1\ 0) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \quad -(1\ 0) \right)$$

$$v_2 \overline{v_2} = |N|^2 \cdot \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 0) \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) \end{pmatrix}$$

$$v_1 \overline{v_1} + v_2 \overline{v_2} = |N|^2 \cdot \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} \overbrace{\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)}^{\mathbb{I}_2} \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \mathbb{I}_2 \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\mathbb{I}_2 \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{k})^2 = |\vec{k}|^2$$

$$= |N|^2 \cdot \begin{pmatrix} \frac{|\vec{k}|^2}{(E+m)^2} & -\frac{\vec{\sigma}\cdot\vec{k}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m} & -\mathbb{I}_2 \end{pmatrix} \overset{|N|^2 = E+m}{=} \begin{pmatrix} |\vec{k}|^2 / E+m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -E-m \end{pmatrix}$$

$$= \frac{E^2-m^2}{E+m} = \frac{(E+m)(E-m)}{E+m} = E-m \qquad (E = k^0)$$

$$= k^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + k^i \gamma^i - m\, I_4 = \not{k} - m$$

//

$A_\mu(x)^{ext}_{\cdot} \sim (\text{Coulomb}) = \left(-\frac{Ze}{4\pi|\vec{x}|}, \vec{0}\right)$ con $e<0$

Nota: $\vec{p}_{in} \neq \vec{p}_{fin}$
La Δ se la lleva el núcleo $(m_N \to \infty$

$$\mathcal{L}_I = -e\bar{\psi}\gamma^\mu\psi(\underset{\text{interno}}{A_\mu(x)} + A_\mu^{ext}(x))$$

A primer orden, el elemento de matriz es, para la dispersión de Coulomb con 1 e

$$S_c = \langle 0 | a_{r'}(\vec{p}')T(-ie\int d^4x\, \bar{\psi}(x)\gamma^\mu\psi(x)(A_\mu + A_\mu^{ext}(x)))\, a_r^+(\vec{p})|0\rangle$$

El A_μ interno no contribuye $\underset{\text{a 1er orden}}{}$ porque no tiene con quien contraerse

Lo ese factor da 0. El A_μ externo sale como escalar, no contiene operador

$$S_c = -ie\int d^4x\, \bar{u}_{r'}(\vec{p}')\gamma^\mu u_r(\vec{p})\, e^{i(p'-p)x} A_\mu^{ext}(x) \underset{\llcorner A_i^{ext}=0 \Rightarrow \mu \to 0 \text{ es el único que sobrevive}}{=}$$

$$= -ie\int d^4x\, \bar{u}_{r'}(p')\gamma^0 u_r(\vec{p})\, \underset{\text{Integro}}{e^{i(E'-E)x_0}} \cdot A_0^{ext}(x) \cdot e^{-i\underbrace{(\vec{p}'-\vec{p})}_{\vec{q}}\vec{x}}$$

$$= -ie(2\pi)\cdot\delta(E'-E)\cdot\bar{u}_{r'}(p')\gamma^0 u_r(p)\int d^3x\,\frac{(-Ze)}{4\pi|\vec{x}|}\,e^{-i\vec{q}\cdot\vec{x}}$$

$$\equiv iZe^2(2\pi)\delta(E'-E)\cdot A_0(\vec{q})\cdot\underbrace{\bar{u}_{r'}(\vec{p}')\gamma^0 u_r(p)}_{\mathcal{M}}$$

Probabilidad de transición:

$$P_{i\to f} = \frac{|\langle S_c\rangle|^2}{\langle i|i\rangle}\cdot\frac{1}{T} = \frac{Z^2 e^4(2\pi)^2\,\delta^{(2)}(E'-E)\cdot\mathcal{M}^+\mathcal{M}\cdot|A_0|^2(\vec{q})}{\langle i|i\rangle\, T}$$

$\langle i|i\rangle =$

$$\langle i|i\rangle = \langle p|p\rangle = \langle 0|a_r(\vec{p})a_r^+(\vec{p})|0\rangle = \langle 0|\{a_r(\vec{p}), a_r^+(\vec{p})\}|0\rangle - \langle 0|\underbrace{a_r^+(\vec{p})a_r(\vec{p})|0}_{=0}$$

$$= \{a_r(\vec{p}), a_r^+(\vec{p})\}\overset{=1}{\langle 0|0\rangle} = \Delta_{\vec{p}\vec{p}} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p})\cdot 2E$$

$$P_{i\to f} = \frac{(2\pi)^2 Z^2 e^4\,\delta^{(2)}(E-E')\,\mathcal{M}^+\mathcal{M}\,|A_0|^2(\vec{q})}{2E\cdot\underbrace{\delta^{(3)}(\vec{0})\cdot(2\pi)^3}_{V}\, T}$$

$\to$ Una de las δ las escribimos como $\overset{\text{representación de la }\delta}{}$

$$\delta(E-E') = \int\frac{dx_0}{2\pi} = \lim_{T\to\infty}\left(\frac{T}{2\pi}\right)$$

Para calcular σ hay que dividir por el flujo incidente ϕ, promediar sobre estados iniciales y sumar sobre estados finales.

$$\phi = \frac{v}{V} = \frac{|\vec{p}|/E}{V}$$

$$\sum_{\vec{p}'} = \left(\frac{1}{2\pi}\right)^3\int\frac{d^3\vec{p}'}{2E'}$$

$\Downarrow$

$$\sigma = \sum_{\vec{p}'}\cdot\sum_{r'}\cdot\left(\overset{\text{promedio iniciales}}{\frac{1}{2}\sum_r}\right)\cdot\frac{P_{i\to f}}{\phi}$$

$$= \sum_{r',r}\frac{1}{2}\cdot\frac{1}{(2\pi)^3}\int\frac{d^3\vec{p}'}{2E'}\cdot\frac{(2\pi)^2 Z^2 e^4\,\delta(E-E')\,\mathcal{M}^+\mathcal{M}\,|A_0|^2(\vec{q})}{2EV\cdot\frac{|\vec{p}|}{EV}\,\cancel{T}}\cdot\frac{\cancel{T}}{2\pi}$$

$$= \frac{1}{2|\vec{p}|}\cdot\frac{1}{(2\pi)^2}\int\frac{d^3p'}{2E'}\delta(E'-E)\cdot\frac{1}{2}\sum_{r,r'}\mathcal{M}^+\mathcal{M}\;|A_0|^2(\vec{q})$$

$\mathcal{M}^+\mathcal{M} = \bar{u}_r(p)\gamma^0 u_{r'}(\vec{p}')\,\bar{u}_{r'}(\vec{p}')\gamma^0 u_r(\vec{p})$ $\to$ Al sumar salen trazas

$\sum \mathcal{M}^+\mathcal{M} = \text{Tr}((\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0) = T$

$$\mathrm{Tr}(\mathbb{1}_4) = 4$$

$$T = \mathrm{Tr}(\not p' \gamma^0 \not p \gamma^0) + m^2\, \mathrm{Tr}(\underset{\underset{1}{\parallel}}{\gamma^{0^2}}) = \mathrm{Tr}(\not p' \underbrace{\gamma^0 \not p \gamma^0}_{T_1}) + 4m^2$$

$$\not p \gamma^\nu = p_\mu \gamma^\mu \gamma^\nu = p_\mu (\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu \gamma^\mu) = 2 p_\mu g^{\mu\nu} - \gamma^\nu \not p = 2p^\nu - \gamma^\nu \not p$$

$\hookrightarrow \nu = 0$

$$\not p \gamma^0 = 2p^0 - \gamma^0 \not p$$

$$\hookrightarrow \mathrm{Tr}(\not p' \gamma^0 \not p \gamma^0) = \mathrm{Tr}(\not p'(2p^0 - \not p \gamma^0)\gamma^0) = 2\,\mathrm{Tr}(\not p' \gamma^0) \cdot p^0 - \mathrm{Tr}(\not p \not p')$$

$$= 2p^0 p'_\mu \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^0)}_{4 g^{\mu 0}} - 4(pp') = 8EE' - 4(pp') = 4EE' + 4\underset{\underset{|\vec p| \cdot |\vec p'| \cdot \cos\theta}{\parallel}}{\vec p \cdot \vec p'}$$

$$T = 4(m^2 + EE' + |\vec p||\vec p'|\cos\theta)$$

Hacemos cambio de variable a esféricas:

$d^3p' = |\vec p'|^2\, d|\vec p'|\, d\Omega_{p'}$

$E' = \sqrt{m^2 + |\vec p'|^2}$

$|\vec p'|\, d|\vec p'| = E'\, dE'$

$\Downarrow$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2|\vec p|} \frac{1}{(2\pi)^2} \int \frac{d|\vec p'|\, |\vec p'|^2 \cdot \delta(E - E')}{2E'} \cdot \frac{1}{2} (A_0(\vec q))^2 \cdot 4(m^2 + EE' + |\vec p| \cdot |\vec p'| \cos\theta) Z^2 e^4$$

$$= \frac{1}{2|\vec p|} \frac{Z^2 e^4}{(2\pi)^2} \int |\vec p'| \cdot dE' \cdot \delta(E - E') \cdot (m^2 + EE' + |\vec p||\vec p'|\cos\theta) \cdot |A_0(\vec q)|^2$$

$$A_0(\vec q) = \int \frac{d^3x\, e^{-i\vec q \vec x}}{4\pi|\vec x|} \overset{\text{esféricas con } \hat z \parallel \vec q}{=} \int \frac{dV\, e^{-iqr\cos\theta}}{4\pi r} = \int \frac{r^2\, dr\, \sin\theta\, d\theta\, d\phi}{4\pi r} e^{-iqr\cos\theta}$$

$$= \int r\, dr \cdot \frac{2\pi}{4\pi} \left[\frac{e^{-iqr\cos\theta}}{iqr}\right]_0^\pi = \frac{1}{2iq} \int_0^\infty dr \cdot (e^{iqr} - e^{-iqr}) = \frac{1}{q} \int_0^\infty dr\, \sin(qr)$$

$$= \frac{1}{q} \cdot \frac{1}{q} \left[-\cos qr\right]_0^\infty = \frac{1}{|\vec q|^2} \cdot (1 - \cos(\infty \cdot q))$$

$\rightarrow$ Como luego integramos sobre $\int dp'$, el $\frac{\cos(\infty \cdot q)}{|\vec q|^2}$ promediará cero.

$$A_0(\vec q) \sim \frac{1}{|\vec q|^2} = \frac{1}{\vec p^2 + \vec p'^2 - 2|\vec p||\vec p'|\cos\theta}$$

La $\delta(E - E')$ fuerza que $E = E'$, y como $m = m' \longrightarrow |\vec p| = |\vec p'|$

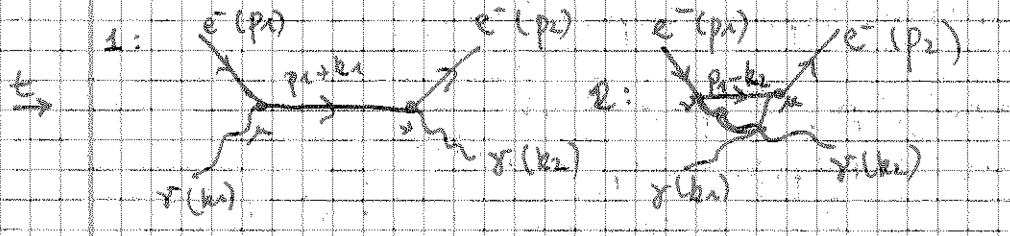
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2|\vec p|} \cdot \frac{Z^2 e^4}{(2\pi)^2} \frac{|\vec p| \cdot (m^2 + E^2 + |\vec p|^2 \cos\theta)}{4|\vec p|^4 \cdot (1 - \cos\theta)^2}}$$

$$= \frac{1}{4} \frac{Z^2 e^4}{(2\pi)^2} \frac{1}{|\vec p|^4} \cdot \frac{(2E^2 + |\vec p^2|(\cos\theta - 1))}{4 \cdot 4 \cdot \sin^4\frac{\theta}{2}} = \frac{Z^2 \alpha^2}{4|\vec p^2|} \cdot \frac{E^2}{|\vec p^2|} \cdot \frac{(1 - v^2 \sin^2\frac{\theta}{2})}{\sin^4\frac{\theta}{2}}$$

$$= \boxed{Z^2 \alpha^2 \left(1 - v^2 \sin^2\frac{\theta}{2}\right)}$$ Límite no relativista: $v \rightarrow 0$ Mandl 8.35)

$\gamma e^- \to \gamma e^-$

2 diagramas de Feynman:



Aplicamos reglas de Feynman. $\mathcal{M}$ tendrá una estructura de tipo:

$\mathcal{M} = e^2\, T^{\mu\nu} \cdot \epsilon_\mu(\vec{k}_1, s_1)\cdot \epsilon^*_\nu(\vec{k}_2, s_2)$

signo +, cruce de líneas bosónicas

$$T^{\mu\nu} = \bar{u}_2\left[\gamma^\nu \frac{\not{p}_1+\not{k}_1+m}{(p_1+k_1)^2-m^2}\gamma^\mu + \gamma^\mu \frac{\not{p}_1-\not{k}_2+m}{(p_1-k_2)^2-m^2}\gamma^\nu\right]u_1$$

Utilizamos que la ecuación de movimiento del fermión es:

$(\not{p}_1 - m)u_1 = 0$; $\to \not{p}_1\gamma^\nu = 2p_1^\nu - \gamma^\nu\not{p}_1$ (ver pág. 4)

$\hookrightarrow (\not{p}_1+m)\gamma^\mu u_1 = 2p_1^\mu u_1 - \gamma^\mu(\not{p}_1-m)u_1 = 2p_1^\mu u_1$

$$\overset{\Downarrow}{T^{\mu\nu}} = \bar{u}_2\left[\gamma^\nu \frac{2p_1^\mu + \not{k}_1\gamma^\mu}{\underset{m^2}{\underset{\shortparallel}{p_1^2}}+\underset{0}{\underset{\shortparallel}{k_1^2}}+2p_1k_1-m^2} + \gamma^\mu\frac{2p_1^\nu - \not{k}_2\gamma^\nu}{m^2+0-2p_1k_2-m^2}\right]u_1$$

$$= \frac{1}{2}\bar{u}_2\left[\gamma^\nu\cdot\frac{2p_1^\mu+\not{k}_1\gamma^\mu}{(p_1k_1)} + \gamma^\mu\cdot\frac{2p_1^\nu-\not{k}_2\gamma^\nu}{-(p_1k_2)}\right]u_1$$

$$A = \overline{\sum_{r_i,s_i}}|\mathcal{M}|^2 = e^2\left(\frac{1}{2}\right)^2\sum_{r_i s_i} T^{\tilde\mu\tilde\nu +}\epsilon^*_{\tilde\mu 1}\,\epsilon_{\tilde\nu 2}\cdot T^{\mu\nu}\epsilon_{\mu,1}\,\epsilon^*_{\nu,2}$$

ver Pich, 4º grado, p. 100

Como la corriente EM se conserva, podemos sustituir $\sum_{s_i}\epsilon^\mu_i\,\epsilon^{*\sigma}_i \to -g^{\mu\sigma}$

$$\downarrow$$

$$A = \frac{e^4}{4}\sum_{r_i}\underset{T^{\mu\nu}T^+_{\mu\nu}}{\underset{\shortparallel}{T^{\mu\nu +}T_{\mu\nu}}} \overset{\text{salen trazas}}{=} \frac{e^2}{16}\mathrm{Tr}\Big\{(\not{p}_2+m)\left[\frac{\gamma^\nu(2p_1^\mu+\not{k}_1\gamma^\mu)}{(p_1k_1)} + \frac{\gamma^\mu(2p_1^\nu-\not{k}_2\gamma^\nu)}{-(p_1k_2)}\right]$$
$$\cdot(\not{p}_1+m)\cdot\left[\frac{(2p_{1\mu}+\gamma_\mu\not{k}_1)}{(p_1k_1)}\gamma_\nu + \frac{(2p_{1\nu}-\gamma_\nu k_2)}{-(p_1k_2)}\gamma_\mu\right]\Big\}$$

$$= \frac{e^4}{16}\cdot\Big\{\frac{\overbrace{\mathrm{Tr}[(\not{p}_2+m)\gamma^\nu(2p_1^\mu+\not{k}_1\gamma^\mu)(\not{p}_1+m)(2p_{1\mu}+\gamma_\mu\not{k}_1)\gamma_\nu]}^{=T_1}}{(p_1k_1)^2}$$

$$- \overbrace{\mathrm{Tr}[(\not{p}_2+m)\gamma^\nu(2p_1^\mu+k_1\gamma^\mu)(\not{p}_1+m)(2p_{1\nu}-\gamma_\nu\not{k}_2)\gamma_\mu]}^{=T_2}/(p_1k_1)(p_1k_2)$$
$$- \underbrace{\mathrm{Tr}[(\not{p}_2+m)\gamma^\mu(2p_1^\nu-\not{k}_2\gamma^\nu)(\not{p}_1+m)(2p_{1\mu}+\gamma_\mu\not{k}_1)\gamma_\nu]}_{=T_3^*}/(p_1k_1)(p_1k_2)$$

$$+ \overbrace{\mathrm{Tr}[(\not{p}_2+m)\gamma^\mu(2p_1^\nu-k_2\gamma^\nu)(\not{p}_1+m)(2p_{1\nu}-\gamma_\nu\not{k}_2)\gamma_\mu]}^{T_4 =}/(p_1k_1)^2\Big\}$$

$T_1 = \mathrm{Tr}\big[(2p_2^\nu - \not{p}_2\gamma^\nu + m\gamma_\nu)\gamma^\nu(2p_1^\mu + \not{k}_1\gamma^\mu)(\not{p}_1 + m)(2p_{1\mu} + \gamma_\mu\not{k}_1)\big]$

$= 2\mathrm{Tr}[\not{p}_2(2p_1^\mu + \not{k}_1\gamma^\mu)(\not{p}_1 + m)(2p_{1\mu} + \gamma_\mu\not{k}_1)]$

$- 4\mathrm{Tr}[(\not{p}_2 - m)(2p_1^\mu + \not{k}_1\gamma^\mu)(\not{p}_1 + m)(2p_{1\mu} + \gamma_\mu\not{k}_1)]$

$= -2\mathrm{Tr}[\not{p}_2(2p_1^\mu + \not{k}_1\gamma^\mu)(\not{p}_1 + m)(2p_{1\mu} + \gamma_\mu\not{k}_1)]$

$+ 4m\,\mathrm{Tr}[(\not{p}_1 + m)\cdot(4m^2 + 2\not{p}_1\not{k}_1 + \underset{2(2k_1p_1) - \not{p}_1\not{k}_1}{2\not{k}_1\not{p}_1} + \underset{\gamma_\mu \not{k}_1^2\gamma^\mu}{[\text{illegible}]})]$ $\hookrightarrow \not{k}_1^2 = k_1^2 = 0$

$= -2\mathrm{Tr}[\ldots] + 4m(4m^2 + 4k_1p_1)\cdot\mathrm{Tr}[\not{p}_1 + m]$

$= \underbrace{\;''\;}_{\hat{T}_1} + 64m^2(p_1k_1 + m^2)$

nº impar γs: fuera, $= 0$

$\hat{T}_1 = -2\mathrm{Tr}[\not{p}_2\not{k}_1\gamma^\mu\not{p}_1\gamma_\mu\not{k}_1] - 2\mathrm{Tr}[\not{p}_2 2p_1^\mu\not{p}_1\gamma_\mu\not{k}_1]$ -1 (5)

$-2\mathrm{Tr}[\not{p}_2\not{k}_1\gamma^\mu\not{p}_1 2p_{1\mu}] - 2\mathrm{Tr}[\not{p}_2 2p_1^\mu\not{p}_1 2p_{1\mu}]$ -2 (3)

$= -2\mathrm{Tr}[\not{p}_2\not{k}_1 2p_1^\mu\gamma_\mu\not{k}_1] + 2\mathrm{Tr}[\not{p}_2\not{k}_1\cdot 4\cdot\not{p}_1\not{k}_1]$ -1 (1)

$-4\mathrm{Tr}[\not{p}_2\underset{m^2}{\underset{\parallel}{\not{p}_1^2}}\not{k}_1] - 4\mathrm{Tr}[\not{p}_2\not{k}_1\not{p}_1^2] - 8\mathrm{Tr}[\not{p}_2\not{p}_1\cdot p_1^2]$

$= 4\mathrm{Tr}[\not{p}_2\underbrace{\not{k}_1\not{p}_1}_{=2k_1p_1 - \not{p}_1\not{k}_1}\not{k}_1] - 8m^2\underbrace{\mathrm{Tr}[\not{p}_2\not{k}_1]}_{4p_2k_1} - 8m^2\mathrm{Tr}[\not{p}_2\not{p}_1]$

$= 8(k_1p_1)\mathrm{Tr}[\not{p}_2\not{k}_1] - 0 - 32m^2(p_2k_1) - 32m^2(p_1p_2)$ ✓

$= 32\big[(p_1k_1)(\underset{(p_1k_2)}{\underset{\parallel}{p_2k_1}}) - m^2((p_1p_2) + (p_2k_1))\big]$ $\ldots\ldots = ((p_1 + k_1)\overset{=p_2+k_2}{}p_2) = m^2 + (\overset{=k_1p_1}{k_2p_2})$

$T_1 = 32\big[(p_1k_1)(p_1k_2) + m^2(2(p_1k_1) + 2m^2 - m^2 - (k_1p_1))\big]$

$= 32\big[(p_1k_1)(\underset{p_2k_1}{\underset{\parallel}{p_1k_2}}) + m^2\cdot(p_1k_1) + m^4\big] = 32\big[(k_1p_1)(k_2p_1) + m^2(k_1p_1) + m^4\big]$

$T_4 \Rightarrow$ por simetría respecto a T_1: $T_4 = T_1(k_2 \leftrightarrow -k_1;\ \mu \leftrightarrow \nu)$.

$T_4 = 32\big[(p_1k_2)(p_1k_1)\cdot(-1)^2 - m^2(p_1k_2) + m^4\big]$

$= 32\big[(k_1p_1)(k_2p_1) - m^2(k_2p_1) + m^4\big]$

$-T_2 = \mathrm{Tr}[(\not{p}_2 + m)\gamma^\nu(2p_1^\mu + \not{k}_1\gamma^\mu)(\not{p}_1 + m)(2p_{1\nu} - \gamma_\nu\not{k}_2)\gamma_\mu]$

$= 4\mathrm{Tr}[(\not{p}_2 + m)\not{p}_1(\not{p}_1 + m)\not{p}_1] \equiv t_1$

$- 2\mathrm{Tr}[(\not{p}_2 + m)\gamma^\nu(\not{p}_1 + m)\gamma_\nu\not{k}_2\not{p}_1] \equiv t_2$

$+ 2\mathrm{Tr}[(\not{p}_2 + m)\not{p}_1\not{k}_1\gamma^\mu(\not{p}_1 + m)\gamma_\mu] \equiv t_3$

$- \mathrm{Tr}[(\not{p}_2 + m)\gamma^\nu\not{k}_1\gamma^\mu(\not{p}_1 + m)\gamma_\nu\not{k}_2\gamma_\mu] \equiv t_4$

$= m^2$

$= 16m^2 \cdot ((p_2p_1) + m^2) = 16m^2(p_1p_2) + 16m^4$

$t_2 = -2\left(\mathrm{Tr}[\not{p}_2 \gamma^\nu \not{p}_1 \gamma_\nu \not{k}_2 \not{p}_1] + m^2 \mathrm{Tr}[\gamma^\nu \gamma_\nu \not{k}_2 \not{p}_1]\right)$

$= -2\left(\mathrm{Tr}[\not{p}_2 (2p_1^\nu - \not{p}_1 \gamma^\nu)\gamma_\nu \not{k}_2 \not{p}_1] + 4m^2 \mathrm{Tr}[\not{k}_2 \not{p}_1]\right)$

$= -2\cdot\left(2\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_2 \not{p}_1] - 4\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_2 \not{p}_1] + 16m^2(k_2p_1)\right)$

$= -2\left(-2\,\mathrm{Tr}[\not{p}_2 (2(p_1k_2) - \not{k}_2\not{p}_1)\not{p}_1] + 16m^2(k_2p_1)\right)$

$= -2\left(-4(p_1k_2)\mathrm{Tr}[\not{p}_2\not{p}_1] + 2\mathrm{Tr}[\not{p}_2\not{k}_2]\cdot m^2 + 16m^2(k_2p_1)\right)$

$= -2\left(-16(p_1k_2)(p_1p_2) + 8m^2(p_2k_2) + 16m^2(k_2p_1)\right)$

$= -16\left(-2(p_1p_2)(p_1k_2) + m^2(p_2k_2) + 2m^2(k_2p_1)\right)$

$t_3 = 2\left(\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_1 \gamma^\mu \not{p}_1 \gamma_\mu] + m^2 \mathrm{Tr}[\not{p}_1 \not{k}_1 \gamma^\mu \cdot \gamma_\mu]\right)$

$= 2\left(\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_1 (2p_1^\mu - \not{p}_1\gamma_\mu)\gamma_\mu] + m^2\mathrm{Tr}[\not{p}_1\not{k}_1]\cdot 4\right)$

$= 2\left(\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_1 (2\not{p}_1 - 4\not{p}_1)] + m^2\, 4 \cdot 4 \cdot (p_1k_1)\right)$

$= 2\left(-2\mathrm{Tr}[\not{p}_2 \not{p}_1 \underbrace{\not{k}_1 \not{p}_1}_{2(k_1p_1) - \not{p}_1\not{k}_1}] + 16m^2(p_1k_1)\right)$

$= 2\left(-4(k_1p_1)\cdot\mathrm{Tr}[\not{p}_2\not{p}_1] + 2m^2 \overset{\mathrm{Tr}[\not{p}_2\not{k}_1]}{\surd} + 16m^2(p_1k_1)\right)$

$= 2\left(-16(k_1p_1)\cdot(p_2p_1) + 8m^2(k_1p_2) + 16m^2(p_1k_1)\right)$

$= 16\left(-2(k_1p_1)(p_2p_1) + m^2(k_1p_2) + 2m^2(p_1k_1)\right)$

$t_4 = -\left(\underbrace{\mathrm{Tr}[\not{p}_2 \gamma^\nu \not{k}_1 \gamma^\mu \not{p}_1 \gamma_\nu \not{k}_2 \gamma_\mu]}_{= t_{4a}} + m^2 \mathrm{Tr}[\gamma^\nu \not{k}_1 \gamma^\mu \underset{= t_{4b}}{\gamma_\nu} \not{k}_2 \gamma_\mu]\right)$

$t_{4a} = \mathrm{Tr}[\not{p}_2 (2k_1^\nu - \not{k}_1\gamma^\nu)\gamma^\mu \not{p}_1 \gamma_\nu \not{k}_2 \gamma_\mu] = 2\mathrm{Tr}[\not{p}_2 \gamma^\mu \not{p}_1 \not{k}_1 \not{k}_2 \gamma_\mu]$

$- \mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \gamma^\mu (2p_1^\nu - \gamma^\nu \not{p}_1)\not{k}_2 \gamma_\mu] = 2\mathrm{Tr}[\gamma_\mu (2p_2^\mu - \gamma^\mu \not{p}_2)\not{p}_1 \not{k}_1 \not{k}_2]$

$- 2\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \gamma^\mu \not{k}_2 \gamma_\mu] + \mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \gamma^\mu \gamma_\nu \not{p}_1 \not{k}_2 \gamma_\mu]$

$= 4\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_1 \not{k}_2] + 4\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] + 2\mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \not{k}_2 \gamma_\nu \not{p}_1]$

$- \mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \gamma^\mu \gamma_\nu \not{p}_1 \gamma_\mu \not{k}_2]$

$= 16\big((p_2k_1)(p_1k_2) - (p_1p_2)(k_1k_2) + \cancel{(p_2k_2)(k_1p_1)} - (p_1p_2)(k_1k_2) - \cancel{(p_2k_2)(p_1k_1}$

$+ (p_2k_1)(p_1k_2)\big) - 4\,\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{k}_2 \not{p}_1]$

$- 2\mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \not{p}_1 \gamma_\nu \not{k}_2] + \mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu \gamma^\mu \gamma_\nu \gamma_\mu \not{p}_1 \not{k}_2]$

$= 16\big(2(p_2k_1)(p_1k_2) - 2(p_1p_2)(k_1k_2) - (p_2k_1)(k_2p_1) + (p_2k_2)(p_2p_1) - (p_2p_1)(k_1$

$+ 4\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] + \mathrm{Tr}[\not{p}_2 \not{k}_1 \gamma^\nu (2g^\mu{}_\nu - \gamma_\nu \gamma^\mu)\gamma_\mu \not{p}_1 \not{k}_2]$

$= 16\cdot\big(2(p_2k_1)(p_1k_2) - 3(p_1p_2)(k_1k_2) + (p_2k_2)(p_2p_1) + (p_2k_2)(k_1p_1) - (p_2p_1)(p_1k_2)$

$+ 8\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] - 16\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2]$

~~M~~ Mejor sin sustituir las trazas:

$$t_{4a} = -4\,\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{k}_1 \not{k}_2] + 4\,\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] - 4\,\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{k}_2 \not{p}_1]$$

$$+ 4\,\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] - 8\,\mathrm{Tr}[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2\,]$$

$\not{a}\not{b} = 2ab - \not{b}\not{a}$ ciclica

$$= -8(p_2p_1)\,\mathrm{Tr}[\not{k}_1\not{k}_2] + 4\,\mathrm{Tr}[\not{p}_1\not{p}_2\not{k}_1\not{k}_2] - 4\,\mathrm{Tr}[\not{p}_1\not{p}_2\not{k}_1\not{k}_2]$$

$$+ 8\,\mathrm{Tr}[\not{p}_2\not{k}_1\not{p}_1\not{k}_2] - 8\,\mathrm{Tr}[\not{p}_2\not{k}_1\not{p}_1\not{k}_2] = -32(p_1p_2)(k_1k_2)$$

$$t_{4b} = \mathrm{Tr}[\gamma^\nu \not{k}_1 \gamma^\mu \gamma_\nu \not{k}_2 \gamma_\mu] = 2\,\mathrm{Tr}[\gamma^\nu \not{k}_1 \not{k}_2 \gamma_\nu] - \mathrm{Tr}[\gamma^\nu \not{k}_1 \underbrace{\gamma^\mu \gamma_\nu \gamma_\mu}_{=-2\gamma_\nu} \not{k}_2]$$

$$= 8\,\mathrm{Tr}[\not{k}_1\not{k}_2] + 2\,\mathrm{Tr}[\underbrace{\gamma^\nu \not{k}_1 \gamma_\nu}_{-2\not{k}_1} \not{k}_2] = 4\,\mathrm{Tr}[\not{k}_1\not{k}_2] = 16(k_1k_2)$$

$$t_4 = -(32(p_1p_2)(k_1k_2) + m^2 16(k_1k_2))$$

$$T_2 = 16m^2(p_1p_2) + 16m^4 + 32(p_1p_2)(p_1k_2) - 16m^2(p_2k_2) - 32m^2(k_2p_1)$$

$$- 32(k_1p_1)(p_2p_1) + 16m^2(k_1p_2) + 32m^2(p_1k_1) + 32(p_1p_2)(k_1k_2) - 16m^2(k_1k_2)$$

Uso que: $k_1k_2 = p_1p_2 - m^2$; $k_1p_2 = k_2p_1$; $k_1p_1 = k_2p_2$; $p_1p_2 = m^2 + k_1p_1 - k_1p_1$

$$= 16m^4 + 16m^2((p_1p_2) - (p_2k_2) - 2(k_2p_1) + (k_1p_2) + 2(p_1k_1) - \overset{\overset{(p_1p_2 - m^2)}{\shortparallel}}{(k_1k_2)})$$

$$+ 32((p_1p_2)(p_1k_2) - (k_1p_1)(p_2p_1) + (p_1p_2)(k_1k_2))$$

$$= 16m^4 + 16m^2((p_1k_1) - (k_1p_2))$$

$$+ 32((p_1p_2)((p_1k_2) - (p_1k_1) + \underset{p_1p_2 - m^2}{\underset{\shortparallel}{(k_1k_2)}}))$$

$$= 32m^4 + 16m^2((k_1p_1) - (k_1p_2)) + 32((p_1p_2)(p_1(\underbrace{k_2 - k_1 + p_2}_{=p_1}))) - 32m^2(p_1p_2)$$

$=m^2$ $=0$

$$= 32m^4 + 16m^2(k_1p_1 - k_1p_2) //$$

$$\hookrightarrow T_2 = -16m^2(2m^2 + k_1p_1 - k_1p_2) = 16m^2(k_1p_2 - k_1p_1 - 2m^2)$$

Por simetría, $T_3 = T_2(k_1 \leftrightarrow -k_2,\ \mu \leftrightarrow \nu) = 16m^2(-k_2p_2 + k_2p_1 - 2m^2)$

$$= 16m^2((p_2k_1) - k_1p_1 - 2m^2) = T_2$$

Juntando $T_2 + T_3 = 32m^2(k_2p_1 - k_1p_1 - 2m^2) //$

Juntando todo:

$$A = \frac{e^4}{16}\left\{32\,\frac{(k_1p_1)(k_2p_1) + m^2(k_1p_1) + m^4}{(p_1k_1)^2} + 32m^2\,\frac{(k_2p_1) - (k_1p_1) - 2m^2}{(p_1k_2)(p_1k_1)}\right.$$

$$\left. + 32\,\frac{(k_1p_1)(k_2p_1) - m^2(k_2p_1) + m^4}{(p_1k_2)^2}\right\}$$

$$= 2e^4\left\{\frac{(k_2p_1)}{(k_1p_1)} + \frac{(k_1p_1)}{(k_2p_1)} + m^2\left(\frac{1}{(k_1p_1)} - \frac{1}{(k_2p_1)} + \frac{m^2}{(p_1k_1)^2} + \frac{m^2}{(k_2p_1)^2}\right.\right.$$

$$+ \frac{1}{\ } - \frac{1}{\ } - \frac{2m^2}{(\ )(\ k_1)}\ \}$$

$$= 2e^4\left\{ \frac{}{(k_1 p_1)} + \frac{}{(k_2 p_1)} + m\left(\frac{}{(k_1p_1)^2} + \frac{}{(k_2p_1)^2} - \frac{}{(k_1p_1)(k_2p_1)} + \frac{}{(k_1p_1)} \quad \frac{}{(k_2p_1)}\right)\right.$$

$$= 2e^4\left\{ \;'' \;+\; '' + m^2\left(\left[\frac{1}{(k_1p_1)} - \frac{1}{(k_2p_1)}\right]^2 m^2 + 2\left[\frac{1}{(k_1p_1)} - \frac{1}{(k_2p_1)}\right] + \underbrace{\frac{1}{m^2} - \frac{1}{m^2}}_{''0}\right)\right]$$

$$= 2e^4\left\{ \;'' \;+\; '' + m^2\left[\left(\frac{1}{(k_1p_1)} - \frac{1}{(k_2p_1)}\right)m + \frac{1}{m}\right]^2 - 1\right\}$$

$$A = 2e^4\left\{\frac{(k_2p_1)}{(k_1p_1)} + \frac{(k_1p_1)}{(k_2p_1)} + \left[1 + \frac{m^2}{(k_1p_1)} - \frac{m^2}{(k_2p_1)}\right]^2 - 1\right\}$$

Calcularemos la sección eficaz en el S. LAB definido por el e en reposo:

$p_1^{\mu} = (m, \vec{0})$

$$\sigma = \frac{1}{2\lambda^{1/2}(s,m^2,0)} \cdot \int dQ_2 \cdot A \qquad \lambda^{1/2}(s,m^2,0) = \sqrt{s^2+m^4-2sm^2} = (s-m^2)$$

$$s = (p_1+k_1)^2 = m^2 + 2mE_1 \rightarrow (s-m^2) = 2mE_1$$

$$\hookrightarrow \int dQ_2 = \frac{1}{(2\pi)^{3\cdot 2-4}}\int \frac{d^3\vec{p}_2}{2p_2^0}\frac{d^3\vec{k}_2}{2k_2^0}\,\delta^{(4)}(p_1+k_1-p_2-k_2)$$

$$= \frac{1}{(2\pi)^2\cdot 4}\int \frac{d^3\vec{p}_2}{p_2^0}\frac{d^3\vec{k}_2}{k_2^0}\cdot\delta(m+\underset{E_1}{\overset{\shortparallel}{k_1^0}}-p_2^0-\underset{E_2}{\overset{\shortparallel}{k_2^0}})\cdot\delta^{(3)}(\vec{k}_1-\vec{p}_2-\vec{k}_2)$$

↳ Sustituir en A:

① $\vec{p}_2 = \vec{k}_1 - \vec{k}_2$

$$= \frac{1}{(2\pi)^2 4}\int \frac{d^3\vec{k}_2}{E_2\, p_2^0}\,\delta(m+E_1-p_2^0-E_2) =$$

↳ en esféricas $d^3\vec{k}_2 = |\vec{k}_2|^2\, d|\vec{k}_2|\, d(\cos\theta)\, d\varphi$
(θ: ángulo entre $\vec{k}_1$ y $\vec{k}_2$) $\quad |\vec{k}_2| \overset{\shortparallel}{} E_2$ (porque masa fotón es nula)

$$= \frac{1}{(2\pi)^2 4}\int \frac{E_2^2\, dE_2\, d(\cos\theta)\, d\varphi}{E_2\, p_2^0}\,\delta(m+E_1-E_2-p_2^0)$$

② sustituir $p_2^0 = m + E_1 - E_2$ en A

$$= \frac{1}{8\pi}\int d\cos\theta\, \frac{E_2\, dE_2}{\sqrt{m^2+(\vec{k}_1-\vec{k}_2)^2}}\,\delta(m+E_1-E_2-\sqrt{m^2+\underset{E_1^2+E_2^2-2E_1E_2\cos\theta}{\underbrace{(\vec{k}_1-\vec{k}_2)^2}}})$$

Transformar la δ:

$$\delta(f(x)) = \frac{\delta(x-a)}{\left|\frac{\partial f}{\partial x}\right|_{x=a}} \quad \text{con } f(a) = 0$$

$$\hookrightarrow m + E_1 - a = \sqrt{m^2+E_1^2+a^2-2E_1 a\cos\theta}$$

$$\Downarrow (\;)^2$$

$$\cancel{m^2} + \cancel{E_1^2} + \cancel{a^2} + 2mE_1 - 2ma - 2E_1 a$$
$$\shortparallel \; \cancel{m^2} + \cancel{E_1^2} + \cancel{a^2} - 2E_1 a\cos\theta$$

$$\Downarrow$$

$$a(\cancel{2}m + \cancel{2}E_1 - \cancel{2}E_1\cos\theta) = \cancel{2}mE_1$$

$$\Downarrow$$

$$\delta(f(E_2)) = \frac{\delta(E_2-a)}{\left|\frac{\partial f}{\partial E_2}\right|_{E_2=a}}$$

$$\downarrow$$

$$\frac{\partial f}{\partial E_2} = -1 - \frac{1\,(2E_2-2E_1\cos\theta)}{2\sqrt{E_1^2+E_2^2-2E_1E_2\cos\theta}} \quad ; \quad a = \frac{mE_1}{m+E_1(1-\cos\theta)} \quad \left(\text{Condición de Compton}\right)$$

$$\left.\frac{\partial f}{\partial E_2}\right|_{E_2=a} = -1 - \frac{\dfrac{mE_1 - mE_1\cos\theta - E_1^2(1-\cos\theta)\cos\theta}{m+E_1(1-\cos\theta)}}{m+E_1 - \dfrac{mE_1}{m+E_1(1-\cos\theta)}}$$

$$= -1 - \frac{mE_1(1-\cos\theta) - E_1^2(1-\cos\theta)\cos\theta}{m^2+E_1m+\cancel{E_1m}+E_1^2(1-\cos\theta)-\cancel{mE_1}}$$

$$= -1 - (1-\cos\theta)\frac{mE_1 - E_1^2(-\cos\theta)}{mE_1+E_1^2(1-\cos\theta)+m^2}$$

$$= -\left(\frac{mE_1 + E_1^2(1-\cos\theta)+m^2 - mE_1(1-\cos\theta) - E_1^2\cos\theta(1-\cos\theta)}{mE_1+E_1^2(1-\cos\theta)+m^2}\right)$$

$$= -\left(\frac{E_1^2 - E_1^2\cos\theta + m^2 + mE_1\cos\theta - E_1^2\cos\theta + E_1^2\cos^2\theta}{mE_1+E_1^2(1-\cos\theta)+m^2}\right)$$

$$= -\left(\frac{E_1(E_1(1-\cos\theta)+m\cos\theta - E_1\cos\theta(1-\cos\theta))+m^2}{mE_1+E_1^2(1-\cos\theta)+m^2}\right) =$$

NO

$\Downarrow$

$\chi = \left|\frac{\partial f_2}{\partial E_2}\right|$ Era más fácil ↙

$$Q_2 = \frac{1}{8\pi}\int d\cos\theta\,\frac{E_2\,dE_2}{m+E_1-a}\cdot\frac{\delta(E_2-a)}{\chi}$$

↘ la ⌐ equivale a esto

$$\hookrightarrow (m+E_1-a)\chi = (m+E_1-a)\left(1+\frac{E_2-E_1\cos\theta}{m+E_1-a}\right)$$

$$= (m+E_1-a+a-E_1\cos\theta) = (E_1(1-\cos\theta)+m) = mE_1/a$$

$$Q_2 = \frac{1}{8\pi}\int d\cos\theta\,\frac{a^2\,\delta(E_2-a)}{mE_1}$$

$\Downarrow$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4mE_1}\,\frac{1}{mE_1}\,E_2^2\,\frac{1}{8\pi}\cdot A$$

complicando ① y ②

y con $E_2 = a = \dfrac{E_1}{1+\frac{E_1}{m}(1-\cos\theta)}$

$A \Rightarrow \quad p_1k_1 = mE_1$

$p_1k_2 = mE_2$

$4\pi\alpha = e^2$

$$= \frac{1}{32\pi}\,\frac{E_2^2}{E_1^2}\,\frac{1}{m^2}\cdot 2e^4\left\{\frac{E_2}{E_1}+\frac{E_1}{E_2}+\left[1+\frac{m}{E_1}-\frac{m}{E_2}\right]^2-1\right\} =$$

$$= \frac{\pi\alpha^2}{m^2}\,\frac{E_2^2}{E_1^2}\cdot\left\{\frac{E_2}{E_1}+\frac{E_1}{E_2}+\left[\underbrace{1+\frac{m-m-E_1(1-\cos\theta)}{E_1}}_{=\cos\theta}\right]^2-1\right\}$$

$\cos^2\theta \qquad \sin^2\theta$

$\Downarrow$

$$\boxed{\left.\frac{d\sigma}{d\cos\theta}\right|_{LAB} = \frac{\pi\alpha^2}{m^2}\cdot\frac{E_2^2}{E_1^2}\cdot\left[\frac{E_2}{E_1}+\frac{E_1}{E_2}-\sin^2\theta\right]}$$

Fórmula de Klein-Nishina (no polarizada)
(Mandl 8.74)